Chapter 1

Elliptic operators in $L^p(\mathbb{R}^N)$: characterization of the domain

In this chapter we consider the following linear second order elliptic operator in divergence form

$$Au := A_0 u + \langle F, Du\rangle - Vu, \tag{1.0.1}$$

where

$$A_0 u := \sum_{i,j=1}^{N} D_i(q_{ij} D_j u)\,.$$

As usual, we will refer to F and V as the drift and the potential term, respectively, and neither F nor V will be assumed to be bounded.

Our aim is to prove a generation result for A in $L^p(\mathbb{R}^N)$ $(1 < p < +\infty)$ with respect to the Lebesgue measure, providing an explicit description of the domain of the generator. Precisely, we show that such a domain is the intersection of the domains of each addend of A in (1.0.1).

This problem is classical and well-known in the case of elliptic operators with regular and bounded coefficients. We refer to the book of Lunardi [32] for a detailed analysis of the subject. On the other hand, there are several approaches to show that elliptic operators with unbounded coefficients generate strongly continuous semigroups in L^p (see [11], [12], [19], [35], [37], [41] and the list of references therein), but only some of them give a precise description of the domain. Besides, in some cases the problem is investigated only for $p = 2$ (see [17], [18] and in [50]).

Here we prove that if $(\mathcal{D}_p, \|\cdot\|_{\mathcal{D}_p})$, with $1 < p < +\infty$, is the Banach space defined as

$$\mathcal{D}_p := \{u \in W^{2,p}(\mathbb{R}^N) \,:\, \langle F, Du\rangle \in L^p(\mathbb{R}^N), Vu \in L^p(\mathbb{R}^N)\}\,,$$
$$\|u\|_{\mathcal{D}_p} := \|u\|_{W^{2,p}(\mathbb{R}^N)} + \|\langle F, Du\rangle\|_{L^p(\mathbb{R}^N)} + \|Vu\|_{L^p(\mathbb{R}^N)}\,,$$

then $(A, \mathcal{D}_p)$ generates a C_0-semigroup in $L^p(\mathbb{R}^N)$, if suitable growth conditions on F, V and their first order derivatives are assumed. As a by-product, one can deduce regularity results for the solution of the elliptic equation associated with A.

The precise description of the domain relies on a priori estimates of the form

$$\|u\|_{2,p} + \|\langle F, Du\rangle\|_p + \|Vu\|_p \le C(\|u\|_p + \|Au\|_p), \tag{1.0.2}$$

for every $p \in (1,\infty)$ and every test function u and for some constant $C > 0$ independent of u. We prove the estimates for $\|Vu\|_p$ and $\|Du\|_p$ using basically integrations by parts and other elementary tools. In the particular case $p = 2$, we also get an estimate for the second order derivatives of u (see Section 1.3). For $p \neq 2$, the variational method fails to estimate $\|D^2u\|_p$ and we have to employ a different technique, which works under stronger assumptions. This is done in Section 1.4, where we use an a priori estimate for the second order derivatives in the case where the involved operator has globally Lipschitz drift coefficient and bounded potential term (we prove such an estimate together with a generation result as a preliminary step in Section 1.2). Once the second order derivatives are estimated, the last term $\|\langle F, Du\rangle\|_p$ in (1.0.2) can be estimated easily by difference.

Using a density argument, (1.0.2) turns out to be true also for functions in $\mathcal{D}_p$. As a consequence, we establish the closedness of $(A, \mathcal{D}_p)$ in $L^p(\mathbb{R}^N)$. Moreover, it is easily seen that $(A, \mathcal{D}_p)$ is quasi dissipative in $L^p(\mathbb{R}^N)$. Therefore, in order to apply the Hille-Yosida generation theorem and to get the desired result, it remains to prove that $\lambda - A$ is surjective from $\mathcal{D}_p$ onto $L^p(\mathbb{R}^N)$, for λ large. Sections 1.5 and 1.6 are devoted to this aim. We proceed differently in the case $p = 2$ and $p \neq 2$. In the first case, we find the solution of the equation $\lambda u - Au = f$ in the whole space as the limit of a sequence of solutions of the same equation in balls with increasing radii and Dirichlet boundary conditions. In the second case, we check the surjectivity of $\lambda - A$ by approximating A with a family of operators whose drift coefficient is globally Lipschitz and whose potential term is bounded. We note that, once again, the first method works under weaker assumptions and this is the reason why we treat the case $p = 2$ separately.

Finally, in Section 1.7 we describe some properties of the above semigroups. We prove that they are positive, not analytic in general, consistent with respect to p. Moreover if V tends to $+\infty$ as $|x| \to +\infty$, then $(A, \mathcal{D}_p)$ has compact resolvent.

1.1 Assumptions and statement of the main results

In the following $q(x) = (q_{ij}(x))$ is a $N \times N$ symmetric real matrix such that $q_{ij} \in C_b^1(\mathbb{R}^N)$ and

(1.1.1) $$\langle q(x)\xi, \xi\rangle := \sum_{i,j=1}^{N} q_{ij}(x)\, \xi_i\xi_j \geq \nu_0|\xi|^2, \qquad \nu_0 > 0,$$

for every $x, \xi \in \mathbb{R}^N$. Moreover, we consider $F \in C^1(\mathbb{R}^N; \mathbb{R}^N)$ and $V \in C^1(\mathbb{R}^N)$ and we assume that V is bounded from below. Without loss of generality, we suppose that $V \geq 1$. We deal with the elliptic operator

(1.1.2) $$Au := A_0u + \langle F, Du\rangle - Vu,$$

where $A_0u(x) := \sum_{i,j=1}^{N} D_i(q_{ij}(x)D_ju(x))$.

For $1 < p < +\infty$, we define the space $(\mathcal{D}_p, \|\cdot\|_{\mathcal{D}_p})$ as

(1.1.3) $$\mathcal{D}_p := \{u \in W^{2,p}(\mathbb{R}^N) : \langle F, Du\rangle \in L^p(\mathbb{R}^N), Vu \in L^p(\mathbb{R}^N)\},$$

(1.1.4) $$\|u\|_{\mathcal{D}_p} := \|u\|_{2,p} + \|\langle F, Du\rangle\|_p + \|Vu\|_p .$$

We endow $\mathcal{D}_p$ also with the graph norm of the operator A, namely

$$\|u\|_A := \|Au\|_p + \|u\|_p .$$

In the case $p = 2$, besides the previous assumptions on the coefficients, we require that the following growth conditions hold

(H1) $|DV| \leq \alpha V^{3/2} + c_\alpha$, (H2) $\mathrm{div}F+\beta V\geq -c_\beta\,,\quad \sum_{i,j=1}^N D_iF_j(x)\xi_i\xi_j\leq \tau\, V(x)|\xi|^2+c_\tau|\xi|^2,\quad \xi,x\in\mathbb{R}^N,$

(H3) $\langle F,DV\rangle+\gamma V^2\geq -c_\gamma,$

(H4) $|F(x)|\leq \theta(1+|x|^2)^{1/2}V(x)+c_\theta\ ,$

with $\alpha,\beta,\gamma,\tau,\theta>0$ and $c_\alpha,c_\beta,c_\gamma,c_\tau,c_\theta\geq 0$ satisfying

$$1-\frac{\beta}{2}-\tau>0\,, \tag{1.1.5}$$

and

$$\frac{M}{4}\alpha^2+\frac{\beta}{2}+\frac{\gamma}{2}<1\,, \tag{1.1.6}$$

where $M:=\sup_{x\in\mathbb{R}^N}\max_{|\xi|=1}\langle q(x)\xi,\xi\rangle$. We note that the second inequality in (H2) is a dissipativity condition for the function F.

The following generation result holds.

Theorem 1.1.1 (p=2) *Suppose that (H1), (H2), (H3), (H4), (1.1.5) and (1.1.6) hold. Then the operator $(A,\mathcal{D}_2)$ generates a C_0-semigroup on $L^2(\mathbb{R}^N)$. If $c_\beta=0$, then the semigroup is contractive.*

In Section 1.6 we prove an analogous result in the general case $p>1$. To this aim we use a different technique, which works under more restrictive assumptions on the coefficients of A. Precisely, we replace assumptions (H1), (H2) and (H4) with the following ones

(H1') $|DV(x)|\leq \alpha\dfrac{V^{2-\sigma}(x)}{(1+|x|^2)^{\mu/2}},$

(H2') $|DF|\leq \frac{1}{\sqrt{N}}(\beta V+c_\beta),$

(H4') $|F(x)|\leq\theta(1+|x|^2)^{\mu/2}V^\sigma(x),$

respectively, where DF denotes the Jacobian matrix of F and $|DF|^2=\sum_{k,i=1}^N|D_kF_i|^2$, $\alpha,\beta,\theta>0$, $c_\beta\geq 0$, $\frac{1}{2}\leq\sigma\leq 1$ and $0\leq\mu\leq 1$. Moreover, we suppose that for every $x\in\mathbb{R}^N$

(H5) $|\langle F(x),Dq_{ij}(x)\rangle|\leq\kappa\, V(x)+c_\kappa,$

holds, with constants $\kappa>0$ and $c_\kappa\geq 0$.

Analogously to the case $p=2$, also in this case a smallness condition on the coefficients is required. Let

$$\omega:=\begin{cases}\frac{M}{4}(p-1)\alpha^2\,, & \text{if } (\sigma,\mu)=\left(\frac{1}{2},0\right),\\ 0\,, & \text{otherwise.}\end{cases}$$

Then we assume that

$$\begin{aligned} &\omega+\sqrt{2}\,\frac{\beta+\sqrt{N}\alpha\theta}{p}+\alpha\theta\frac{p-1}{p}<1\,, && \text{if } 1<p<2\,,\\ &\omega+\sqrt{2}\left(\beta+\sqrt{N}\alpha\theta\right)\left(\frac{1}{p}+\frac{1}{\sqrt{N}}\right)<1\,, && \text{if } p\geq 2\,. \end{aligned} \tag{1.1.7}$$

The following generation result holds.

Theorem 1.1.2 (1<p<+∞) *Suppose that (H1'), (H2'), (H4'), (H5) and (1.1.7) are satisfied, for some* $1 < p < \infty$. *Then the operator* $(A, \mathcal{D}_p)$ *generates a* C_0*-semigroup on* $L^p(\mathbb{R}^N)$*, which turns out to be contractive if* $c_\beta = 0$.

Remark 1.1.3 We observe that (1.1.7) for $p \geq 2$ implies (1.1.7) for $1 < p < 2$, since

$$\sqrt{2}\,\frac{\beta + \sqrt{N}\alpha\theta}{p} + \alpha\theta\frac{p-1}{p} \leq \sqrt{2}\left(\beta + \sqrt{N}\alpha\theta\right)\left(\frac{1}{p} + \frac{1}{\sqrt{N}}\right), \qquad p > 1\,.$$

Moreover, we note that when $p = 2$, (1.1.7) is not equivalent to (1.1.6), but it is stronger. This fact relies on the different technique employed in the general case and, in particular, on the fact that we need that other suitable operators verify our assumptions. For further details we refer to Section 1.6. In any case, the two methods yield the same semigroup in $L^2(\mathbb{R}^N)$.
Finally, we point out that in Theorem 1.1.2 we do not explicitly assume (H3), since (H1') and (H4') imply

$$|\langle F, DV\rangle| \leq \alpha\theta V^2\,. \tag{1.1.8}$$

Remark 1.1.4 Hypothesis (H1) is essential to determine the domain. In fact in [41, Example 3.7] the authors exhibit a Schrödinger operator $A = \Delta - V$ in $L^2(\mathbb{R}^3)$ such that (H1) holds with a too large constant α and the domain is not $W^{2,2}(\mathbb{R}^3) \cap D(V)$. Moreover in [41] it is observed that (H1) holds for example for any polynomial whose homogenous part of maximal degree is positive definite. (H1) fails for the function $U = 1 + x^2y^2$.

Remark 1.1.5 We note that making particular choices of the parameters μ and σ, we may cover cases already known or discuss new ones. For example, if $\mu = 0$ and $\sigma = \frac{1}{2}$, then we get exactly the framework of [41]

$$|F| \leq \theta V^{1/2}, \qquad |DV| \leq \alpha V^{3/2}$$

and therefore of [12]. If we take V constant, then we reduce to the case where F is globally Lipschitz continuous studied in [37]. Setting $\mu = 0$ and $\sigma = 1$ we have the case

$$|F| \leq \theta V, \qquad |DV| \leq \alpha V,$$

which, according to our knowledge, seems to be new. From the second condition above, one deduces that V grows at most exponentially. In particular, we can treat in this way polynomials V as in Remark 1.1.4.

If we optimize assumption (H4') choosing $\mu = \sigma = 1$, analogously to (H4) in the case $p = 2$, then (H1') becomes $|DV(x)| \leq \alpha\frac{V(x)}{(1+|x|^2)^{1/2}}$, which is much more restrictive than (H1). This shows that the cases $p = 2$ and $p \neq 2$ are quite different. Such a difference is also confirmed by the fact that when $p = 2$ we do not require any condition on $\langle Dq_{ij}, F\rangle$.

The assumptions for $p \neq 2$ are determined by our approach to estimate the second order derivatives of a test function u in terms of u and Au. The idea is to get first local estimates. To this aim we change variables and localize the equation $Au = f$ in certain balls $B(x_0, r(x_0))$. The new operator produced by this technique (see (1.4.14)) has a globally Lipschitz continuous drift term and a bounded potential. The radius $r(x_0)$ has to grow at most linearly with respect to $|x_0|$ in order to use a covering argument and to obtain global estimates. So, roughly speaking, we must require that $r(x_0) \leq 1 + |x_0|$ and that $V(x)$ is "close" to $V(x_0)$ if $|x - x_0| < r(x_0)$. This is exactly guaranteed by assumptions (H4') (see (1.4.2)) and (H1') (see Lemma 1.4.3). The Lipschitz continuity of the transformed drift coefficient follows from (H2'). All the details are given in Section 1.6.

1.2 Operators with globally Lipschitz drift coefficient and bounded potential term

In this section we collect all the results concerning operators with globally Lipschitz drift coefficient and bounded potential term that will be used in the sequel. We first prove an a priori estimate for the second order derivatives of a test function u, using the same technique of [37] but specifying how the constants involved depend on the operator. Then, we show a generation result, giving an explicit description of the domain.

Let

$$B = \sum_{i,j=1}^{N} D_i(a_{ij}D_j) + \sum_{i=1}^{N} b_iD_i - c \tag{1.2.1}$$

and assume that

(i) $a_{ij} = a_{ji} \in C_b^1(\mathbb{R}^N)$, $\quad \sum_{i,j=1}^{N} a_{ij}\xi_i\xi_j \geq \nu_0|\xi|^2$,

(ii) $b = (b_1, ..., b_N)$ is globally Lipschitz in $\mathbb{R}^N$,

(iii) $c \in L^\infty(\mathbb{R}^N)$,

(iv) $\sup_{x\in\mathbb{R}^N} |\langle Da_{ij}(x), b(x)\rangle| < +\infty\,,\quad i,j = 1,...,N.$

The following a priori estimate is a crucial point for our aims.

Theorem 1.2.1 *There exists a constant $C > 0$ depending on $p, N, \nu_0, \|a_{ij}\|_\infty, \|Da_{ij}\|_\infty, \|\langle Da_{ij}, b\rangle\|_\infty$, $\|c\|_\infty$ and the Lipschitz constant of b, denoted by $[b]_1$, such that for all $u \in C_c^\infty(\mathbb{R}^N)$*

$$\int_{\mathbb{R}^N} |D^2u|^p\,dx \leq C\int_{\mathbb{R}^N} (|Bu|^p + |u|^p)\,dx. \tag{1.2.2}$$

Proof. We split the proof in two steps.

Step 1. We assume that the operator B is written in the non-divergence form

$$B = \sum_{i,j=1}^{N} a_{ij}D_{ij} + \sum_{i=1}^{N} b_iD_i - c$$

and that $b \in C^2(\mathbb{R}^N;\mathbb{R}^N)$ with bounded first and second order derivatives, besides assumptions (i), (ii), (iii) and (iv).

Let $u \in C_c^\infty(\mathbb{R}^N)$. Then u solves the equation

$$D_tu - Bu = f \quad \text{in } \mathbb{R}^{N+1},$$

with $f = -Bu$. Let us consider the ordinary Cauchy problem in $\mathbb{R}^N$

$$\begin{cases} \dfrac{d\xi}{dt} = b(\xi), & t \in \mathbb{R} \\ \xi(0) = x. \end{cases} \tag{1.2.3}$$

Since b is globally Lipschitz, for all $x \in \mathbb{R}^N$ there is a unique global solution $\xi(t,x)$ of (1.2.3) and the identity

$$x = \xi(t, \xi(-t,x)), \qquad t \in \mathbb{R},\ x \in \mathbb{R}^N \tag{1.2.4}$$

holds. Moreover, from [36, Section 2.1] it follows that if ξ_x denotes the Jacobian matrix of the derivatives of ξ with respect to x, then

$$
\begin{aligned}
&|\xi_x(t,x)| \le e^{|t|\,\|Db\|_\infty}, \qquad t\in\mathbb{R},\ x\in\mathbb{R}^N\\
(1.2.5)\qquad &|\xi_{tx}(t,x)| \le \|Db\|_\infty e^{|t|\,\|Db\|_\infty}, \qquad t\in\mathbb{R},\ x\in\mathbb{R}^N\\
&\left|\frac{\partial}{\partial t}\xi_x(t,\xi(-t,x))\right| \le \|Db\|_\infty e^{3|t|\,\|Db\|_\infty}, \qquad t\in\mathbb{R},\ x\in\mathbb{R}^N.
\end{aligned}
$$

With analogous notation we have also that
(1.2.6)

$$
\begin{aligned}
&|\xi_{xx}(t,x)| \le e^{|t|\,\|Db\|_\infty}(e^{|t|\|Db\|_\infty}-1)\frac{\|D^2b\|_\infty}{\|Db\|_\infty}, \qquad t\in\mathbb{R},\ x\in\mathbb{R}^N\\
&\left|\frac{\partial}{\partial x_i}\xi_x(t,\xi(-t,x))\right| \le e^{3|t|\,\|Db\|_\infty}(e^{|t|\|Db\|_\infty}-1)\frac{\|D^2b\|_\infty}{\|Db\|_\infty}, \qquad t\in\mathbb{R},\ x\in\mathbb{R}^N,\ i=1,...N.
\end{aligned}
$$

In the case where b is constant, one should replace $\frac{e^{|t|\|Db\|_\infty}-1}{\|Db\|_\infty}$ by $|t|$. In particular, all the above functions are bounded in $[-T,T]\times\mathbb{R}^N$, for every $T>0$. Finally, the matrix ξ_x is invertible with determinant bounded away from zero in every strip $[-T,T]\times\mathbb{R}^N$.
Setting $v(t,y)=u(\xi(-t,y))$, a straightforward computation shows that

$$D_t v - \widetilde{B}v = \widetilde{f}, \quad \text{in } \mathbb{R}^{N+1}$$

with $\widetilde{f}(t,y)=f(\xi(-t,y))$ and

$$
\begin{aligned}
\widetilde{B} &= \sum_{i,j=1}^N \widetilde{a}_{ij}(t,y)D_{y_iy_j} + \sum_{i=1}^N \widetilde{b}_i(t,y)D_{y_i} - \widetilde{c},\\
\widetilde{a}_{ij}(t,y) &= \sum_{h,k=1}^N D_{x_h}\xi_i(t,\xi(-t,y))a_{hk}(\xi(-t,y))D_{x_k}\xi_j(t,\xi(-t,y))\\
\widetilde{b}_i(t,y) &= \sum_{h,k=1}^N D_{x_hx_k}\xi_i(t,\xi(-t,y))a_{hk}(\xi(-t,y)),\\
\widetilde{c}(t,y) &= c(\xi(-t,y)).
\end{aligned}
$$

Since the coefficients a_{ij} belong to $C_b^1(\mathbb{R}^N)$ and satisfy (iv), then $(t,y)\to a_{ij}(\xi(-t,y))$ is bounded and differentiable with bounded derivatives in $[-T,T]\times\mathbb{R}^N$. Taking into account (1.2.5) and (1.2.6) it follows that for all $(t,y)\in[-T,T]\times\mathbb{R}^N$ we have

$$|\widetilde{a}_{ij}(t,y)| + |D_t\widetilde{a}_{ij}(t,y)| + |D_{y_k}\widetilde{a}_{ij}(t,y)| + |\widetilde{b}_i(t,y)| \le L, \quad i,j,k=1,...N,$$

where L depends on $T, N, \|a_{ij}\|_\infty, \|Da_{ij}\|_\infty, \|\langle Da_{ij},b\rangle\|_\infty, \|Db\|_\infty, \|D^2b\|_\infty$. Moreover

$$\sum_{i,j=1}^N \widetilde{a}_{ij}(t,y)\eta_i\eta_j \ge \widetilde{\nu}_0|\eta|^2, \quad \eta,\, y\in\mathbb{R}^N,\, t\in[-T,T],$$

with $\widetilde{\nu}_0$ depending on $\nu_0, T, \|Db\|_\infty$. Finally, the modulus of continuity of $\widetilde{a}_{ij}$ depends only on $T, N, \|a_{ij}\|_\infty, \|Da_{ij}\|_\infty, \|\langle Da_{ij},b\rangle\|_\infty, \|Db\|_\infty, \|D^2b\|_\infty$. Therefore $D_t-\widetilde{B}$ is a uniformly parabolic operator in $[-T,T]\times\mathbb{R}^N$, for every $T>0$. Applying the classical L^p-estimates available from the theory of uniformly parabolic operators (see e.g. [30, Section IV.10]) we have that

$$(1.2.7)\qquad \int_{-1/2}^{1/2}\int_{\mathbb{R}^N}(|D_yv(t,y)|^p + |D_y^2v(t,y)|^p)dy\,dt \le K\int_{-1}^{1}\int_{\mathbb{R}^N}(|\widetilde{f}(t,y)|^p + |v(t,y)|^p)dy\,dt$$

where K depends on $p, N, \widetilde{\nu}_0, \|\widetilde{a}_{ij}\|_\infty, \|D\widetilde{a}_{ij}\|_\infty, \|D_t\widetilde{a}_{ij}\|_\infty, \|\widetilde{b}_i\|_\infty, \|\widetilde{c}\|_\infty$, hence on p, N, ν_0, $\|a_{ij}\|_\infty$, $\|Da_{ij}\|_\infty$, $\|\langle Da_{ij}, b\rangle\|_\infty, \|Db\|_\infty, \|D^2b\|_\infty, \|c\|_\infty$.
In order to come back to the function u, we observe that, setting $(S(t)\varphi)(x) = \varphi(\xi(t,x))$ then, for every fixed t, $S(t)$ maps $W^{2,p}(\mathbb{R}^N)$ into itself and

$$\begin{aligned}
\int_{\mathbb{R}^N} |(S(t)\varphi)(x)|^p dx &\leq \alpha_1(t) \int_{\mathbb{R}^N} |\varphi(y)|^p dy, \\
\int_{\mathbb{R}^N} |D_x(S(t)\varphi)(x)|^p dx &\leq \alpha_2(t) \int_{\mathbb{R}^N} |D_y\varphi(y)|^p dy, \\
\int_{\mathbb{R}^N} |D_x^2(S(t)\varphi)(x)|^p dx &\leq \alpha_3(t) \int_{\mathbb{R}^N} (|D_y^2\varphi(y)|^p + |D_y\varphi(y)|^p) dy,
\end{aligned}$$

with $\alpha_1(t), \alpha_2(t), \alpha_3(t)$ depending on $t, p, N, \sup_{\mathbb{R}^N} |\xi_x(-t,\cdot)|$ and $\alpha_3(t)$ depending also on $\sup_{\mathbb{R}^N}$ $|\xi_{xx}(-t,\cdot)|$. It follows that $t \mapsto \alpha_i(t)$, $i = 1,2,3$, are uniformly bounded in t in the interval $[-1,1]$. In the sequel we denote by α_i the respective upper bounds. Moreover, by (1.2.4) each $S(t)$ is invertible with $S(t)^{-1} = S(-t)$. Now, recalling that $u = S(t)v$, for every t, we have

$$\int_{\mathbb{R}^N} |D_x^2 u(x)|^p dx \leq \alpha_3 \int_{\mathbb{R}^N} (|D_y^2 v(t,y)|^p + |D_y v(t,y)|^p) dy.$$

Integrating from $-1/2$ to $1/2$ and taking into account (1.2.7) we obtain

$$\begin{aligned}
\int_{\mathbb{R}^N} |D_x^2 u(x)|^p dx &\leq \alpha_3\, K \int_{-1}^{1} \int_{\mathbb{R}^N} (|\widetilde{f}(t,y)|^p + |v(t,y)|^p) dy\, dt \\
&\leq 2\,\alpha_1\alpha_3 K \int_{\mathbb{R}^N} (|f(x)|^p + |u(x)|^p) dx,
\end{aligned}$$

which is the claim.

Step 2. Take B in the general form (1.2.1) and assume that the coefficients satisfy $(i), (ii), (iii)$ and (iv). Then we can write

$$B = \sum_{i,j=1}^N a_{ij} D_{ij} + \sum_{j=1}^N \left(\sum_{i=1}^N D_i a_{ij} + b_j \right) D_j - c.$$

Let $\eta \in C_c^\infty(\mathbb{R}^N)$, supp $\eta \subset B_1$, $\eta \geq 0$, $\int_{\mathbb{R}^N} \eta = 1$ and set $\hat{b} = b * \eta$. If we define

$$\hat{B} = \sum_{i,j=1}^N a_{ij} D_{ij} + \sum_{j=1}^N \hat{b}_j D_j - c,$$

then $\hat{B}$ satisfies all the assumptions of the previous step. Indeed, since b is Lipschitz continuous, $b - \hat{b}$ is bounded:

$$|b(x) - \hat{b}(x)| \leq \int_{\mathbb{R}^N} |b(x) - b(x-y)|\eta(y) dy \leq [b]_1 \int_{\mathbb{R}^N} |y|\eta(y) dy = c_\eta [b]_1.$$

Then

$$\begin{aligned}
|\langle Da_{ij}(x), \hat{b}(x)\rangle| &\leq |\langle Da_{ij}(x), b(x)\rangle| + |\langle Da_{ij}(x), b(x) - \hat{b}(x)\rangle| \\
&\leq \|\langle Da_{ij}, b\rangle\|_\infty + \|Da_{ij}\|_\infty c_\eta [b]_1,
\end{aligned}$$

and

$$\begin{aligned}
&\|D\hat{b}\|_\infty \leq [b]_1 \\
&\|D^2\hat{b}\|_\infty \leq [b]_1 \|D\eta\|_1.
\end{aligned}$$

From the first step it follows that there exists a constant $C > 0$ depending on N,p,ν_0,$\|a_{ij}\|_\infty$, $\|Da_{ij}\|_\infty$, $\|\langle a_{ij}, b\rangle\|_\infty$,$[b]_1$,$\|c\|_\infty$ such that for all $u \in C_c^\infty(\mathbb{R}^N)$

$$\|D^2u\|_p \leq C(\|\hat{B}u\|_p + \|u\|_p).$$

Therefore

$$\|D^2u\|_p \leq C(\|Bu\|_p + \|Bu - \hat{B}u\|_p + \|u\|_p) \leq C_1(\|Bu\|_p + \|Du\|_p + \|u\|_p),$$

with C_1 depending on the stated quantities. Using the interpolatory estimate $\|Du\|_p \leq C_2\|u\|_p^{1/2} \cdot \|D^2u\|_p^{1/2}$ we conclude the proof. □

Next, we show that the operator B endowed with the domain

$$\mathcal{D} = \{u \in W^{2,p}(\mathbb{R}^N) \,:\, \langle b, Du\rangle \in L^p(\mathbb{R}^N)\}$$

generates a C_0-semigroup on $L^p(\mathbb{R}^N)$, $1 < p < +\infty$ (see also [37]). The following lemma is useful (see [37, Lemma 2.1]).

Lemma 1.2.2 *Let $1 < p < +\infty$ and $u \in W^{2,p}(B_R) \cap W_0^{1,p}(B_R)$. If $\eta \in C^1(\overline{B}_R)$ is nonnegative, then*

$$(p-1)\int_{B_R} \eta|u|^{p-2} \sum_{i,j=1}^N a_{ij}D_iuD_ju\; \chi_{\{u\neq 0\}} + \int_{B_R} u|u|^{p-2} \sum_{i,j=1}^N a_{ij}D_iuD_j\eta \leq -\int_{B_R} \eta u|u|^{p-2} \sum_{i,j=1}^N D_i(a_{ij}D_ju). \tag{1.2.8}$$

Proof. Suppose first $p \geq 2$. In this case the function $u|u|^{p-2}$ belongs to $W^{1,q}(B_R)$, where q is the conjugate exponent of p. Indeed, it is obvious that $u|u|^{p-2}$ is in $L^q(B_R)$. Concerning the first order derivatives, we have $D(u|u|^{p-2}) = (p-1)|u|^{p-2}Du$. Then, using Hölder's inequality with exponent $\frac{p}{q} \geq 1$ we get

$$\begin{aligned}\int_{B_R} |u|^{q(p-2)}|Du|^q &\leq \left(\int_{B_R} |Du|^p\right)^{\frac{q}{p}} \left(\int_{B_R} |u|^{\frac{pq(p-2)}{p-q}}\right)^{1-\frac{q}{p}} \\ &= \left(\int_{B_R} |Du|^p\right)^{\frac{q}{p}} \left(\int_{B_R} |u|^p\right)^{1-\frac{q}{p}}.\end{aligned}$$

Therefore, integration by parts is allowed in the right hand side of (1.2.8) and the statement is verified with equality.

Assume now $1 < p < 2$. Let first $u \in C^2(\overline{B}_R) \cap C_0(B_R)$. For every $\delta > 0$ we have

$$\begin{aligned}-\int_{B_R} \eta\, u(u^2+\delta)^{\frac{p}{2}-1} \sum_{i,j=1}^N D_i(a_{ij}D_ju) =&\int_{B_R} \eta(u^2+\delta)^{\frac{p}{2}-2}((p-1)u^2+\delta) \sum_{i,j=1}^N a_{ij}D_iuD_ju \\ &+\int_{B_R} u(u^2+\delta)^{\frac{p}{2}-1} \sum_{i,j=1}^N a_{ij}D_iuD_j\eta.\end{aligned} \tag{1.2.9}$$

Then, from Fatou's Lemma we have

$$
\begin{aligned}
&(p-1)\int_{B_R}\eta|u|^{p-2}\sum_{i,j=1}^N a_{ij}D_iuD_ju\ \chi_{\{u\neq 0\}}\\
&\le \liminf_{\delta\to 0}\Big(-\int_{B_R}\eta\, u(u^2+\delta)^{\frac{p}{2}-1}\sum_{i,j=1}^N D_i(a_{ij}D_ju)-\int_{B_R}u(u^2+\delta)^{\frac{p}{2}-1}\sum_{i,j=1}^N a_{ij}D_iuD_j\eta\Big)\\
&=-\int_{B_R}\eta\, u|u|^{p-2}\sum_{i,j=1}^N D_i(a_{ij}D_ju)-\int_{B_R}u|u|^{p-2}\sum_{i,j=1}^N a_{ij}D_iuD_j\eta.
\end{aligned}
$$

It follows that the function $\eta|u|^{p-2}\sum_{i,j=1}^N a_{ij}D_iuD_ju\ \chi_{\{u\neq 0\}}$ belongs to $L^1(B_R)$ and, letting $\delta\to 0$ in (1.2.9), by dominated convergence (1.2.8) holds with equality. In the general case where $u\in W^{2,p}(B_R)\cap W_0^{1,p}(B_R)$, we can consider a sequence (u_n) in $C^2(\overline{B}_R)\cap C_0(B_R)$ such that u_n converges to u in $W^{2,p}(B_R)$. In particular, we can find a subsequence (u_{n_k}) and functions $h_1,h_2,h_3\in L^p(B_R)$ such that $u_{n_k}, Du_{n_k}, D^2u_{n_k}$ converge to u, Du and D^2u, respectively, almost everywhere and

$$
\begin{aligned}
|u_{n_k}(x)| &\le h_1(x),\\
|Du_{n_k}(x)| &\le h_2(x),\\
|D^2u_{n_k}(x)| &\le h_3(x)
\end{aligned}
$$

(see [10, Teorema IV.9]. Taking the previous step into account and applying again Fatou's Lemma, we get

$$
\begin{aligned}
&(p-1)\int_{B_R}\eta|u|^{p-2}\sum_{i,j=1}^N a_{ij}D_iuD_ju\ \chi_{\{u\neq 0\}}\\
&\le \liminf_{k\to+\infty}\Big(-\int_{B_R}\eta\, u_{n_k}|u_{n_k}|^{p-2}\sum_{i,j=1}^N D_i(a_{ij}D_ju_{n_k})\\
&\qquad -\int_{B_R}u_{n_k}|u_{n_k}|^{p-2}\sum_{i,j=1}^N a_{ij}D_iu_{n_k}D_j\eta\Big).
\end{aligned}\tag{1.2.10}
$$

Using Young's inequality one has

$$
\begin{aligned}
\Big|u_{n_k}|u_{n_k}|^{p-2}\sum_{i,j=1}^N D_i(a_{ij}D_ju_{n_k})\Big| &\le c_1|u_{n_k}|^{p-1}\Big(|Du_{n_k}|+|D^2u_{n_k}|\Big)\\
&\le c_2\Big(|u_{n_k}|^p+\big(|Du_{n_k}|+|D^2u_{n_k}|\big)^p\Big)\\
&\le c_3\Big(|u_{n_k}|^p+|Du_{n_k}|^p+|D^2u_{n_k}|^p\Big)\\
&\le c_3\Big(h_1^p+h_2^p+h_3^p\Big)\in L^1(B_R),
\end{aligned}
$$

where c_3 depends on $\|a_{ij}\|_\infty, \|Da_{ij}\|_\infty$ and p. In the same way, one can estimate the remaining term, hence estimate (1.2.8) follows from (1.2.10) using dominated convergence. □

Proposition 1.2.3 $(B,\mathcal{D})$ *generates a strongly continuous semigroup* $T(t)$ *in* $L^p(\mathbb{R}^N)$, $1<p<\infty$. *Moreover, setting* $\lambda_p := -\inf_{x\in\mathbb{R}^N}\Big(\frac{1}{p}\operatorname{div} b(x)+c(x)\Big)$, *for all* $\lambda>\lambda_p$ *and* $f\in L^p(\mathbb{R}^N)$, *there exists a unique solution* $u\in\mathcal{D}$ *of* $\lambda u-Bu=f$ *and the estimate*

$$\|u\|_p\le(\lambda-\lambda_p)^{-1}\|f\|_p \tag{1.2.11}$$

is satisfied.

PROOF. It is sufficient to prove the statement when c is equal to zero, since in the general case, the thesis easily follows from a perturbation argument (see [21, III.1.3]).

Let us consider $(B, C_c^\infty(\mathbb{R}^N))$. Proceeding as in the forthcoming Lemma 1.3.1, it can be proved that $C_c^\infty(\mathbb{R}^N)$ is dense in $\mathcal{D}$ with respect to its natural norm

$$\|u\|_{\mathcal{D}} = \|u\|_{2,p} + \|\langle b, Du\rangle\|_p.$$

The interpolatory estimate $\|Du\|_p \le k(\|u\|_p + \|D^2u\|_p)$ and estimate (1.2.2) yield immediately

$$\|Du\|_p \le C(\|u\|_p + \|Bu\|_p), \quad u \in C_c^\infty(\mathbb{R}^N).$$

Therefore, we have

$$\|\langle b, Du\rangle\|_p = \left\| \sum_{i,j=1}^N D_i(a_{ij}D_ju) - Bu \right\|_p \le C(\|D^2u\|_p + \|Du\|_p + \|Bu\|_p) \le C(\|u\|_p + \|Bu\|_p).$$

Collecting all the estimates so far, we have established that for every $u \in C_c^\infty(\mathbb{R}^N)$, hence, by density, for every $u \in \mathcal{D}$

$$\|u\|_{2,p} + \|\langle b, Du\rangle\|_p \le C(\|u\|_p + \|Bu\|_p).$$

Since the other inequality is obvious, we have that $\|\cdot\|_{\mathcal{D}}$ and the graph norm of B, $\|\cdot\|_B$, are equivalent. Therefore, $(\mathcal{D}, \|\cdot\|_B)$ is complete and as a consequence $(B, \mathcal{D})$ is closed in $L^p(\mathbb{R}^N)$.

Let us prove that $(B - \lambda_0, C_c^\infty(\mathbb{R}^N))$ is dissipative in $L^p(\mathbb{R}^N)$, where

$$\lambda_0 = -\frac{1}{p} \inf_{\mathbb{R}^N} \operatorname{div} b.$$

In this case, we say that $(B, C_c^\infty(\mathbb{R}^N))$ is *quasi-dissipative*. Let $\lambda > \lambda_0$ and $u \in C_c^\infty(\mathbb{R}^N)$ be fixed. Multiplying the equation $\lambda u - Bu = f$ by $u|u|^{p-2}$ and integrating by parts we deduce

$$\lambda \int_{\mathbb{R}^N} |u|^p dx + (p-1) \int_{\mathbb{R}^N} |u|^{p-2} \sum_{i,j=1}^N a_{ij} D_iuD_ju \, dx + \frac{1}{p}\int_{\mathbb{R}^N} \operatorname{div} b\, |u|^p dx = \int_{\mathbb{R}^N} f\, u|u|^{p-2} dx$$

and then

$$\begin{aligned}(\lambda - \lambda_0)\int_{\mathbb{R}^N} |u|^p dx \;&\le\; \int_{\mathbb{R}^N} \left(\lambda + \frac{1}{p}\operatorname{div} b\right)|u|^p dx + \nu_0(p-1)\int_{\mathbb{R}^N} |Du|^2|u|^{p-2}dx \\ &\le\; \left(\int_{\mathbb{R}^N} |f|^p dx\right)^{\frac{1}{p}} \left(\int_{\mathbb{R}^N} |u|^p dx\right)^{1-\frac{1}{p}}.\end{aligned}$$

Dividing by $\|u\|_p^{p-1}$ we get $(\lambda - \lambda_0)\|u\|_p \le \|\lambda u - Bu\|_p$, as claimed. Therefore, the operator $(B, C_c^\infty(\mathbb{R}^N))$ is quasi-dissipative.

The next step is to show that $(\lambda - B)C_c^\infty(\mathbb{R}^N)$ is dense in $L^p(\mathbb{R}^N)$ for some large λ. Let q be the conjugate exponent of p and let $w \in L^q(\mathbb{R}^N)$ be such that

$$\int_{\mathbb{R}^N} (\lambda\varphi - B\varphi) w\, dx = 0, \qquad \forall\, \varphi \in C_c^\infty(\mathbb{R}^N). \tag{1.2.12}$$

We claim that $w = 0$. By a classical result concerning local regularity of distributional solutions to elliptic equations (see [5] and the references therein), it turns out that $w \in W^{2,q}_{\mathrm{loc}}(\mathbb{R}^N)$. Therefore we are allowed to integrate by parts in (1.2.12) and we deduce that

$$\int_{\mathbb{R}^N} \lambda w\, \varphi\, dx - \int_{\mathbb{R}^N} \sum_{i,j=1}^N D_i(a_{ij}D_jw)\varphi\, dx + \int_{\mathbb{R}^N} \operatorname{div} b\, w\, \varphi\, dx + \int_{\mathbb{R}^N} \langle b, Dw\rangle \varphi\, dx = 0. \tag{1.2.13}$$

Using an approximation argument, it can be seen that the equation in this form is satisfied also by any function φ of $L^p(\mathbb{R}^N)$ with compact support. Indeed, if φ is such a function, set $\varphi_n = \varrho_n * \varphi$, where ϱ_n is a standard sequence of mollifiers. Then $\varphi_n \in C_c^\infty(\mathbb{R}^N)$ and φ_n converges to φ in $L^p(\mathbb{R}^N)$, as $n \to \infty$. Moreover, we can find $R > 0$ sufficiently large in such a way that $\text{supp}\varphi_n$ and $\text{supp}\varphi$ are contained in B_R, for every $n \in \mathbb{N}$. Each φ_n satisfies (1.2.13) and letting $n \to \infty$, we obtain that φ verifies (1.2.13), too.

Now, let η be in $C_c^\infty(\mathbb{R}^N)$ such that $\eta \equiv 1$ in B_1, $0 \le \eta \le 1$, $\eta \equiv 0$ in $\mathbb{R}^N \setminus B_2$ and set $\eta_n(x) = \eta(\frac{x}{n})$. Plugging $w|w|^{q-2}\,\eta_n^2$ into (1.2.13) and using (1.2.8) we deduce

$$\begin{aligned}
&\int_{\mathbb{R}^N} \lambda |w|^q\,\eta_n^2 + (p-1)\int_{\mathbb{R}^N} \eta_n^2 |w|^{q-2} \sum_{i,j=1}^N a_{ij} D_i w D_j w\ \chi_{\{w\neq 0\}}\\
&\qquad +2\int_{\mathbb{R}^N} w|w|^{q-2}\eta_n \sum_{i,j=1}^N a_{ij} D_i w D_j \eta_n + \int_{\mathbb{R}^N} \text{div} b\,|w|^q\,\eta_n^2 + \int_{\mathbb{R}^N} \langle b, Dw\rangle w|w|^{q-2}\,\eta_n^2\\
&\quad \le \int_{\mathbb{R}^N} \lambda |w|^q\,\eta_n^2 - \int_{\mathbb{R}^N} \sum_{i,j=1}^N D_i(a_{ij}D_j w) w|w|^{q-2}\,\eta_n^2 + \int_{\mathbb{R}^N} \text{div} b\,|w|^q\,\eta_n^2\\
&\qquad + \int_{\mathbb{R}^N} \langle b, Dw\rangle w|w|^{q-2}\,\eta_n^2 = 0.
\end{aligned}$$

Then, using the ellipticity condition and integrating by parts we get

$$\begin{aligned}
&\int_{\mathbb{R}^N} \lambda |w|^q\,\eta_n^2 + \nu_0(p-1)\int_{\mathbb{R}^N} \eta_n^2 |w|^{q-2}|Dw|^2\ \chi_{\{w\neq 0\}} + 2\int_{\mathbb{R}^N} w|w|^{q-2}\eta_n \sum_{i,j=1}^N a_{ij} D_i w D_j \eta_n\\
&\qquad + \int_{\mathbb{R}^N} \text{div} b\,|w|^q\,\eta_n^2 - \frac{1}{q}\int_{\mathbb{R}^N} \text{div} b\,|w|^q\,\eta_n^2 - \frac{2}{q}\int_{\mathbb{R}^N} \langle b, D\eta_n\rangle |w|^q\,\eta_n \le 0.
\end{aligned}$$

Therefore

$$\int_{\mathbb{R}^N} \left(\lambda + \frac{1}{p}\,\text{div} b\right) |w|^q\,\eta_n^2 + \nu_0(p-1)\int_{\mathbb{R}^N} \eta_n^2 |w|^{q-2}|Dw|^2\ \chi_{\{w\neq 0\}} \le I_1 + I_2, \tag{1.2.14}$$

where

$$\begin{aligned}
I_1 &= -2\int_{\mathbb{R}^N} w|w|^{q-2}\eta_n \sum_{i,j=1}^N a_{ij} D_i w D_j \eta_n\,dx\\
I_2 &= \frac{2}{q}\int_{\mathbb{R}^N} \langle b, D\eta_n\rangle |w|^q\,\eta_n\,dx.
\end{aligned}$$

From Hölder's inequality it follows that

$$\begin{aligned}
|I_1| &\le 2\,NK \int_{\mathbb{R}^N} \eta_n\,|Dw|\,|D\eta_n|\,|w|^{q-1}\,dx\\
&\le \frac{2\,N\,\|D\eta\|_\infty K}{n} \int_{\mathbb{R}^N} \eta_n\,|Dw|\,|w|^{(q-2)/2}|w|^{q/2}\ \chi_{\{w\neq 0\}}\,dx \qquad (1.2.15)\\
&\le \frac{\|D\eta\|_\infty\,NK}{n} \int_{\mathbb{R}^N} \eta_n^2 |Dw|^2 |w|^{q-2}\ \chi_{\{w\neq 0\}}\,dx + \frac{\|D\eta\|_\infty\,NK}{n} \int_{\mathbb{R}^N} |w|^q\,dx,
\end{aligned}$$

where $K = \max_{i,j}\|a_{ij}\|_\infty$. Concerning I_2, we observe that since b is Lipschitz continuous in $\mathbb{R}^N$, there exists a constant $L > 0$ such that $|b(x)| \le L(1 + |x|)$, for every $x \in \mathbb{R}^N$. Therefore

$$\begin{aligned}
|I_2| &\le \frac{2}{q}\int_{n\le|x|\le 2n} \eta_n(x)|b(x)|\,|D\eta_n(x)|\,|w(x)|^q\,dx \qquad (1.2.16)\\
&\le \frac{2\|D\eta\|_\infty L}{q}\int_{n\le|x|\le 2n} \frac{(1+|x|)}{n}|w(x)|^q\,dx\\
&\le \frac{6\,\|D\eta\|_\infty L}{q}\int_{n\le|x|\le 2n} |w(x)|^q\,dx.
\end{aligned}$$

Taking (1.2.15) and (1.2.16) into account, (1.2.14) gives

$$\int_{\mathbb{R}^N}\left(\lambda+\frac{1}{p}\,\mathrm{div}b\right)|w|^q\,\eta_n^2+\left(\nu_0(p-1)-\frac{\|D\eta\|_\infty\,N\,K}{n}\right)\int_{\mathbb{R}^N}\eta_n^2|w|^{q-2}|Dw|^2\;\chi_{\{w\neq0\}}$$
$$\leq\frac{\|D\eta\|_\infty\,N\,K}{n}\int_{\mathbb{R}^N}|w|^q\,dx+\frac{6\,\|D\eta\|_\infty L}{q}\int_{n\leq|x|\leq2n}|w|^q\,dx.$$

For n large $\nu_0(p-1)-\frac{\|D\eta\|_\infty\,N\,K}{n}>0$ and if $\lambda>\lambda_0$ we have

$$(\lambda-\lambda_0)\int_{\mathbb{R}^N}|w|^q\,\eta_n^2\leq\frac{\|D\eta\|_\infty\,N\,K}{n}\int_{\mathbb{R}^N}|w|^q\,dx+\frac{6\,\|D\eta\|_\infty L}{q}\int_{n\leq|x|\leq2n}|w|^q\,dx.$$

Letting $n\to+\infty$ we infer $w=0$.

From the Lumer Phillips Theorem [21, Theorem II.3.15] it follows that the closure $(\mathcal{B},D(\mathcal{B}))$ of $(B,\,C_c^\infty(\mathbb{R}^N))$ generates a strongly continuous semigroup in $L^p(\mathbb{R}^N)$. Since $(B,\mathcal{D})$ is closed and $C_c^\infty(\mathbb{R}^N)\subseteq\mathcal{D}$, we find that $(B,\mathcal{D})$ extends $(\mathcal{B},D(\mathcal{B}))$. Conversely, if $f\in\mathcal{D}$, then there exists a sequence (f_n) in $C_c^\infty(\mathbb{R}^N)$ such that f_n converges to f with respect to $\|\cdot\|_{\mathcal{D}}$, which is equivalent to $\|\cdot\|_B$. This implies, by definition, that $f\in D(\mathcal{B})$ and $Bf=\mathcal{B}f$. Therefore $(\mathcal{B},D(\mathcal{B}))$ coincides with $(B,\mathcal{D})$.

As far as the last part of the statement is concerned, we observe that as a consequence of the generation result, for λ large, the resolvent equation $\lambda u-Bu=f$ admits a unique solution $u\in\mathcal{D}$, for every $f\in L^p(\mathbb{R}^N)$. In order to determine the lower bound of λ, as before we have to multiply the equation $\lambda u-Bu=f$ by $u|u|^{p-2}$ and to integrate by parts. In this way we find that λ has to be strictly larger than $\lambda_p=-\inf\left(\frac{1}{p}\,\mathrm{div}\,b+c\right)$ and that estimate (1.2.11) holds, as stated. □

1.3 A priori estimates of $\|Vu\|_p$, $\|Du\|_p$ and $\|D^2u\|_2$

From now on, for clarity of exposition, we assume that $c_\alpha=c_\beta=c_\gamma=c_\tau=c_\theta=0$ in conditions (H1), (H2), (H3) and (H4). This is always possible, keeping the same constants α,β,γ,τ, just replacing V with $V+\lambda$ and choosing λ large enough (this implies possibly different constants in the statements).
In this section we provide, as a preliminary step, some a priori estimates for the solutions of the elliptic equation $\lambda u-Au=f$. Precisely, via integrations by parts and other elementary tools, we prove that for all $u\in\mathcal{D}_p$, the L^p-norms of Vu and Du may be estimated by the L^p-norms of Au and u itself, with constants independent of u. If $p=2$, we also deduce an analogous estimate for the second order derivatives of u.

Let us first show that $C_c^\infty(\mathbb{R}^N)$ is dense in $(\mathcal{D}_p,\|\cdot\|_{\mathcal{D}_p})$, $1<p<+\infty$, so that all our estimates will be proved on test-functions.

Lemma 1.3.1 *Suppose that (H4) holds. Then $C_c^\infty(\mathbb{R}^N)$ is dense in $(\mathcal{D}_p,\|\cdot\|_{\mathcal{D}_p})$.*

Proof. Let η be a cut-off function such that $0\leq\eta\leq1$, $\eta\equiv1$ in B_1, $\mathrm{supp}\,\eta\subset\mathrm{B}_2$ and $|D\eta|^2+|D^2\eta|\leq L$. We write $\eta_n(x)$ in place of $\eta(x/n)$.

Suppose that $u\in\mathcal{D}_p$. It is easy to see that $\|\eta_nu-u\|_{\mathcal{D}_p}$, as $n\to\infty$. In fact, $\eta_nu\to u$ in $W^{2,p}(\mathbb{R}^N)$ and $V\eta_nu\to Vu$ in $L^p(\mathbb{R}^N)$, by dominated convergence. Moreover,

$$\langle F,D(\eta_nu)\rangle=\eta_n\langle F,Du\rangle+u\langle F,D\eta_n\rangle\,.$$

As before, the first term in the right hand side converges to $\langle F, Du\rangle$ in $L^p(\mathbb{R}^N)$, as n goes to infinity. The second term tends to 0 since from (H4) it follows that

$$
\begin{aligned}
\int_{\mathbb{R}^N} |u|^p |\langle F, D\eta_n\rangle|^p\,dx &\le L^{p/2}\theta^p \int_{B_{2n}\setminus B_n} |Vu|^p \left(\frac{1+4n^2}{n^2}\right)^{p/2} dx \\
&\le 5^{p/2} L^{p/2}\,\theta^p \int_{\mathbb{R}^N\setminus B_n} |Vu|^p\,dx. \qquad (1.3.1)
\end{aligned}
$$

This shows that the set of functions in $\mathcal{D}_p$ having compact support, denoted by $\mathcal{D}_{p,c}$, is dense in $\mathcal{D}_p$.

Suppose now that $u \in \mathcal{D}_{p,c}$. A standard convolution argument shows the existence of a sequence of smooth functions with compact support converging to u in $\mathcal{D}_p$. Thus, the density of $C_c^\infty(\mathbb{R}^N)$ in $(\mathcal{D}_p, \|\cdot\|_{\mathcal{D}_p})$ follows. □

We state that, under rather weak assumptions, the operator $(A, C_c^\infty(\mathbb{R}^N))$ is dissipative in $L^p(\mathbb{R}^N)$, for any $1 < p < +\infty$.

Lemma 1.3.2 *Suppose that*

$$\operatorname{div} F + p\,V \ge 0. \qquad (1.3.2)$$

Then $(A, C_c^\infty(\mathbb{R}^N))$ *is dissipative in* $L^p(\mathbb{R}^N)$.

PROOF. We have to prove that for all $\lambda > 0$ and for all $u \in C_c^\infty(\mathbb{R}^N)$ one has

$$\|u\|_p \le \frac{1}{\lambda}\|\lambda u - Au\|_p. \qquad (1.3.3)$$

Let $\lambda > 0$ be fixed. If $u \in C_c^\infty(\mathbb{R}^N)$ we set $u^* = u|u|^{p-2}$ and recall that

$$D(u^*) = (p-1)|u|^{p-2}Du, \quad D(|u|^p) = pu^*Du\,. \qquad (1.3.4)$$

Set $\lambda u - Au = f$. Multiplying both sides of this equation by u^* and integrating by parts, we obtain

$$\lambda \int_{\mathbb{R}^N} |u|^p + (p-1)\int_{\mathbb{R}^N} \langle qDu, Du\rangle |u|^{p-2}\,dx + \frac{1}{p}\int_{\mathbb{R}^N} \operatorname{div} F |u|^p\,dx + \int_{\mathbb{R}^N} V|u|^p\,dx = \int_{\mathbb{R}^N} fu^*\,dx\,.$$

By (1.1.1) we get

$$(p-1)\int_{\mathbb{R}^N} \langle qDu, Du\rangle |u|^{p-2}\,dx \ge (p-1)\nu_0 \int_{\mathbb{R}^N} |Du|^2|u|^{p-2}\,dx \ge 0$$

and taking (1.3.2) into account it turns out that

$$\lambda \int_{\mathbb{R}^N} |u|^p \le \int_{\mathbb{R}^N} fu^*\,dx \le \left(\int_{\mathbb{R}^N} |f|^p\,dx\right)^{\frac{1}{p}} \left(\int_{\mathbb{R}^N} |u|^p\,dx\right)^{1-\frac{1}{p}}.$$

Multiplying by $\|u\|_p^{1-p}$ we get (1.3.3). □

Remark 1.3.3 It is noteworthy observing that if (1.3.2) holds, $1 < p \le 2$ and $u \in C_c^\infty(\mathbb{R}^N)$ then

$$\int_{\mathbb{R}^N} |Du|^p \le c\int_{\mathbb{R}^N} (|Au|^p + |u|^p)\,dx\,, \qquad (1.3.5)$$

where $c = c(\nu_0, p) > 0$. In fact, from the proof of Lemma 1.3.2, with $\lambda = 1$, we deduce that

$$(1.3.6) \qquad \begin{aligned} \int_{\mathbb{R}^N} |Du|^2 |u|^{p-2}\, dx &\le \frac{1}{\nu_0(p-1)} \left(\int_{\mathbb{R}^N} |u - Au|^p\, dx \right)^{\frac{1}{p}} \left(\int_{\mathbb{R}^N} |u|^p\, dx \right)^{1-\frac{1}{p}} \\ &\le c \int_{\mathbb{R}^N} (|Au|^p + |u|^p)\, dx\,, \end{aligned}$$

where $c = c(\nu_0, p) > 0$. If $p = 2$, we are done. If $1 < p < 2$, Young's inequality with exponent $2/p$ yields

$$\int_{\{u\neq 0\}} |Du|^p\, dx = \int_{\{u\neq 0\}} \left(|Du|^p |u|^{\frac{p(p-2)}{2}} \right) |u|^{-\frac{p(p-2)}{2}}\, dx \le c_p \int_{\{u\neq 0\}} (|Du|^2 |u|^{p-2} + |u|^p)\, dx$$

and (1.3.5) follows by (1.3.6).

Remark 1.3.4 We note that condition (H2'), with $c_\beta = 0$, together with (1.1.7) implies condition (1.3.2), so that Lemma 1.3.2 still holds. If $c_\beta \neq 0$, then the same computations of Lemma 1.3.2 show that $(A - \frac{c_\beta}{p}, C_c^\infty(\mathbb{R}^N))$ is dissipative in $L^p(\mathbb{R}^N)$, which means that operator $(A, C_c^\infty(\mathbb{R}^N))$ is quasi-dissipative. Explicitly, one has

$$(1.3.7) \qquad \|u\|_p \le \left(\lambda - \frac{c_\beta}{p} \right)^{-1} \|(\lambda - A)u\|_p\,, \quad u \in C_c^\infty(\mathbb{R}^N).$$

In the following lemma we prove an estimate of the L^p-norm of Vu.

Lemma 1.3.5 *Let $1 < p < +\infty$. Assume that (H1), (H3) and*

$$(1.3.8) \qquad \mathrm{div}F + \beta V \ge 0$$

hold with

$$(1.3.9) \qquad \frac{M}{4}(p-1)\alpha^2 + \frac{\beta}{p} + \gamma\frac{p-1}{p} < 1\,,$$

where $M := \sup_{x\in\mathbb{R}^N} \max_{|\xi|=1} \langle q(x)\xi, \xi\rangle$.
If $u \in C_c^\infty(\mathbb{R}^N)$, then

$$(1.3.10) \qquad \int_{\mathbb{R}^N} |Vu|^p\, dx \le c \int_{\mathbb{R}^N} (|Au|^p + |u|^p)\, dx$$

for some $c > 0$ depending only on p, M, ν_0 and on the constants in (H1), (H3) and (1.3.8).

Proof. Let $u \in C_c^\infty(\mathbb{R}^N)$. We recall that if $u^* = u|u|^{p-2}$, then (1.3.4) holds.
Integrating by parts one deduces

$$\begin{aligned} &\int_{\mathbb{R}^N} (A_0 u) V^{p-1} u^*\, dx = -\int_{\mathbb{R}^N} \langle qDu, D(V^{p-1}u^*)\rangle\, dx \\ &= -(p-1) \int_{\mathbb{R}^N} \langle qDu, Du\rangle V^{p-1} |u|^{p-2}\, dx - (p-1) \int_{\mathbb{R}^N} \langle qDu, DV\rangle V^{p-2} |u|^{p-2} u\, dx \end{aligned}$$

and

$$\begin{aligned} &\int_{\mathbb{R}^N} V^{p-1} \langle F, Du\rangle\, u^*\, dx = \frac{1}{p} \int_{\mathbb{R}^N} V^{p-1} \langle F, D(|u|^p)\rangle\, dx \\ &= -\frac{1}{p} \int_{\mathbb{R}^N} V^{p-1} \mathrm{div}F |u|^p\, dx - \frac{p-1}{p} \int_{\mathbb{R}^N} V^{p-2} \langle F, DV\rangle |u|^p\, dx\,. \end{aligned}$$

Thus, multiplying (1.1.2) by $V^{p-1}u^*$ and integrating, we obtain

$$(1.3.11)\qquad (p-1)\int_{\mathbb{R}^N}\langle qDu,Du\rangle V^{p-1}|u|^{p-2}\,dx+\int_{\mathbb{R}^N}|Vu|^p\,dx$$
$$=-\int_{\mathbb{R}^N}(Au)V^{p-1}u^*\,dx-\frac{1}{p}\int_{\mathbb{R}^N}V^{p-1}\mathrm{div}F|u|^p\,dx$$
$$-\frac{p-1}{p}\int_{\mathbb{R}^N}V^{p-2}\langle F,DV\rangle|u|^p\,dx-(p-1)\int_{\mathbb{R}^N}\langle qDu,DV\rangle V^{p-2}|u|^{p-2}u\,dx.$$

Now, assumptions (1.3.8) and (H3) imply

$$(1.3.12)\qquad -\int_{\mathbb{R}^N}V^{p-1}\mathrm{div}F|u|^p\,dx\le\beta\int_{\mathbb{R}^N}|Vu|^p\,dx$$

and

$$(1.3.13)\qquad -\int_{\mathbb{R}^N}V^{p-2}\langle F,DV\rangle|u|^p\,dx\le\gamma\int_{\mathbb{R}^N}|Vu|^p\,dx\,,$$

respectively.
By (1.1.1) and (H1) the last term in (1.3.11) can be estimated as follows

$$(1.3.14)\qquad \int_{\mathbb{R}^N}\langle qDu,DV\rangle V^{p-2}|u|^{p-2}u\,dx \le \int_{\mathbb{R}^N}\langle qDu,Du\rangle^{1/2}\langle qDV,DV\rangle^{1/2}V^{p-2}|u|^{p-1}\,dx$$
$$\le \alpha\sqrt{M}\int_{\mathbb{R}^N}\langle qDu,Du\rangle^{1/2}V^{p-1/2}|u|^{p-1}\,dx\,.$$

Setting $Q^2:=\int_{\mathbb{R}^N}\langle qDu,Du\rangle V^{p-1}|u|^{p-2}\,dx$ and $R^2:=\int_{\mathbb{R}^N}|Vu|^p\,dx$, from Hölder's inequality it follows

$$(1.3.15)\qquad \int_{\mathbb{R}^N}\langle qDu,Du\rangle^{1/2}V^{p-1/2}|u|^{p-1}\,dx\le QR.$$

Thus, collecting (1.3.11)–(1.3.14) we obtain

$$(p-1)Q^2+\left(1-\frac{\beta}{p}-\frac{\gamma(p-1)}{p}\right)R^2 \le \alpha(p-1)\sqrt{M}QR+\left|\int_{\mathbb{R}^N}(Au)V^{p-1}u^*\,dx\right|$$
$$\le (p-1)Q^2+\frac{(p-1)\alpha^2M}{4}R^2+\left|\int_{\mathbb{R}^N}(Au)V^{p-1}u^*\,dx\right|.$$

Since

$$\left|\int_{\mathbb{R}^N}(Au)V^{p-1}u^*\,dx\right|\le\int_{\mathbb{R}^N}|Au||Vu|^{p-1}\,dx\le\varepsilon R^2+c_\varepsilon\int_{\mathbb{R}^N}|Au|^p\,dx\,,$$

the thesis follows from (1.3.9) and by choosing ε small enough. □

The next result provides an L^p-estimate of $V|Du|$, with $p\ge 2$. In particular, since $V\ge 1$, it extends estimate (1.3.5) to the case $p>2$. We explicitly notice that we need a further assumption on F, namely the dissipativity condition.

Lemma 1.3.6 *Let $p\ge 2$. Assume that (H1), (H2), (H3) and (1.3.9) hold and that β satisfies also the inequality*

$$(1.3.16)\qquad 1-\frac{\beta}{p}-\tau>0\,.$$

If $u \in C_c^\infty(\mathbb{R}^N)$, *then*

$$\int_{\mathbb{R}^N} V|Du|^p\,dx + \int_{\mathbb{R}^N} |Du|^{p-2}|D^2u|^2\,dx \le c\int_{\mathbb{R}^N} (|Au|^p + |u|^p)\,dx\,, \tag{1.3.17}$$

with c *depending on* $N, p, \nu_0, \alpha, \beta, \tau, M, \|Dq_{ij}\|_\infty$.

Proof. We divide the proof in two steps: in the first step we consider the supplementary assumption that $q_{ij} \in C^2(\mathbb{R}^N)$, in the second one we remove this condition via an approximation procedure.

Step 1. Suppose that $q_{ij} \in C^2(\mathbb{R}^N) \cap C_b^1(\mathbb{R}^N)$, for every $1 \le i,j \le N$. Let $u \in C_c^\infty(\mathbb{R}^N)$ and define $f = \lambda u - Au$, with $\lambda > 0$ to be chosen later. With a fixed $k \in \{1,...,N\}$, we differentiate with respect to x_k, so that

$$\begin{aligned}\lambda D_k u - \sum_{i,j=1}^N D_i(D_k q_{ij} D_j u) - \sum_{i,j=1}^N D_i(q_{ij} D_{jk} u) - \sum_{i=1}^N D_k F_i D_i u \\ - \sum_{i=1}^N F_i D_{ik} u + u D_k V + V D_k u = D_k f.\end{aligned} \tag{1.3.18}$$

Multiplying (1.3.18) by $D_k u|Du|^{p-2}$, summing over $k = 1,...,N$ and integrating on $\mathbb{R}^N$ we get

$$\lambda\int_{\mathbb{R}^N} |Du|^p\,dx + I_1 + I_2 + I_3 + I_4 + I_5 + \int_{\mathbb{R}^N} V|Du|^p\,dx = \int_{\mathbb{R}^N} \langle Df, Du\rangle |Du|^{p-2}\,dx, \tag{1.3.19}$$

where

$$\begin{aligned}
I_1 &= -\int_{\mathbb{R}^N} \sum_{i,j,k=1}^N D_i(D_k q_{ij} D_j u)\, D_k u|Du|^{p-2}\,dx,\\
I_2 &= -\int_{\mathbb{R}^N} \sum_{i,j,k=1}^N D_i(q_{ij} D_{jk} u) D_k u|Du|^{p-2}\,dx,\\
I_3 &= -\int_{\mathbb{R}^N} \sum_{i,k=1}^N D_k F_i\; D_i u\; D_k u|Du|^{p-2}\,dx,\\
I_4 &= -\int_{\mathbb{R}^N} \sum_{i,k=1}^N F_i\; D_{ik} u\; D_k u|Du|^{p-2}\,dx,\\
I_5 &= \int_{\mathbb{R}^N} \langle DV, Du\rangle u|Du|^{p-2}\,dx\,.
\end{aligned}$$

Let us estimate the integrals above. Since $t \mapsto t|t|^{p-2}$ is in $C^1(\mathbb{R}^N;\mathbb{R}^N)$, integrating by parts and applying Hölder's and Young's inequalities we have

$$\begin{aligned}
|I_1| &= \Bigg|\int_{\mathbb{R}^N} \sum_{i,j,k=1}^N D_k q_{ij}\; D_j u D_{ik} u |Du|^{p-2} \\
&\quad + (p-2)\int_{\mathbb{R}^N} \sum_{i,j,k,h=1}^N D_k q_{ij}\; D_j u D_k u D_h u D_{ih} u |Du|^{p-4}\Bigg| \\
&\le c_1 \int_{\mathbb{R}^N} |Du|^{p-1}|D^2u|\,dx = c_1\int_{\mathbb{R}^N} |Du|^{p/2}(|Du|^{(p-2)/2}|D^2u|)\,dx \\
&\le \frac{c_1}{\varepsilon}\int_{\mathbb{R}^N} |Du|^p\,dx + c_1\,\varepsilon\int_{\mathbb{R}^N} |Du|^{p-2}|D^2u|^2\,dx\,,
\end{aligned}$$

where $c_1 = c_1(p, N, \|Dq_{ij}\|_\infty)$ and $\varepsilon > 0$ is arbitrary. Consequently

$$I_1 \geq -\frac{c_1}{\varepsilon}\int_{\mathbb{R}^N} |Du|^p\,dx - c_1\,\varepsilon\int_{\mathbb{R}^N} |Du|^{p-2}|D^2u|^2\,dx\,. \tag{1.3.20}$$

Assumption (1.1.1) allows to estimate the second integral, after an integration by parts; indeed

$$\begin{aligned} I_2 &= \int_{\mathbb{R}^N} \sum_{i,j,k=1}^N q_{ij}\; D_{jk}u\; D_{ik}u|Du|^{p-2}\,dx \\ &\quad + \frac{p-2}{4}\int_{\mathbb{R}^N} \sum_{i,j=1}^N q_{ij}\; D_j(|Du|^2)\,D_i(|Du|^2)|Du|^{p-4}\,dx \\ &\geq \nu_0 \int_{\mathbb{R}^N} |D^2u|^2|Du|^{p-2}\,dx + \nu_0\frac{p-2}{4}\int_{\mathbb{R}^N} \Big|D\big(|Du|^2\big)\Big|^2\,|Du|^{p-4}\,dx\,. \end{aligned}$$

Since the last term is nonnegative we deduce that

$$I_2 \geq \nu_0 \int_{\mathbb{R}^N} |Du|^{p-2}|D^2u|^2\,dx. \tag{1.3.21}$$

From (H2) it follows immediately that

$$I_3 \geq -\tau\int_{\mathbb{R}^N} V|Du|^p\,dx. \tag{1.3.22}$$

As far as I_4 is concerned, integrating by parts, it turns out that

$$\begin{aligned} I_4 &= \int_{\mathbb{R}^N} \sum_{i,k=1}^N D_iF_i\;(D_ku)^2\;|Du|^{p-2}\,dx + \int_{\mathbb{R}^N} \sum_{i,k=1}^N F_i\;D_ku\;D_{ik}u\;|Du|^{p-2}\,dx \\ &\quad +(p-2)\int_{\mathbb{R}^N} \sum_{i,k,h=1}^N F_i\;(D_ku)^2\;D_hu\;D_{ih}u\;|Du|^{p-4}\,dx \\ &= \int_{\mathbb{R}^N} \mathrm{div}F\,|Du|^p\,dx - I_4 - (p-2)I_4 \end{aligned}$$

which implies by (H2) that

$$I_4 = \frac{1}{p}\int_{\mathbb{R}^N} \mathrm{div}F\,|Du|^p\,dx \geq -\frac{\beta}{p}\int_{\mathbb{R}^N} V|Du|^p\,dx. \tag{1.3.23}$$

Applying (H1) and Young's inequality, we get

$$\begin{aligned} |I_5| &\leq \alpha\int_{\mathbb{R}^N} V^{\frac{3}{2}}|u||Du|^{p-1}\,dx = \alpha\int_{\mathbb{R}^N} (V\,|u|\,|Du|^{\frac{p-2}{2}})(V^{\frac{1}{2}}\;|Du|^{\frac{p}{2}})\,dx \\ &\leq \frac{\alpha}{\varepsilon}\int_{\mathbb{R}^N} |Vu|^2|Du|^{p-2}\,dx + \varepsilon\alpha\int_{\mathbb{R}^N} V|Du|^p\,dx \\ &\leq c_2\int_{\mathbb{R}^N} |Vu|^p\,dx + c_2\int_{\mathbb{R}^N} |Du|^p\,dx + \varepsilon\alpha\int_{\mathbb{R}^N} V|Du|^p\,dx \end{aligned}$$

with $c_2 = c_2(\varepsilon, p, \alpha)$. Then

$$I_5 \geq -c_2\int_{\mathbb{R}^N} |Vu|^p\,dx - c_2\int_{\mathbb{R}^N} |Du|^p\,dx - \varepsilon\alpha\int_{\mathbb{R}^N} V|Du|^p\,dx. \tag{1.3.24}$$

We are left to estimate the integral in the right hand side in (1.3.19). Integrating by parts and arguing as before we obtain

$$
\begin{aligned}
\left|\int_{\mathbb{R}^N}\langle Df,Du\rangle|Du|^{p-2}\,dx\right| &\le (p-1)\sum_{h,k=1}^N\int_{\mathbb{R}^N}|f|\,|Du|^{p-2}|D_{hk}u|\,dx\\
&= (p-1)\int_{\mathbb{R}^N}|f|\,|Du|^{\frac{p-2}{2}}|Du|^{\frac{p-2}{2}}\sum_{h,k=1}^N|D_{hk}u|\,dx\\
&\le c_3\int_{\mathbb{R}^N}|f|^2\,|Du|^{p-2}\,dx+\varepsilon(p-1)\int_{\mathbb{R}^N}|Du|^{p-2}|D^2u|^2\,dx\,,
\end{aligned}
$$

with $c_3=c_3(p,N,\varepsilon)$. Applying Young's inequality we have finally

$$
\begin{aligned}
\left|\int_{\mathbb{R}^N}\langle Df,Du\rangle|Du|^{p-2}\,dx\right| &\le c_4\int_{\mathbb{R}^N}|f|^p\,dx+c_4\int_{\mathbb{R}^N}|Du|^p\,dx\\
&\quad+\varepsilon(p-1)\int_{\mathbb{R}^N}|Du|^{p-2}|D^2u|^2\,dx\,,
\end{aligned}
\tag{1.3.25}
$$

with $c_4=c_4(p,N,\varepsilon)$. Collecting (1.3.20)–(1.3.25) from (1.3.19) we obtain

$$
\begin{aligned}
&\Big(\lambda-\frac{c_1}{\varepsilon}-c_2-c_4\Big)\int_{\mathbb{R}^N}|Du|^p\,dx\\
&+\Big(\nu_0-(c_1+p-1)\varepsilon\Big)\int_{\mathbb{R}^N}|Du|^{p-2}|D^2u|^2\,dx\\
&+\left(1-\frac{\beta}{p}-\tau-\varepsilon\alpha\right)\int_{\mathbb{R}^N}V|Du|^p\,dx\\
&\le c_2\int_{\mathbb{R}^N}|Vu|^p\,dx+c_4\int_{\mathbb{R}^N}|f|^p\,dx\,.
\end{aligned}
$$

From (1.3.16) and (1.3.10), choosing first a small ε and then a large λ, we deduce that

$$
\int_{\mathbb{R}^N}(|Du|^p+V|Du|^p)\,dx+\int_{\mathbb{R}^N}|Du|^{p-2}|D^2u|^2\,dx\le c\int_{\mathbb{R}^N}(|Au|^p+|u|^p)\,dx\,,
$$

where the constant c depends on $p,N,\nu_0,M,\|Dq_{ij}\|_\infty$ and the constants in (H1), (H2), (H3).

Step 2. Let φ be a standard mollifier and set, as usual, $\varphi_\varepsilon(x)=\varepsilon^{-N}\varphi\left(\frac{x}{\varepsilon}\right)$. If $q_{ij}^\varepsilon=q_{ij}*\varphi_\varepsilon$ and

$$
A^\varepsilon u=\sum_{i,j=1}^N D_i(q_{ij}^\varepsilon D_ju)+\langle F,Du\rangle-Vu\,,
$$

then by Step 1, noticing that $\|q_{ij}^\varepsilon\|_\infty\le\|q_{ij}\|_\infty$, $\|Dq_{ij}^\varepsilon\|_\infty\le\|Dq_{ij}\|_\infty$ and that (q_{ij}^ε) satisfy (1.1.1) with the same constant ν_0, it follows that

$$
\int_{\mathbb{R}^N}(|Du|^p+V|Du|^p)\,dx+\int_{\mathbb{R}^N}|Du|^{p-2}|D^2u|^2\,dx\le c\int_{\mathbb{R}^N}(|A^\varepsilon u|^p+|u|^p)\,dx\,,
$$

with c independent of ε. Since $\|A^\varepsilon u-Au\|_p\to 0$ as ε goes to 0, we get the thesis. □

1.4 A priori estimates of $\|D^2u\|_p$, $\|\langle F,Du\rangle\|_p$

In the present section, we estimate the L^p norm of the second order derivatives of a solution $u\in\mathcal{D}_p$ of $Au=f$, $f\in L^p(\mathbb{R}^N)$. The proof is more involved than that of the case $p=2$ given in Section 1.3, since the variational method fails. Thus, we employ a different technique, which works under more restrictive assumptions on the coefficients of A, precisely we replace assumptions (H1) and (H4) with (H1') and (H4'), respectively. As noticed in Section 1.1, these assumptions imply (1.1.8). Moreover, (H5) is assumed.

The estimate of the second order derivatives is proved in Proposition 1.4.5. The idea is to define, via a change of variables and a localization argument, a family of operators, say $\{A_{x_0}\}_{x_0\in\mathbb{R}^N}$, with a globally Lipschitz drift coefficient and a bounded potential term. Then we apply Theorem 1.2.1 to each A_{x_0} to obtain local estimates of the L^p-norm of the second order derivatives of u. In order to get global estimates, we use a covering argument based on Besicovitch's Covering Theorem (see Proposition 1.4.1 below). We just note that the transformed operators $\{A_{x_0}\}$ turn out to be uniformly elliptic if and only if we require that $|F| \le \theta V^{1/2}$, which is the case of [41].

Once that the estimate of the second order derivatives is available, by difference we get the estimate for $\langle F, Du\rangle$.

Proposition 1.4.1 *Let $\mathcal{F} = \{B(x,\rho(x))\}_{x\in\mathbb{R}^N}$ be a collection of balls such that*

$$|\rho(x) - \rho(y)| \le L|x-y|, \quad x,y \in \mathbb{R}^N, \tag{1.4.1}$$

with $L < \frac{1}{2}$. Then there exist a countable subcovering $\{B(x_n,\rho(x_n))\}$ and a natural number $\zeta = \zeta(N,L)$ such that at most ζ among the doubled balls $\{B(x_n, 2\rho(x_n))\}$ overlap.

The above proposition relies on the following version of the Besicovitch covering theorem, (see e.g. [4, Theorem 2.18]).

Proposition 1.4.2 *There exists a natural number $\xi(N)$ satisfying the following property. If $\Omega \subset \mathbb{R}^N$ is a bounded set and $\rho : \Omega \to (0,+\infty)$, then there is a set $S \subset \Omega$, at most countable, such that $\Omega \subset \bigcup_{x\in S} B(x,\rho(x))$ and every point of $\mathbb{R}^N$ belongs at most to $\xi(N)$ balls $B(x,\rho(x))$ centered at points of S.*

We turn now to the proof of Proposition 1.4.1.

Proof of Proposition 1.4.1. If $L = 0$ then the radii are constant and the statement easily follows.

If $L > 0$, we consider the sets

$$\begin{aligned} \Omega_n &:= B\Big(0, 2\rho(0)(1+L)^n\Big) \setminus B\Big(0, 2\rho(0)(1+L)^{n-1}\Big), \quad n \ge 1 \\ \Omega_0 &:= B(0, 2\rho(0)). \end{aligned}$$

Applying Proposition 1.4.2 we have that for all $n \in \mathbb{N}_0$ there exists a (at most) countable subset $S_n \subset \Omega_n$, such that $\Omega_n \subset \bigcup_{x\in S_n} B(x,\rho(x)) =: C_n$. Since (1.4.1) implies $\rho(x) \le \rho(0) + L|x|$, it is easy to prove that

$$C_n \subset B\Big(0, \rho(0)(2(1+L)^{n+1}+1)\Big) \setminus B\Big(0, \rho(0)(2(1-L)(1+L)^{n-1}-1)\Big), \quad n \ge 1.$$

Note that $2(1+L)^{n-1}(1-L) - 1 > 0$ for all $n \ge 1$ because $L < \frac{1}{2}$. Since $1 + L > 1$, there exists $k = k(L) \in \mathbb{N}$ such that for all $n \ge k$

$$2(1-L)(1+L)^{n-1} - 1 > 2(1+L)^{n-k+1} + 1,$$

which implies that $C_n \cap C_{n-k} = \emptyset$. Hence the intersection of at most k among the sets C_n can be non-empty. Moreover, at most $\xi(N)$ among the balls centered at points of S_n overlap. It turns

out that $\mathcal{F}' = \{B(x, \rho(x)) : x \in S_n,\, n \in \mathbb{N}_0\} =: \{B(x_j, \rho_j)\}$ is a countable subcovering of $\mathbb{R}^N$ and if $\xi' = k\,\xi(N)$ then at most ξ' balls of $\mathcal{F}'$ overlap.
To estimate the number of overlapping doubled balls $\{B(x_j, 2\rho_j)\}$ we proceed as in [41, Lemma 2.2]. Let $B(x_i, \rho_i) \in \mathcal{F}'$ be fixed and set $J(i) = \{j \in \mathbb{N} : B(x_i, 2\rho_i) \cap B(x_j, 2\rho_j) \neq \emptyset\}$. If $j \in J(i)$ it turns out that $|\rho_i - \rho_j| \leq 2L(\rho_i + \rho_j)$, because $|x_i - x_j| \leq 2(\rho_i + \rho_j)$, yielding $\frac{1-2L}{1+2L}\rho_i \leq \rho_j \leq \frac{1+2L}{1-2L}\rho_i$. Thus, the balls $B(x_j, \rho_j)$, $j \in J(i)$, are contained in $B(x_i, \frac{5+2L}{1-2L}\rho_i)$. Since at most ξ' of the balls $B(x_j, \rho_j)$ overlap, we obtain

$$\left(\frac{1-2L}{1+2L}\right)^N \rho_i^N \,\mathrm{card}\, J(i) \leq \sum_{j \in J(i)} \rho_j^N \leq \xi' \left(\frac{5+2L}{1-2L}\right)^N \rho_i^N ,$$

which implies $\mathrm{card}\, J(i) \leq \xi' \left(\frac{(5+2L)(1+2L)}{(1-2L)^2}\right)^N$, so that the number of overlapping doubled balls is an integer ζ, with $\zeta \leq 1 + \xi' \left(\frac{(5+2L)(1+2L)}{(1-2L)^2}\right)^N$. □

The following simple lemma is a straightforward consequence of assumption (H1') and it will be useful to prove Proposition 1.4.5 below.

Lemma 1.4.3 *Assume that (H1') holds. Then there exist $\varepsilon > 0$ and two constants $a, b > 0$, depending on α, σ, μ, such that for all $x_0 \in \mathbb{R}^N$*

$$aV(x) \leq V(x_0) \leq bV(x), \quad \text{for every } x \in B(x_0, 3\varepsilon r(x_0)),$$

with

$$r(x_0) := (1 + |x_0|^2)^{\mu/2} V^{\sigma-1}(x_0). \tag{1.4.2}$$

Proof. We remark that from the choice of the parameters μ and σ and since $V \geq 1$ then

$$(1 + |x|^2)^{\mu/2} V^{\sigma-1}(x) \leq 1 + |x|, \tag{1.4.3}$$

for every $x \in \mathbb{R}^N$. Moreover, (H1') is equivalent to one of the following inequalities

$$\begin{aligned} |DV^{\sigma-1}(x)| &\leq \frac{\alpha(1-\sigma)}{(1+|x|^2)^{\mu/2}}, \qquad \sigma < 1, \\ |D \log V(x)| &\leq \frac{\alpha}{(1+|x|^2)^{\mu/2}}, \qquad \sigma = 1. \end{aligned} \tag{1.4.4}$$

We prove the thesis assuming $\sigma < 1$, the case $\sigma = 1$ being analogous.

Fix $x_0 \in \mathbb{R}^N$ and write r in place of $r(x_0)$.

Suppose first that $|x_0| < 1$. From (1.4.3) and (1.4.2) it follows that $B(x_0, 3\varepsilon r) \subset B(0, 2)$, for every $0 < \varepsilon \leq 1/6$. Moreover, since V is a continuous function and $V \geq 1$, we have also that there exist $\omega_1, \omega_2 > 0$, independent of x_0, such that

$$\omega_1 = \inf_{y \in B(0,2)} \frac{1}{V(y)} \leq \inf_{y \in B(x_0, 3\varepsilon r)} \frac{1}{V(y)} \leq \frac{V(x_0)}{V(x)} \leq \sup_{y \in B(0,2)} V(y) = \omega_2, \quad x \in B(x_0, 3\varepsilon r).$$

Let us now deal with the case $|x_0| \geq 1$. By (1.4.3) one has $r(y) \leq 1 + |y|$, $y \in \mathbb{R}^N$, so that for every $0 < \varepsilon \leq 1/6$

$$\sup_{|y| \geq 1} \frac{1 + |y|^2}{1 + (|y| - 3\varepsilon r)^2} < +\infty .$$

Therefore, there exist $\varepsilon \leq 1/6$ and τ both independent of x_0, such that

$$\frac{3\varepsilon\alpha(1-\sigma)(1+|x_0|^2)^{\mu/2}}{(1 + (|x_0| - 3\varepsilon r)^2)^{\mu/2}} \leq \tau < 1,$$

where α and σ are as in (H1'). Thus, by the mean value theorem and (1.4.4) it follows that for every $x \in B(x_0, 3\varepsilon r)$

$$V^{\sigma-1}(x_0)(1-\tau) \le V^{\sigma-1}(x) \le V^{\sigma-1}(x_0)(1+\tau)$$

and, multiplying by $V^{1-\sigma}(x)V^{1-\sigma}(x_0)$,

$$V^{1-\sigma}(x)(1-\tau) \le V^{1-\sigma}(x_0) \le V^{1-\sigma}(x)(1+\tau)\,. \tag{1.4.5}$$

Therefore the statement is proved with $a = \inf\{\omega_1, (1-\tau)^{\frac{1}{1-\sigma}}\}$ and $b = \sup\{\omega_2, (1+\tau)^{\frac{1}{1-\sigma}}\}$. □

The following algebraic lemma is useful to prove Proposition 1.4.5.

Lemma 1.4.4 *If (H1') holds, with $(\sigma, \mu) \neq (\frac{1}{2}, 0)$, then for every $\delta > 0$ there exists $c_\delta > 0$ such that*

$$|DV| \le \delta V^{3/2} + c_\delta\,. \tag{1.4.6}$$

Proof. If $\frac{1}{2} < \sigma \le 1$, then (1.4.6) trivially follows by Young's inequality, with c_δ depending only on σ, α and c_α. If instead $\sigma = \frac{1}{2}$, then by assumption $\mu > 0$. For all $\delta > 0$ choose $R_\delta > 0$ such that $(1+|x|^2)^{\mu/2} \ge \alpha/\delta$ for every $x \in \mathbb{R}^N \setminus B_{R_\delta}$. Hence

$$|DV| \le \alpha \frac{V^{3/2}}{(1+|x|^2)^{\mu/2}} \le \delta V^{3/2} + \alpha \sup_{x \in B_{R_\delta}} V^{3/2}(x)\,.$$

□

In the following proposition we extend to the case $p \neq 2$ the estimate of the second order derivatives stated in (1.5.1) in the case $p = 2$.

Proposition 1.4.5 *Assume (H1'), (H2'), (H4'), (H5) with constants satisfying (1.1.7). If $u \in \mathcal{D}_p$ then*

$$\int_{\mathbb{R}^N} (|Vu|^p + |\langle F, Du\rangle|^p + |D^2u|^p)\,dx \le c\int_{\mathbb{R}^N} (|Au|^p + |u|^p)\,dx\,, \tag{1.4.7}$$

with c depending only on N, p, ν_0, M, $\|q_{ij}\|_\infty$, $\|Dq_{ij}\|_\infty$ and the constants in (H1'), (H2'), (H4') and (H5).

Proof. By Lemma 1.3.1 we may reduce to consider $u \in C_c^\infty(\mathbb{R}^N)$. Moreover, for the sake of simplicity and without loss of generality, we can prove the statement assuming $c_\beta = 0$.
Set $f = Au$. We claim that the assumptions of Lemma 1.3.5 hold. Since $|\operatorname{div} F| \le \sqrt{N}|DF|$ then (H2') implies

$$\operatorname{div} F + \beta V \ge 0 \tag{1.4.8}$$

with $\beta < p$ because of (1.1.7).
Moreover, (H1') and (H4') imply (1.1.8), that is

$$|\langle F, DV\rangle| \le \alpha\theta V^2\,.$$

If $(\sigma, \mu) = (\frac{1}{2}, 0)$, then (H1) trivially follows from (H1') and (1.1.8) implies (1.3.9). If instead $\sigma > \frac{1}{2}$ or $\mu > 0$, then by Lemma 1.4.4 (H1) holds, with α and c_α replaced by δ and c_δ, respectively, with δ arbitrarily small. Choose δ, depending only on N, p, M and on the constants in (H1'), (H2'), (H4') and (H5), such that

$$\frac{M}{4}(p-1)\delta^2 + \frac{\beta}{p} + \alpha\theta\frac{p-1}{p} < 1\,. \tag{1.4.9}$$

Thus, (1.3.9) holds and Lemma 1.3.5 implies

$$\int_{\mathbb{R}^N} |Vu|^p \, dx \le c \int_{\mathbb{R}^N} (|f|^p + |u|^p) \, dx \, . \tag{1.4.10}$$

It remains to estimate the L^p-norms of $|D^2 u|$ and $\langle F, Du\rangle$. We begin by considering the second order derivatives of u. Then, by difference, we obtain the estimate of $\langle F, Du\rangle$.
For every $x_0 \in \mathbb{R}^N$, let ε and $r = r(x_0)$ be as in Lemma 1.4.3. We point out that ε is independent of x_0.
Define y_0 equal to λx_0, with $\lambda := V^{1/2}(x_0)$. We consider two cut-off functions η and φ in $C_c^\infty(\mathbb{R}^N)$, $0 \le \eta, \varphi \le 1$, satisfying the following conditions

$$\begin{aligned} &\eta \equiv 1 \text{ in } B(y_0, \varepsilon\lambda r)\,, \quad \operatorname{supp} \eta \subset B(y_0, 2\varepsilon\lambda r)\,, \\ &\varphi \equiv 1 \text{ in } B(y_0, 2\varepsilon\lambda r)\,, \quad \operatorname{supp} \varphi \subset B(y_0, 3\varepsilon\lambda r)\,, \\ &|D\eta|^2 + |D^2\eta| + |D\varphi|^2 + |D^2\varphi| \le \frac{L}{\lambda^2 r^2}\,, \end{aligned} \tag{1.4.11}$$

for some $L > 0$, depending on ε, but neither on x_0 nor on y_0. For every $x \in \mathbb{R}^N$, define $y = \lambda x$ and consider $v(y) = u\left(\frac{y}{\lambda}\right)$. Then v satisfies the equation

$$\sum_{i,j=1}^N D_{y_i}(\tilde{q}_{ij} D_{y_j} v)(y) + \frac{1}{\lambda}\langle \tilde{F}(y), D_y v(y)\rangle - \frac{1}{\lambda^2}\tilde{V}(y)v(y) = \frac{1}{\lambda^2}\tilde{f}(y), \quad y \in \mathbb{R}^N$$

with $\tilde{q}_{ij}(y) = q_{ij}\left(\frac{y}{\lambda}\right)$, $\tilde{F}(y) = F\left(\frac{y}{\lambda}\right)$, $\tilde{V}(y) = V\left(\frac{y}{\lambda}\right)$ and $\tilde{f}(y) = f(\frac{y}{\lambda})$.
Setting $w(y) = \eta(y)v(y)$ we deduce that

$$\sum_{i,j=1}^N D_{y_i}(\tilde{q}_{ij}(y) D_{y_j} w(y)) + \frac{1}{\lambda}\langle \tilde{F}(y), D_y w(y)\rangle - \frac{1}{\lambda^2}\tilde{V}(y)w(y) = g(y) \tag{1.4.12}$$

with g defined as follows

$$g(y) := \frac{1}{\lambda^2}\eta(y)\tilde{f}(y) + 2\langle \tilde{q}(y)D\eta(y), Dv(y)\rangle + \operatorname{div}(\tilde{q}D\eta)(y)v(y) + \frac{1}{\lambda}\langle \tilde{F}(y), D\eta(y)\rangle v(y), \tag{1.4.13}$$

$y \in \mathbb{R}^N$. Since $\operatorname{supp} w \subset B(y_0, 2\varepsilon\lambda r)$, equation (1.4.12) is equivalent to

$$\sum_{i,j=1}^N D_{y_i}(\tilde{q}_{ij}(y) D_{y_j} w(y)) + \frac{1}{\lambda}\varphi(y)\langle \tilde{F}(y), D_y w(y)\rangle - \frac{1}{\lambda^2}\varphi(y)\tilde{V}(y)w(y) = g(y), \quad y \in \mathbb{R}^N.$$

Now, let us define the operator

$$\tilde{A} = \sum_{i,j=1}^N D_{y_i}(\tilde{q}_{ij} D_{y_j}) + \frac{1}{\lambda}\varphi\,\langle \tilde{F}, D_y\rangle - \frac{1}{\lambda^2}\varphi\,\tilde{V}. \tag{1.4.14}$$

Claim 1. $\frac{1}{\lambda^2}\varphi\,\tilde{V}$ and $\left|\langle \frac{1}{\lambda}\varphi\tilde{F}, D\tilde{q}_{ij}\rangle\right|$ are bounded in $\mathbb{R}^N$ and $\frac{1}{\lambda}\varphi\,\tilde{F}$ is globally Lipschitz in $\mathbb{R}^N$ with $\left\|\frac{1}{\lambda^2}\varphi\,\tilde{V}\right\|_\infty$, $\left\|\langle \frac{1}{\lambda}\varphi\tilde{F}, D\tilde{q}_{ij}\rangle\right\|_\infty$ and the Lipschitz constant of $\frac{1}{\lambda}\varphi\,\tilde{F}$ independent of x_0.

Proof of claim 1. The main tool is Lemma 1.4.3. Recalling the definition of λ, $\tilde{V}$ and the relationship between y and x, from Lemma 1.4.3 it follows that

$$\sup_{y\in\mathbb{R}^N} \frac{1}{\lambda^2}\varphi(y)\,\tilde{V}(y) \le \sup_{x \in B(x_0, 3\varepsilon r)} \frac{V(x)}{V(x_0)} \le \frac{1}{a},$$

Taking into account assumptions (H2'), (H4') and (1.4.11), we have that

$$\begin{aligned}
\sup_{y\in\mathbb{R}^N}\left|\frac{1}{\lambda}D_y\big(\varphi(y)\,\tilde F(y)\big)\right| &= \sup_{y\in B(y_0,3\varepsilon\lambda r)}\left|\frac{1}{\lambda^2}(D_xF)\left(\frac{y}{\lambda}\right)\varphi(y)+\frac{1}{\lambda}F\left(\frac{y}{\lambda}\right)D_y\varphi(y)\right|\\
&\le \sup_{x\in B(x_0,3\varepsilon r)}\frac{\beta V(x)}{V(x_0)}+L\sup_{x\in B(x_0,3\varepsilon r)}\frac{|F(x)|}{r\,V(x_0)}\\
&\le \beta\sup_{x\in B(x_0,3\varepsilon r)}\frac{V(x)}{V(x_0)}+L\theta\sup_{x\in B(x_0,3\varepsilon r)}\frac{(1+|x|^2)^{\frac{\mu}{2}}V^\sigma(x)}{(1+|x_0|^2)^{\frac{\mu}{2}}V^\sigma(x_0)}
\end{aligned}$$

Using Lemma 1.4.3 and equation (1.4.3) we infer that

$$\begin{aligned}
\sup_{y\in\mathbb{R}^N}\left|\frac{1}{\lambda}D_y\big(\varphi(y)\,\tilde F(y)\big)\right| &\le \frac{\beta}{a}+\frac{L\theta}{a^\sigma}\frac{[1+(|x_0|+3\varepsilon r)^2]^{\frac{\mu}{2}}}{(1+|x_0|^2)^{\frac{\mu}{2}}}\\
&\le \frac{\beta}{a}+\frac{L\,\theta\,8^{\frac{\mu}{2}}}{a^\sigma}
\end{aligned}$$

which implies that $\frac{1}{\lambda}\varphi\,\tilde F$ is globally Lipschitz in $\mathbb{R}^N$, uniformly with respect to x_0.
Finally, assumption (H5) yields

$$\begin{aligned}
\sup_{y\in\mathbb{R}^N}\left|\langle\frac{1}{\lambda}\varphi(y)\tilde F(y),D_y\tilde q_{ij}(y)\rangle\right| &\le \sup_{y\in B(y_0,3\varepsilon\lambda r)}\left|\langle\frac{1}{\lambda}\tilde F(y),D_y\tilde q_{ij}(y)\rangle\right|\\
&\le \sup_{x\in B(x_0,3\varepsilon r)}\frac{1}{\lambda^2}\left|\langle F(x),Dq_{ij}(x)\rangle\right|\\
&\le \kappa\sup_{x\in B(x_0,3\varepsilon r)}\frac{V(x)}{V(x_0)}+c_\kappa\sup_{x\in B(x_0,3\varepsilon r)}\frac{1}{V(x_0)}\le\frac{\kappa}{a}+c_\kappa\,,
\end{aligned}$$

because of Lemma 1.4.3 and $V\ge 1$.

Claim 2. The function g in (1.4.13) satisfies the estimate

(1.4.15)
$$\int_{\mathbb{R}^N}|g(y)|^p\,dy\le\frac{C}{\lambda^{2p-N}}\int_{B(x_0,2\varepsilon r)}\Big(|u(x)|^p+|f(x)|^p+|V(x)u(x)|^p+|V^{1/2}(x)Du(x)|^p\Big)\,dx\,,$$

for some C depending on ε, but not on x_0.

Proof of claim 2. We separately consider each term of g. The constants occurring in the estimates may depend on ε.
The first term in (1.4.13) is the easiest to estimate, in fact

(1.4.16)
$$\int_{\mathbb{R}^N}\left|\frac{1}{\lambda^2}\eta(y)f\left(\frac{y}{\lambda}\right)\right|^p\,dy\le\frac{1}{\lambda^{2p}}\int_{B(y_0,2\varepsilon\lambda r)}\left|f\left(\frac{y}{\lambda}\right)\right|^p\,dy=\frac{1}{\lambda^{2p-N}}\int_{B(x_0,2\varepsilon r)}|f(x)|^p\,dx\,.$$

Using (1.4.11) we can estimate the L^p-norm of the next two terms as follows

$$\begin{aligned}
&\int_{\mathbb{R}^N}|2\langle\tilde q(y)D_y\eta(y),D_yv(y)\rangle|^p\,dy\le\frac{C_1}{\lambda^{2p}r^p}\int_{B(y_0,2\varepsilon\lambda r)}\left|Du\left(\frac{y}{\lambda}\right)\right|^p\,dy\\
&=\frac{C_1}{\lambda^{2p-N}r^p}\int_{B(x_0,2\varepsilon r)}|Du(x)|^p\,dx=\frac{C_1}{\lambda^{2p-N}}\int_{B(x_0,2\varepsilon r)}\frac{V^{p(1-\sigma)}(x_0)}{(1+|x_0|^2)^{p\mu/2}}|Du(x)|^p\,dx
\end{aligned}$$

and

$$\begin{aligned}
&\int_{\mathbb{R}^N}|\mathrm{div}(\tilde qD\eta)(y)v(y)|^p\,dy\le\frac{C_2}{\lambda^{2p}\,r^{2p}}\int_{B(y_0,2\varepsilon\lambda r)}|v(y)|^p\,dy\\
&=\frac{C_2}{\lambda^{2p-N}r^{2p}}\int_{B(x_0,2\varepsilon r)}|u(x)|^p\,dx=\frac{C_2}{\lambda^{2p-N}}\int_{B(x_0,2\varepsilon r)}\frac{V^{2p(1-\sigma)}(x_0)}{(1+|x_0|^2)^{p\mu}}|u(x)|^p\,dx\,,
\end{aligned}$$

with C_1 and C_2 independent of x_0.
Recalling that $V \geq 1$, $\sigma \geq \frac{1}{2}$, $\mu \geq 0$ and using Lemma 1.4.3, we obtain

$$\begin{aligned}\int_{B(x_0,2\varepsilon r)} \frac{V^{p(1-\sigma)}(x_0)}{(1+|x_0|^2)^{p\mu/2}}|Du(x)|^p\,dx &\leq \int_{B(x_0,2\varepsilon r)} |V^{1/2}(x_0)Du(x)|^p\,dx \\ &\leq b^{p/2}\int_{B(x_0,2\varepsilon r)} |V^{1/2}(x)Du(x)|^p\,dx\end{aligned}$$

and

$$\begin{aligned}\int_{B(x_0,2\varepsilon r)} \frac{V^{2p(1-\sigma)}(x_0)}{(1+|x_0|^2)^{p\mu}}|u(x)|^p\,dx &\leq \int_{B(x_0,2\varepsilon r)} |V(x_0)u(x)|^p\,dx \\ &\leq b^{p}\int_{B(x_0,2\varepsilon r)} |V(x)u(x)|^p\,dx\,.\end{aligned}$$

Hence, there exists C_3 independent of x_0 such that the following inequality holds

$$\begin{aligned}(1.4.17)\qquad &\int_{\mathbb{R}^N} \left(|2\langle\tilde{q}(y)D_y\eta(y),D_yv(y)\rangle|^p + |\mathrm{div}(\tilde{q}D\eta)(y)v(y)|^p\right)dy \leq \\ &\leq \frac{C_3}{\lambda^{2p-N}}\int_{B(x_0,2\varepsilon r)} (|V(x)u(x)|^p + |V^{1/2}(x)Du(x)|^p)\,dx\,.\end{aligned}$$

Concerning the last term in (1.4.13), we use again assumption (H4') and we get

$$\begin{aligned}(1.4.18)\qquad &\int_{\mathbb{R}^N}\left|\frac{1}{\lambda}\langle\tilde{F}(y),D\eta(y)\rangle v(y)\right|^p dy \leq \frac{c}{\lambda^{2p-N}}\int_{B(x_0,2\varepsilon r)}\frac{|F(x)|^p|u(x)|^p}{r^p}\,dx \\ &\leq \frac{c\,\theta^p}{\lambda^{2p-N}}\int_{B(x_0,2\varepsilon r)}\left|\frac{(1+|x|^2)^{\mu/2}V^{\sigma-1}(x)}{(1+|x_0|^2)^{\mu/2}V^{\sigma-1}(x_0)}\right|^p|V(x)u(x)|^p\,dx \\ &\leq \frac{C_4}{\lambda^{2p-N}}\int_{B(x_0,2\varepsilon r)}|V(x)u(x)|^p\,dx\end{aligned}$$

where C_4 is not depending on x_0. Thus, the claim is proved since collecting (1.4.16)-(1.4.18), inequality (1.4.15) follows.

Let us now prove (1.4.7). Applying Theorem 1.2.1 with B replaced by $\tilde{A}$, we have

$$\int_{\mathbb{R}^N}|D^2w(y)|^p\,dy \leq K\int_{\mathbb{R}^N}(|w(y)|^p + |g(y)|^p)\,dy\,,$$

with K independent of x_0. By the definition of w it follows that

$$\int_{B(y_0,\varepsilon\lambda r)}|D^2v(y)|^p\,dy \leq K\int_{B(y_0,2\varepsilon\lambda r)}(|v(y)|^p + |g(y)|^p)\,dy$$

and consequently, since $y = \lambda x$,

$$\begin{aligned}&\frac{1}{\lambda^{2p-N}}\int_{B(x_0,\varepsilon r)}|D^2u|^p\,dx \leq \\ &\leq K_1\,\lambda^N\int_{B(x_0,2\varepsilon r)}|u|^p\,dx + K_1\frac{1}{\lambda^{2p-N}}\int_{B(x_0,2\varepsilon r)}\left(|u|^p+|f|^p+|Vu|^p+|V^{1/2}Du|^p\right)dx\,.\end{aligned}$$

Multiplying both sides of the previous inequality by λ^{2p-N} and recalling that $\lambda = V^{1/2}(x_0)$ we obtain

$$\begin{aligned}&\int_{B(x_0,\varepsilon r)}|D^2u|^p\,dx \leq \\ &\leq K_1\int_{B(x_0,2\varepsilon r)}|V(x_0)u(x)|^p\,dx + K_1\int_{B(x_0,2\varepsilon r)}\left(|u|^p+|f|^p+|Vu|^p+|V^{1/2}Du|^p\right)dx\,,\end{aligned}$$

which implies

$$\text{(1.4.19)} \qquad \int_{B(x_0,\varepsilon r)} |D^2u|^p\,dx \le K_2 \int_{B(x_0,2\varepsilon r)} \Big(|u|^p + |f|^p + |Vu|^p + |V^{1/2}Du|^p\Big)\,dx\,,$$

because of Lemma 1.4.3. Now, in order to apply Proposition 1.4.1 we need to verify the Lipschitz continuity of the radius $\varepsilon\, r$ with respect to x_0. To this aim, we remark that from assumption (H1') it follows that

$$\begin{aligned}
|D(\varepsilon r)(x)| &= \varepsilon\left|\mu(1+|x|^2)^{\frac{\mu}{2}-1}xV^{\sigma-1}(x) + (\sigma-1)(1+|x|^2)^{\frac{\mu}{2}}V^{\sigma-2}(x)DV(x)\right| \\
&\le \varepsilon\left\{\frac{1}{(1+|x|^2)^{\frac{1-\mu}{2}}V^{1-\sigma}(x)} + (1-\sigma)(1+|x|^2)^{\frac{\mu}{2}}V^{\sigma-2}(x)|DV(x)|\right\} \\
&\le \varepsilon\,\{1+(1-\sigma)\alpha\}
\end{aligned}$$

which is less than $1/2$, choosing a smaller ε if necessary. Let $\{B(x_j, \varepsilon r_j)\}$ be the covering of $\mathbb{R}^N$ yielded by Proposition 1.4.1. Applying (1.4.19) to each x_j and summing over j, it follows that

$$\begin{aligned}
&\int_{\mathbb{R}^N} |D^2u|^p\,dx \le \sum_{j\in\mathbb{N}} \int_{B(x_j,\varepsilon r_j)} |D^2u|^p\,dx \\
&\le K_2 \sum_{j\in\mathbb{N}} \int_{B(x_j,2\varepsilon r_j)} \Big(|u|^p + |f|^p + |Vu|^p + |V^{1/2}Du|^p\Big)\,dx \\
&= K_2 \int_{\mathbb{R}^N} \Big(|u(x)|^p + |f(x)|^p + |V(x)u(x)|^p + |V^{1/2}(x)Du(x)|^p\Big) \sum_{j\in\mathbb{N}} \chi_{B(x_j,2\varepsilon r_j)}(x)\,dx \\
&\le \zeta\,K_2 \int_{\mathbb{R}^N} \Big(|u|^p + |f|^p + |Vu|^p + |V^{1/2}Du|^p\Big)\,dx\,,
\end{aligned}$$

where ζ is given by Proposition 1.4.1. Now, [41, Proposition 2.3] yields two constants $\gamma_0, c > 0$ (independent of u) such that for all $0 < \gamma \le \gamma_0$

$$\|V^{1/2}Du\|_p \le \gamma\|D^2u\|_p + \frac{c}{\gamma}\|Vu\|_p.$$

Choosing γ sufficiently small and taking into account (1.4.10) it turns out that

$$\int_{\mathbb{R}^N} |D^2u|^p\,dx \le c\int_{\mathbb{R}^N} (|f|^p + |u|^p)\,dx\,,$$

for some $c > 0$ depending on the stated quantities.
Once that the estimate of the second order derivatives is available, by difference we get the estimate for $\langle F, Du\rangle$, that is

$$\int_{\mathbb{R}^N} |\langle F, Du\rangle|^p\,dx \le c\int_{\mathbb{R}^N} (|f|^p + |u|^p)\,dx\,.$$

□

1.5 Generation of a C_0-semigroup in $L^2(\mathbb{R}^N)$

In this section we prove Theorem 1.1.1, which states that the operator $(A, \mathcal{D}_2)$ (see (1.1.3)) generates a C_0-semigroup in $L^2(\mathbb{R}^N)$, which turns out to be contractive if $c_\beta = 0$.
The proof goes as follows. As a by-product of Lemma 1.3.1 we deduce that the a priori estimates proved in Section 1.3, with $p = 2$ extend to $\mathcal{D}_2$. More precisely, it follows from Lemma 1.3.1,

Remark 1.3.3, Lemmas 1.3.5 and 1.3.6 that if $u \in \mathcal{D}_2$ and (H1), (H2), (H3), (H4), (1.1.5) and (1.1.6) hold, then

$$\int_{\mathbb{R}^N} (|Du|^2 + |Vu|^2 + |D^2u|^2)\, dx \leq c \int_{\mathbb{R}^N} (|Au|^2 + |u|^2)\, dx \,, \tag{1.5.1}$$

for some c depending only on $N, \nu_0, \alpha, \beta, \tau, M, \|Dq_{ij}\|_\infty$. By difference, since Au is in $L^2(\mathbb{R}^N)$, then

$$\int_{\mathbb{R}^N} |\langle F, Du\rangle|^2\, dx \leq c \int_{\mathbb{R}^N} (|Au|^2 + |u|^2)\, dx \,, \tag{1.5.2}$$

with a possibly different c.
Estimates (1.5.1) and (1.5.2) allow to prove that $(A, \mathcal{D}_2)$ is closed in $L^2(\mathbb{R}^N)$. Clearly, it is densely defined. If $c_\beta = 0$, then $(A, \mathcal{D}_2)$ is also dissipative. In order to apply the Hille-Yosida Theorem, it remains to prove that $\lambda - A : \mathcal{D}_2 \to L^2(\mathbb{R}^N)$ is bijective for sufficiently large λ. This is proved through a standard procedure, namely by approximating the solution of the elliptic equation $\lambda u - Au = f$, $f \in L^2(\mathbb{R}^N)$, with a sequence of solutions of the same equation in balls with increasing radii and satisfying Dirichlet boundary conditions.

Lemma 1.5.1 *Suppose that (H1), (H2), (H3), (H4), (1.1.5) and (1.1.6) hold. Then $(A, \mathcal{D}_2)$ is closed in $L^2(\mathbb{R}^N)$. Moreover, $(A - \frac{c_\beta}{2}, \mathcal{D}_2)$ is dissipative.*

PROOF. If $u \in \mathcal{D}_2$, then $\|u\|_A \leq c_1\|u\|_{\mathcal{D}_2}$, $\|\cdot\|_A$ being the graph norm of A, for some positive c_1 depending on $\|q_{ij}\|_\infty$ and $\|Dq_{ij}\|_\infty$. Moreover, from (1.5.1) and (1.5.2) there exists $c_2 > 0$ such that $\|u\|_{\mathcal{D}_2} \leq c_2\|u\|_A$. This proves that $\|\cdot\|_{\mathcal{D}_2}$ is equivalent to $\|\cdot\|_A$; since $\mathcal{D}_2$ is obviously complete with respect to the former, it turns out that $\mathcal{D}_2$ is also complete with respect to the latter, which just means that $(A, \mathcal{D}_2)$ is closed.
Finally, taking into account Remark 1.3.4 and Lemma 1.3.1, we conclude that $(A - \frac{c_\beta}{2}, \mathcal{D}_2)$ is dissipative. □

In the proposition below we study the surjectivity of the operator $\lambda - A$, for positive λ. We remark that the injectivity for $\lambda > \frac{c_\beta}{2}$ follows from the dissipativity stated in Lemma 1.5.1.

Proposition 1.5.2 *Suppose that (H1), (H2), (H3), (H4), (1.1.5) and (1.1.6) hold. Then for every $f \in L^2(\mathbb{R}^N)$ and for every $\lambda > c_\beta/2$, there exists a solution $u \in \mathcal{D}_2$ of*

$$\lambda u - Au = f, \quad \text{in } \mathbb{R}^N. \tag{1.5.3}$$

Moreover,

$$\|u\|_2 \leq \left(\lambda - \frac{c_\beta}{2}\right)^{-1} \|f\|_2 \,. \tag{1.5.4}$$

PROOF. We deal with the case $c_\beta = 0$ only, since the remaining case $c_\beta \neq 0$ is analogous.
For each $\rho > 0$ consider the Dirichlet problem

$$\begin{cases} \lambda u - Au = f, & \text{in } B_\rho \\ u = 0, & \text{on } \partial B_\rho \,, \end{cases} \tag{1.5.5}$$

with $\lambda > 0$ and $f \in L^2(\mathbb{R}^N)$. According to [26, Theorem 9.15] there exists a unique solution u_ρ of (1.5.5) in $W^{2,2}(B_\rho) \cap W_0^{1,2}(B_\rho)$. Let us prove that the dissipativity estimate

$$\lambda\|u_\rho\|_{L^2(B_\rho)} \leq \|f\|_{L^2(\mathbb{R}^N)}$$

holds. Multiplying

$$\lambda u_\rho - A u_\rho = f \tag{1.5.6}$$

by u_ρ and integrating by parts with similar estimates as in the proof of Lemma 1.3.2, taking into account that $u_\rho = 0$ on ∂B_ρ, we get

$$\lambda \int_{B_\rho} u_\rho^2\,dx + \nu_0 \int_{B_\rho} |Du_\rho|^2\,dx + \frac{1}{2}\int_{B_\rho} \mathrm{div}F\, u_\rho^2\,dx + \int_{B_\rho} V u_\rho^2\,dx \le \int_{B_\rho} f u_\rho\,dx$$

and by (H2) it follows

$$\lambda \int_{B_\rho} u_\rho^2\,dx + \nu_0 \int_{B_\rho} |Du_\rho|^2\,dx + \left(1 - \frac{\beta}{2}\right)\int_{B_\rho} V u_\rho^2\,dx \le \left(\int_{B_\rho} u_\rho^2\,dx\right)^{1/2} \left(\int_{B_\rho} f^2\,dx\right)^{1/2} .$$

Then we have

$$\|u_\rho\|_{L^2(B_\rho)} \le \lambda^{-1}\|f\|_{L^2(\mathbb{R}^N)}\,, \quad \|Du_\rho\|_{L^2(B_\rho)} \le \nu_0^{-1/2}\lambda^{-1/2}\|f\|_{L^2(\mathbb{R}^N)}\,. \tag{1.5.7}$$

Multiplying (1.5.6) by Vu_ρ, with analogous estimates as in the proof of Lemma 1.3.5 we get the inequality

$$\|Vu_\rho\|_{L^2(B_\rho)} \le c\|f\|_{L^2(\mathbb{R}^N)}\,, \tag{1.5.8}$$

with c independent of ρ.
Let $\rho_1 < \rho_2 < \rho$. By [26, Theorem 9.11] and (1.5.7) we obtain

$$\|u_\rho\|_{W^{2,2}(B_{\rho_1})} \le c_1\left(\|f\|_{L^2(B_{\rho_2})} + \|u_\rho\|_{L^2(B_{\rho_2})}\right) \le c_2\|f\|_{L^2(\mathbb{R}^N)},$$

with c_1 and c_2 independent of ρ. Thus, $\{u_\rho\}$ is bounded in $W^{2,2}_{\mathrm{loc}}(\mathbb{R}^N)$, hence there is a sequence $\{u_{\rho_n}\}$, $\rho_n < \rho_{n+1}$, weakly convergent to u in $W^{2,2}_{\mathrm{loc}}(\mathbb{R}^N)$ and strongly in $L^2_{\mathrm{loc}}(\mathbb{R}^N)$. Actually, $\{u_{\rho_n}\}$ strongly converges to u in $W^{2,2}_{\mathrm{loc}}(\mathbb{R}^N)$. In fact, fixed s and t, $0 < s < t$, for every n, m such that $\rho_n, \rho_m > t$, by [26, Theorem 9.11] again,

$$\|u_{\rho_n} - u_{\rho_m}\|_{W^{2,2}(B_s)} \le c(s,t)\|u_{\rho_n} - u_{\rho_m}\|_{L^2(B_t)}\,,$$

since both u_{ρ_n} and u_{ρ_m} satisfy $\lambda u - Au = f$ in B_t. The convergence of $\{u_{\rho_n}\}$ to u in $L^2(B_t)$ proves that $\{u_{\rho_n}\}$ is a Cauchy sequence in $W^{2,2}(B_s)$ and so the assertion follows. As a consequence, u is a solution of (1.5.3) for a.e. $x \in \mathbb{R}^N$.

In order to conclude, it remains to prove that $u \in \mathcal{D}_2$. First, we prove that $u \in W^{1,2}(\mathbb{R}^N)$ and $Vu \in L^2(\mathbb{R}^N)$, then that $\langle F, Du\rangle \in L^2(\mathbb{R}^N)$. Finally, by difference from (1.5.3) and using classical L^2-regularity, it follows that $u \in W^{2,2}(\mathbb{R}^N)$.
By (1.5.7) and (1.5.8) we get that, fixed $R < \rho_n$,

$$\int_{B_R} u_{\rho_n}^2\,dx \le \int_{B_{\rho_n}} u_{\rho_n}^2\,dx \le \lambda^{-2}\int_{\mathbb{R}^N} f^2\,dx\,,$$

$$\int_{B_R} |Du_{\rho_n}|^2\,dx \le \int_{B_{\rho_n}} |Du_{\rho_n}|^2\,dx \le \nu_0^{-1}\lambda^{-1}\int_{\mathbb{R}^N} f^2\,dx$$

and

$$\int_{B_R} (Vu_{\rho_n})^2\,dx \le \int_{B_{\rho_n}} (Vu_{\rho_n})^2\,dx \le c\int_{\mathbb{R}^N} f^2\,dx\,.$$

Since c does not depend on ρ_n and R, letting first $n \to +\infty$ and then $R \to +\infty$, we get (1.5.4) and

$$\int_{\mathbb{R}^N} (|Du|^2 + |Vu|^2)\, dx \le c \int_{\mathbb{R}^N} f^2\, dx\,.$$

In particular, $u \in W^{1,2}(\mathbb{R}^N)$ and $Vu \in L^2(\mathbb{R}^N)$.
Now, let $\eta \in C_c^\infty(\mathbb{R}^N)$ such that $0 \le \eta \le 1$, $\eta \equiv 1$ in B_1, $\operatorname{supp}\eta \subset \mathrm{B}_2$ and $|D\eta|^2 + |D^2\eta| \le L$. Set $\eta_n(x) = \eta(x/n)$. We have

$$(1.5.9) \qquad A(\eta_n u) - \eta_n Au = \sum_{i,j=1}^N q_{ij} D_j u D_i \eta_n + D_i(q_{ij} u D_j \eta_n) + \langle F, D\eta_n\rangle u\,.$$

Observe that $A(\eta_n u) - \eta_n Au \to 0$ as $n \to +\infty$ in the L^2-norm. In fact, $\sum_{i,j=1}^N (q_{ij} D_j u D_i \eta_n + D_i(q_{ij} u D_j \eta_n))$ goes to 0 in the L^2-norm, since $u \in W^{1,2}(\mathbb{R}^N)$ and, arguing as in (1.3.1), we obtain the convergence to 0 for the last term in (1.5.9), too. Since $\eta_n Au \to Au$ in L^2, then $A(\eta_n u) \to Au$, too. Being $\eta_n u \in \mathcal{D}_2$, by the equivalence of the two norms $\|\cdot\|_{\mathcal{D}_2}$ and $\|\cdot\|_A$ proved in Lemma 1.5.1 we get

$$\|\langle F, Du\rangle \eta_n\|_{L^2(\mathbb{R}^N)} \le c\left(\|A(\eta_n u)\|_{L^2(\mathbb{R}^N)} + \|\eta_n u\|_{L^2(\mathbb{R}^N)}\right) + \|\langle F, D\eta_n\rangle u\|_{L^2(\mathbb{R}^N)}.$$

Letting $n \to +\infty$, one then establishes

$$\|\langle F, Du\rangle\|_{L^2(\mathbb{R}^N)} \le c\left(\|Au\|_{L^2(\mathbb{R}^N)} + \|u\|_{L^2(\mathbb{R}^N)}\right)\,.$$

By difference, $\sum_{i,j=1}^N D_i(q_{ij} D_j u)$ belongs to $L^2(\mathbb{R}^N)$. Thus, by (1.1.1) and L^2 elliptic regularity the second order derivatives of u are in L^2, which implies that $u \in W^{2,2}(\mathbb{R}^N)$ and $u \in \mathcal{D}_2$. □

The proof that the operator $(A, \mathcal{D}_2)$ generates a strongly continuous semigroup in $L^2(\mathbb{R}^N)$ is now a straightforward consequence of the above results.

Proof of Theorem 1.1.1. It is easily seen that $(A, \mathcal{D}_2)$ is densely defined, then the assertion follows from the Hille-Yosida Theorem (see [21, Theorem II.3.5]). If $c_\beta = 0$ then $(A, \mathcal{D}_2)$ is dissipative and therefore the generated semigroup is contractive. □

1.6 Generation of a C_0-semigroup in $L^p(\mathbb{R}^N)$

The present section is devoted to the proof of Theorem 1.1.2. As in the case $p = 2$ treated in Section 1.5, the a priori estimates given by Proposition 1.4.5 allow to prove that $\|\cdot\|_{\mathcal{D}_p}$ and $\|\cdot\|_A$ are equivalent norms. This easily implies the closedness of $(A, \mathcal{D}_p)$. Moreover, it is readily seen that $(A, \mathcal{D}_p)$ is quasi dissipative. It remains to show that $\lambda - A$ is surjective for λ large and this is, actually, the main result of the section. The proof is different from that of Proposition 1.5.2, which does not work for $p \ne 2$. Here we approximate the coefficients of the operator A. Moreover, we clarify the reason why we require assumption (1.1.7), which is stronger than the corresponding one for $p = 2$. In fact, also the operators A_ε defined in the proof of Proposition 1.6.2 must satisfy our hypotheses.

The proof of the following Lemma is the same as the one of Lemma 1.5.1 and we omit it.

Lemma 1.6.1 *Suppose that (H1'), (H2'), (H4') and (H5) hold, with constants satisfying (1.1.7). Then $(A, \mathcal{D}_p)$ is closed in $L^p(\mathbb{R}^N)$. Moreover, $(A - \frac{c_\beta}{p}, \mathcal{D}_p)$ is dissipative.*

Proposition 1.6.2 *Suppose that (H1'), (H2'), (H4') and (H5) hold, with constants satisfying (1.1.7). Then for every $f \in L^p(\mathbb{R}^N)$ and for every $\lambda > \frac{c_\beta}{p}$ a unique solution $u \in \mathcal{D}_p$ of*

$$\lambda u - Au = f, \quad \textit{in } \mathbb{R}^N$$

exists. Moreover,

$$\|u\|_p \le \left(\lambda - \frac{c_\beta}{p}\right)^{-1} \|f\|_p \,. \tag{1.6.1}$$

PROOF. Uniqueness and estimate (1.6.1) immediately follow from (1.3.7). As far as the existence is concerned, for fixed $\varepsilon > 0$, let us define $F_\varepsilon : \mathbb{R}^N \to \mathbb{R}^N$ and $V_\varepsilon : \mathbb{R}^N \to \mathbb{R}$ as

$$F_\varepsilon := \frac{F}{1+\varepsilon V}\,, \qquad V_\varepsilon := \frac{V}{1+\varepsilon V}\,.$$

It is easy to prove that (H1'), (H2'), (H4') and (H5) imply

(H_ε1) $|DV_\varepsilon(x)| \le \alpha \frac{V_\varepsilon^{2-\sigma}(x)}{(1+|x|^2)^{\mu/2}}$,

(H_ε2) $|DF_\varepsilon| \le \sqrt{2}(\frac{\beta}{\sqrt{N}} + \alpha\theta)V_\varepsilon + \sqrt{\frac{2}{N}}\, c_\beta$,

(H_ε4) $|F_\varepsilon(x)| \le \theta(1+|x|^2)^{\mu/2} V_\varepsilon^\sigma(x)$,

(H_ε5) $|\langle F_\varepsilon(x), Dq_{ij}(x)\rangle| \le \kappa\, V_\varepsilon(x) + c_\kappa$,

respectively.
Assumptions (H_ε1), (H_ε2) and (H_ε4) yield

$$\operatorname{div} F_\varepsilon + \sqrt{2}(\beta + \sqrt{N}\alpha\theta)V_\varepsilon + \sqrt{2}\, c_\beta \ge 0\,, \qquad |\langle F_\varepsilon, DV_\varepsilon\rangle| \le \alpha\theta V_\varepsilon^2 \tag{1.6.2}$$

and

$$\sum_{i,j=1}^N D_i F_\varepsilon^j(x)\xi_i\xi_j \le \sqrt{2}\left(\frac{\beta}{\sqrt{N}} + \alpha\theta\right) V_\varepsilon(x)|\xi|^2 + \sqrt{\frac{2}{N}}\, c_\beta|\xi|^2\,, \qquad \xi, x \in \mathbb{R}^N\,.$$

Notice that V_ε is bounded and F_ε is globally Lipschitz in $\mathbb{R}^N$. Precisely,

$$||V_\varepsilon||_\infty \le \frac{1}{\varepsilon}\,, \quad \text{and} \quad ||D_i F_\varepsilon^j||_\infty \le \frac{1}{\varepsilon}\left(\frac{\beta}{\sqrt{N}} + \alpha\theta\right) + \frac{c_\beta}{\sqrt{N}}\,, \qquad 1 \le i,j \le N\,.$$

Moreover, if $(\sigma,\mu) \ne \left(\frac{1}{2}, 0\right)$ arguing as in the proof of Lemma 1.4.4 and observing that $V_\varepsilon \le V$, we have that for every $\delta > 0$ there exists $c_\delta \ge 0$ such that

$$|DV_\varepsilon| \le \delta V_\varepsilon^{3/2} + c_\delta\,, \quad \text{for every } \varepsilon > 0\,. \tag{1.6.3}$$

Therefore, the above inequality and (1.1.7) imply that there exists $\delta > 0$ independent of ε such that

$$\frac{M}{4}(p-1)\delta^2 + \sqrt{2}\frac{\beta + \sqrt{N}\alpha\theta}{p} + \alpha\theta\frac{p-1}{p} < 1\,. \tag{1.6.4}$$

Let us consider the operator

$$A_\varepsilon := A_0 + \langle F_\varepsilon, D\rangle - V_\varepsilon$$

where, as previously defined, A_0 stands for $\sum_{i,j=1}^N D_i(q_{ij}D_j)$.
Define $\mathcal{D}_{p,\varepsilon}$ and its norms $\|\cdot\|_{\mathcal{D}_{p,\varepsilon}}$ and $\|\cdot\|_{A_\varepsilon}$ analogously to $\mathcal{D}_p$, $\|\cdot\|_{\mathcal{D}_p}$ and $\|\cdot\|_A$, respectively, that is

$$\begin{aligned}
\mathcal{D}_{p,\varepsilon} &:= \left\{u \in W^{2,p}(\mathbb{R}^N) \,:\, \langle F_\varepsilon, Du\rangle \in L^p(\mathbb{R}^N)\right\}\,,\\
\|u\|_{\mathcal{D}_{p,\varepsilon}} &:= \|u\|_{2,p} + \|V_\varepsilon u\|_p + \|\langle F_\varepsilon, Du\rangle\|_p\,,\\
\|u\|_{A_\varepsilon} &:= \|A_\varepsilon u\|_p + \|u\|_p\,.
\end{aligned}$$

Since the constants involved in ($H_\varepsilon 1$), ($H_\varepsilon 2$), ($H_\varepsilon 4$), ($H_\varepsilon 5$) and (1.6.4) are independent of ε, from Lemma 1.6.1 we get that there exist k_1 and k_2, independent of ε, such that

$$k_1\|u\|_{A_\varepsilon} \le \|u\|_{\mathcal{D}_{p,\varepsilon}} \le k_2\|u\|_{A_\varepsilon}\,. \tag{1.6.5}$$

Since the operator A_ε satisfies the assumptions of Proposition 1.2.3, for every $\lambda > \sqrt{2}\frac{c_\beta}{p}$ one has $\lambda \in \rho(A_\varepsilon)$ and $\|R(\lambda, A_\varepsilon)\| \le \left(\lambda - \sqrt{2}\frac{c_\beta}{p}\right)^{-1}$. In fact, using the inequality $V_\varepsilon \ge (1+\varepsilon)^{-1}$, the first estimate in (1.6.2) and noting that (1.1.7) implies $\sqrt{2}\,\frac{\beta+\sqrt{N}\alpha\theta}{p} < 1$, we get

$$-\inf_{x\in\mathbb{R}^N}\left(\frac{1}{p}\,\mathrm{div}F_\varepsilon(x) + V_\varepsilon(x)\right) \le \frac{1}{1+\varepsilon}\left(\sqrt{2}\,\frac{\beta+\sqrt{N}\alpha\theta}{p} - 1\right) + \sqrt{2}\frac{c_\beta}{p} < \sqrt{2}\frac{c_\beta}{p}\,.$$

Therefore, if $\lambda > \sqrt{2}\,\frac{c_\beta}{p}$ then for every $f \in L^p(\mathbb{R}^N)$ and for all $\varepsilon > 0$, there exists a unique $u_\varepsilon \in \mathcal{D}_{p,\varepsilon}$ such that

$$\lambda u_\varepsilon - A_\varepsilon u_\varepsilon = f, \quad \text{in } \mathbb{R}^N \tag{1.6.6}$$

and

$$\|u_\varepsilon\|_p \le \left(\lambda - \sqrt{2}\frac{c_\beta}{p}\right)^{-1}\|f\|_p\,. \tag{1.6.7}$$

Using (1.6.5), (1.6.6) and (1.6.7) we obtain

$$\|u_\varepsilon\|_{\mathcal{D}_{p,\varepsilon}} \le k_2\left(\|A_\varepsilon u_\varepsilon\|_p + \|u_\varepsilon\|_p\right) \le k_2\left(1 + \frac{\lambda+1}{\lambda - \sqrt{2}\frac{c_\beta}{p}}\right)\|f\|_p\,. \tag{1.6.8}$$

In particular, we have that $\{u_\varepsilon\}$ is bounded in $W^{2,p}(\mathbb{R}^N)$, thus there exist $u \in W^{2,p}(\mathbb{R}^N)$ and a sequence $\{u_{\varepsilon_n}\}$ converging to u weakly in $W^{2,p}(\mathbb{R}^N)$ and strongly in $W^{1,p}_{\mathrm{loc}}(\mathbb{R}^N)$. Therefore, up to a subsequence, $u_{\varepsilon_n} \to u$ and $Du_{\varepsilon_n} \to Du$ a.e. in $\mathbb{R}^N$. From (1.6.8) we obtain in particular that $\|V_{\varepsilon_n}u_{\varepsilon_n}\|_p + \|\langle F_{\varepsilon_n}, Du_{\varepsilon_n}\rangle\|_p \le c\|f\|_p$, which implies, using Fatou's Lemma, that

$$\|Vu\|_p + \|\langle F, Du\rangle\|_p \le c\|f\|_p\,.$$

Thus, $u \in \mathcal{D}_p$.

It remains to prove that u solves $\lambda u - Au = f$ a.e. in $\mathbb{R}^N$. From (1.6.6) and the definition of A_{ε_n} we infer that

$$\lambda u_{\varepsilon_n} - A_0 u_{\varepsilon_n} = f_{\varepsilon_n},$$

where $f_{\varepsilon_n} = f + \langle F_{\varepsilon_n}, Du_{\varepsilon_n}\rangle - V_{\varepsilon_n}u_{\varepsilon_n} \in L^p(\mathbb{R}^N)$. Applying the classical local L^p-estimates (see [26, Theorem 9.11]) it follows that for every $0 < \rho_1 < \rho_2$

$$\|u_{\varepsilon_n}\|_{W^{2,p}(B_{\rho_1})} \le C(\|f_{\varepsilon_n}\|_{L^p(B_{\rho_2})} + \|u_{\varepsilon_n}\|_{L^p(B_{\rho_2})}), \tag{1.6.9}$$

with C depending on ρ_1, ρ_2 but independent of n. Since u_{ε_n} and f_{ε_n} converge to u and $f + \langle F, Du\rangle - Vu$, respectively, in $L^p_{\mathrm{loc}}(\mathbb{R}^N)$ as $n \to \infty$, by applying (1.6.9) to the difference $u_{\varepsilon_n} - u_{\varepsilon_m}$ we get that $\{u_{\varepsilon_n}\}$ is a Cauchy sequence in $W^{2,p}(B_{\rho_1})$. This implies that u_{ε_n} converges to u in $W^{2,p}_{\mathrm{loc}}(\mathbb{R}^N)$ and then, letting $n \to \infty$ in the equation solved by u_{ε_n}, it follows that u satisfies $\lambda u - Au = f$ a.e. in $\mathbb{R}^N$.

To conclude the proof it remains to show that $\lambda - A$ is surjective also when $\lambda > \frac{c_\beta}{p}$. This follows from the dissipativity of the operator $A - \frac{c_\beta}{p}$, stated in Lemma 1.6.1, and the fact that $\lambda - (A - \frac{c_\beta}{p})$ is surjective for $\lambda > (\sqrt{2}-1)c_\beta/p$. Thus $\lambda - (A - \frac{c_\beta}{p})$ is also surjective for $\lambda > 0$, which means that $\lambda - A$ is surjective for $\lambda > \frac{c_\beta}{p}$, as claimed. □

We are ready to prove Theorem 1.1.2.

Proof of Theorem 1.1.2. Since $C_c^\infty(\mathbb{R}^N) \subset \mathcal{D}_p \subset L^p(\mathbb{R}^N)$, it follows that $\mathcal{D}_p$ is a dense subset in $L^p(\mathbb{R}^N)$. Moreover, $(A, \mathcal{D}_p)$ is closed, by Lemma 1.6.1. By Proposition 1.6.2 and (1.6.1), for every $\lambda > \frac{c_\beta}{p}$, $\lambda - A : \mathcal{D}_p \to L^p(\mathbb{R}^N)$ is bijective and

$$\|(\lambda - A)^{-1} f\|_p \le \left(\lambda - \frac{c_\beta}{p}\right)^{-1} \|f\|_p .$$

The thesis follows from the Hille-Yosida Theorem. □

1.7 Comments and consequences

In this final section we establish some further properties of the semigroup $T_p(\cdot)$ generated by $(A, \mathcal{D}_p)$ on $L^p(\mathbb{R}^N)$. We note that since all the assumptions of Theorem 1.1.2 for $p = 2$ imply those of Theorem 1.1.1, the semigroup $T_2(\cdot)$ is uniquely determined.
We point out that the semigroups given by Theorem 1.1.2 are not analytic, in general. A counterexample is the Ornstein-Uhlenbeck semigroup, as shown below (see e.g. [35, Example 4.4]).

Example 1.7.1 Let $Au = u'' + xu'$ be the Ornstein Uhlenbeck operator in one dimension. We prove that the semigroup $T(t)$ generated by A with domain $D(A) = \{u \in W^{2,p}(\mathbb{R}) \mid xu' \in L^p(\mathbb{R})\}$ in $L^p(\mathbb{R})$ is not differentiable and hence, *a fortiori*, it is not analytic. To this aim it is sufficient to prove that $T(t)$ is not continuous from $L^p(\mathbb{R})$ in $D(A)$. For every $u \in L^p(\mathbb{R})$, $t > 0$ and $x \in \mathbb{R}$, the Ornstein Uhlenbeck semigroup can be represented by

$$(T(t)u)(x) = \frac{1}{\sqrt{2\pi(e^{2t}-1)}} \int_{\mathbb{R}} e^{-\frac{y^2}{2(e^{2t}-1)}} u(e^t x - y) dy.$$

Let $u_n = \chi_{[n,n+1]}$. Then

$$(T(t)u_n)(x) = \frac{1}{\sqrt{2\pi(e^{2t}-1)}} \int_{e^t x - n - 1}^{e^t x - n} e^{-\frac{y^2}{2(e^{2t}-1)}} dy$$

and consequently

$$\frac{d}{dx}(T(t)u_n)(x) = \frac{e^t}{\sqrt{2\pi(e^{2t}-1)}} \Big(e^{-\frac{(e^t x - n)^2}{2(e^{2t}-1)}} - e^{-\frac{(e^t x - n - 1)^2}{2(e^{2t}-1)}} \Big),$$

$$\frac{d^2}{dx^2}(T(t)u_n)(x) = \frac{e^{2t}}{\sqrt{2\pi(e^{2t}-1)^3}} \Big(-(e^t x - n) e^{-\frac{(e^t x - n)^2}{2(e^{2t}-1)}} + (e^t x - n - 1) e^{-\frac{(e^t x - n - 1)^2}{2(e^{2t}-1)}} \Big).$$

It follows that

$$\begin{aligned}
\left\| \frac{d^2}{dx^2}(T(t)u_n) \right\|_p &= \frac{e^{2t}}{\sqrt{2\pi(e^{2t}-1)^3}} \left(\int_{\mathbb{R}} \left| y\, e^{-\frac{y^2}{2(e^{2t}-1)}} - (y-1)\, e^{-\frac{(y-1)^2}{2(e^{2t}-1)}} \right|^p e^{-t} dy \right)^{\frac{1}{p}} \\
&\le \frac{e^{2t} 2^{1-\frac{1}{p}}}{\sqrt{2\pi(e^{2t}-1)^3}} \left(\int_{\mathbb{R}} |y|^p\, e^{-\frac{p\, y^2}{2(e^{2t}-1)}} e^{-t} dy \right. \\
&\quad \left. + \int_{\mathbb{R}} |y-1|^p\, e^{-\frac{p(y-1)^2}{2(e^{2t}-1)}} e^{-t} dy \right)^{\frac{1}{p}} \\
&\le \frac{2\, e^{2t - \frac{t}{p}}}{\sqrt{2\pi(e^{2t}-1)^3}} \left(\int_{\mathbb{R}} |y|^p\, e^{-\frac{p\, y^2}{2(e^{2t}-1)}} dy \right)^{\frac{1}{p}} \\
&= c_p \frac{e^{t\left(2 - \frac{1}{p}\right)}}{(e^{2t}-1)^{1-\frac{1}{2p}}}.
\end{aligned}$$

Hence $\left\|\frac{d^2}{dx^2}(T(t)u_n)\right\|_p$ can be estimated indipendently of n. Moreover we have

$$\left\|x\frac{d}{dx}(T(t)u_n)\right\|_p^p = \frac{1}{(2\pi(e^{2t}-1))^{\frac{p}{2}}}\int_{\mathbb{R}}|y+n|^p\left|e^{-\frac{y^2}{2(e^{2t}-1)}}-e^{-\frac{(y-1)^2}{2(e^{2t}-1)}}\right|^p e^{-t}dy.$$

Since $y^2 \le (y-1)^2$ if $y \le 1/2$, by Fatou's Lemma we deduce that

$$\liminf_{n\to+\infty}\left\|x\frac{d}{dx}(T(t)u_n)\right\|_p^p \ge \frac{e^{-t}}{(2\pi(e^{2t}-1))^{\frac{p}{2}}}\int_{\{0\le y\le\frac{1}{2}\}}\liminf_{n\to+\infty}(y+n)^p\,e^{-\frac{p\,y^2}{2(e^{2t}-1)}}dy = +\infty.$$

Thus we have found a sequence (u_n) in $L^p(\mathbb{R})$ such that $\|u_n\|_p = 1$ but $\lim_{n\to+\infty}\|AT(t)u_n\|_p = +\infty$, for every fixed $t > 0$.

In the following proposition we prove the consistency of $T_p(\cdot)$.

Proposition 1.7.2 *Assume that the assumptions of Theorem 1.1.2 hold for some p and q, with $1 < p, q < +\infty$. If $f \in L^p(\mathbb{R}^N) \cap L^q(\mathbb{R}^N)$ then $T_p(t)f = T_q(t)f$, for all $t \ge 0$.*

Proof. By [21, Corollary III.5.5] we have only to prove that the resolvent operators of $(A, \mathcal{D}_p)$, $(A, \mathcal{D}_q)$ are consistent, for λ large, i.e. that for every $f \in L^p(\mathbb{R}^N) \cap L^q(\mathbb{R}^N)$ there exists $u \in W^{2,p}(\mathbb{R}^N) \cap W^{2,q}(\mathbb{R}^N)$ such that $\lambda u - Au = f$. This follows from the proofs of Proposition 1.6.2 and [37, Theorem 2.2] since the same property holds for uniformly elliptic operators. □

Now we prove the positivity of T_p.

Proposition 1.7.3 *$T_p(\cdot)$ is positive, i.e. if $f \in L^p(\mathbb{R}^N)$, $f \ge 0$, then $T_p(t)f \ge 0$, for all $t \ge 0$.*

Proof. The positivity of the semigroup T_p is equivalent to the positivity of the resolvent $(\lambda - A)^{-1}$ for all λ sufficiently large. By the proof of Proposition 1.6.2 this last property turns out to be true once that each A_ε is shown to have a positive resolvent. From [37, Theorem 2.2] this holds because the operators A_ε can be approximated by uniformly elliptic operators. □

In the following proposition we show the compactness of the resolvent of $(A, \mathcal{D}_p)$ assuming that the potential V tends to infinity as $|x| \to +\infty$. This result is similar to [41, Proposition 6.4] and we give the proof for the sake of completeness.

Proposition 1.7.4 *If $\lim_{|x|\to+\infty} V(x) = +\infty$ then the resolvent of $(A, \mathcal{D}_p)$ is compact.*

Proof. Let us prove that $\mathcal{D}_p$ is compactly embedded into $L^p(\mathbb{R}^N)$. Let $\mathcal{F}$ be a bounded subset of $\mathcal{D}_p$. By the assumption, given $\varepsilon > 0$ there exists $R > 0$ such that $V(x) \ge \varepsilon^{-1}$, if $|x| \ge R$. It follows that

$$\int_{|x|>R}|f(x)|^p\,dx \le \varepsilon^p\int_{|x|>R}|V(x)f(x)|^p\,dx \le \varepsilon^p C = \varepsilon' \tag{1.7.1}$$

for every $f \in \mathcal{F}$. Since the embedding of $W^{2,p}(B_R)$ into $L^p(B_R)$ is compact, the set $\mathcal{F}' = \{f_{|B_R} \,|\, f \in \mathcal{F}\}$, which is bounded in $W^{2,p}(B_R)$, is totally bounded in $L^p(B_R)$. Therefore there exist $r \in \mathbb{N}$ and $g_1, ..., g_r \in L^p(B_R)$ such that

$$\mathcal{F}' \subseteq \bigcup_{i=1}^r \{g \in L^p(B_R) \,|\, \|g - g_i\|_{L^p(B_R)} < \varepsilon'\}. \tag{1.7.2}$$

Set

$$\tilde{g}_i = \begin{cases} g_i & \text{in } B_R \\ 0 & \text{in } \mathbb{R}^N \setminus B_R. \end{cases}$$

Then $\tilde{g}_i \in L^p(\mathbb{R}^N)$ and from (1.7.1) and (1.7.2) it follows that

$$\mathcal{F} \subseteq \bigcup_{i=1}^r \{g \in L^p(\mathbb{R}^N) \,|\, \|g - \tilde{g}_i\|_p < 2\varepsilon'\}.$$

This implies that $\mathcal{F}$ is relatively compact in $L^p(\mathbb{R}^N)$ and the proof is complete. □

Finally, as a corollary of the estimates proved in the previous sections we prove an interpolatory estimate for the functions in $\mathcal{D}_p$.

Corollary 1.7.5 *For every $u \in \mathcal{D}_p$ the following estimate*

$$\|Du\|_p \leq c\|u\|_p^{1/2}\|\lambda u - Au\|_p^{1/2}$$

holds for every λ sufficiently large.

PROOF. By density it is sufficient to consider $u \in C_c^\infty(\mathbb{R}^N)$. The thesis easily follows from (1.4.7), (1.6.1) and the inequality

$$\|Du\|_p \leq c\|u\|_p^{1/2}\|D^2 u\|_p^{1/2}\,.$$

□