Chapter 9

Copula and semicopula transforms

In this chapter, a method will be studied for transforming a copula into another one via a continuous and strictly increasing function. For the first time, this method appeared in the theory of semigroups and it was already used for triangular norms ([141, 83]). Recently, it has been studied in the theory of copulas in [49], where strong conditions on the transformating function are given, and in [87], where the authors are interested, in particular, in the study of the invariance of copulas under such transformations. However, the approach presented here takes into account the ideas presented in [7], where transformations of copulas and semicopulas are a useful tool to investigate bivariate notions of aging.

Therefore, in section 9.1 we study first the transformation of semicopulas; then sections 9.2 and 9.3 are devoted to a characterization of this transformation in the class of copulas and to the study of its properties.

For the results here presented, we can also see [46].

9.1 Transformation of semicopulas

We denote by Θ the set of continuous and strictly increasing functions $h : [0,1] \to [0,1]$ with $h(1) = 1$ and we denote by Θ_i the subset of Θ defined by those $h \in \Theta$ that are invertible. The following theorem is basic for what follows.

Theorem 9.1.1. *For all $h \in \Theta$ and $S \in \mathcal{S}$, the function $S_h : [0,1]^2 \to [0,1]$, defined, for all x and y in $[0,1]$, by*

$$S_h(x,y) := h^{[-1]}\left(S(h(x),h(y))\right) \tag{9.1}$$

is a semicopula. Moreover, if S is continuous, then also S_h is continuous.

Proof. If t is in $[0,1]$, then

$$S_h(t,1) = h^{[-1]}\left(S(h(t),h(1))\right) = h^{[-1]}(h(t)) = t = S_h(1,t).$$

Let x, x', y be in $[0,1]$ with $x \le x'$. Then

$$\begin{aligned} h(x) \le h(x') &\Longrightarrow S(h(x),h(y)) \le S(h(x'),h(y)) \\ &\Longrightarrow h^{[-1]}\left(S(h(x),h(y))\right) \le h^{[-1]}\left(S(h(x'),h(y))\right), \end{aligned}$$

namely $x \mapsto S_h(x,y)$ is increasing; similarly, $y \mapsto S_h(x,y)$ is increasing. □

The function S_h given by (9.1) is said to be the *transformation* of S via h, or the *h–transformation* of S.

Theorem 9.1.1 introduces a mapping $\Psi : \mathcal{S} \times \Theta \to \mathcal{S}$ defined, for all x and y in $[0,1]$, by

$$\Psi(S,h)(x,y) := h^{[-1]}\left(S(h(x),h(y))\right).$$

We shall often set $\Psi_h\, S := \Psi(S,h)$.

The set $\{\Psi_h, h \in \Theta\}$ is closed with respect to the composition $\circ$. Moreover, given $h, g \in \Theta$, for all $S \in \mathcal{S}$ we have

$$\begin{aligned} (\Psi_g \circ \Psi_h)\left(S(x,y)\right) &= \Psi\left(\Psi(S,h),g\right)(x,y) = g^{[-1]}\left(\Psi_h\, S\left(g(x),g(y)\right)\right) \\ &= g^{[-1]}\Big(h^{[-1]}\, S\left((h\circ g)(x),(h\circ g)(y)\right)\Big) \\ &= (h\circ g)^{[-1]}\left(S\left((h\circ g)(x),(h\circ g)(y)\right)\right) = \Psi_{h\circ g} S(x,y). \end{aligned}$$

The identity mapping in $\mathcal{S}$, which coincides with $\Psi_{\mathrm{id}_{[0,1]}}$, is, obviously, the neutral element of the composition operator $\circ$ in $\{\Psi_h, h \in \Theta\}$. Moreover, if $h \in \Theta_i$, then Ψ_h admits an inverse function given by $\Psi_h^{-1} = \Psi_{h^{-1}}$ and the mapping $\Psi : \mathcal{S} \times \Theta_i \to \mathcal{S}$ is the so–called *action* of the group Θ_i on $\mathcal{S}$.

Notice that, given the copula Π, for all $h \in \Theta$ $\Psi_h\Pi$ is an Archimedean and continuous t-norm with additive generator $\varphi(t) = -\ln(h(t))$ (see Theorem 1.4.2). Moreover, for all $h \in \Theta$, we have $\Psi_h M = M$ and $\Psi_h Z = Z$.

Definition 9.1.1. A subset $\mathcal{B}$ of $\mathcal{S}$ is said to be *stable* (or *closed*) with respect to (or under) Ψ if the image of $\mathcal{B} \times \Theta$ under Ψ is contained in $\mathcal{B}$, $\Psi_h\mathcal{B} \subseteq \mathcal{B}$ for every $h \in \Theta$.

It is easily proved that the subsets of commutative and continuous semicopulas are closed under Ψ. Moreover, the following result can be proved (see also [141, 83]).

Proposition 9.1.1. *The class $\mathcal{T}$ of all t–norms is closed under Ψ.*

Proof. For each $h \in \Theta$ and $T \in \mathcal{T}$, it suffices to show that the function $T_h := \Psi_h T$, defined by

$$T_h(x,y) := h^{[-1]}\left(T(h(x),h(y))\right) \qquad \text{for all } x, y \in [0,1],$$

is associative. Set $\delta := h(0) \geq 0$. For all s, t and u all belonging to $[0,1]$, simple calculations lead to the two expressions

$$T_h\left[T_h(s,t),u\right] = h^{[-1]}\left\{T\left[T(h(s),h(t))\vee\delta,h(u)\right]\right\}$$
$$T_h\left[s,T_h(t,u)\right] = h^{[-1]}\left\{T\left[h(s),T(h(t),h(u))\vee\delta\right]\right\}.$$

If $T(h(s),h(t)) \leq \delta$, then

$$T_h\left[T_h(s,t),u\right] = h^{[-1]}\left(T(\delta,h(u))\right) \leq h^{[-1]}(\delta) = 0,$$

and either

$$\begin{aligned} T_h\left[s,T_h(t,u)\right] &= h^{[-1]}\left(T\left(h(s),T(h(t),h(u))\right)\right) \\ &= h^{[-1]}\left(T\left(T(h(s),h(t)),h(u)\right)\right) \leq h^{[-1]}\left(T(\delta,h(u)\right) \leq h^{[-1]}(\delta) = 0, \end{aligned}$$

or

$$T_h\left[s,T_h(t,u)\right] = h^{[-1]}\left(T(h(s),\delta)\right) \leq h^{[-1]}(\delta) = 0.$$

Therefore T_h is associative.

If $T\left(h(s),h(t)\right) > \delta$, then

$$T_h\left[T_h(s,t),u\right] = h^{[-1]}\left\{T\left[T(h(s),h(t)),h(u)\right]\right\}$$

and either

$$\begin{aligned} T_h\left[s,T_h(t,u)\right] &= h^{[-1]}\left(T\left(h(s),T(h(t),h(u))\right)\right) \\ &= h^{[-1]}\left(T\left(T(h(s),h(t)),h(u)\right)\right) = T_h\left[T_h(s,t),u\right], \end{aligned}$$

or

$$T_h\left[s,T_h(t,u)\right] = h^{[-1]}\left(T(h(s),\delta)\right) \leq h^{[-1]}(\delta) = 0,$$

but, in this case, we have also

$$\begin{aligned} T_h\left[T_h(s,t),u\right] &= h^{[-1]}\left\{T\left[T(h(s),h(t)),h(u)\right]\right\} \\ &= h^{[-1]}\left(T\left(h(s),T(h(t),h(u))\right)\right) \leq h^{[-1]}\left(T\left(h(s),\delta\right)\right) \leq h^{[-1]}(\delta) = 0; \end{aligned}$$

which is the desired assertion. □

A t–norm T is said to be *isomorphic* to a t–norm T' if, and only if, there exists $h \in \Theta_i$ such that $T' = T_h$, viz. T' is the h–transformation of T. The following result characterizes in terms of transformations two important subsets of t–norms (see [83]).

Theorem 9.1.2. *Let T be a function from $[0,1]^2$ to $[0,1]$.*

(i) *T is a strict t–norm if, and only if, T is isomorphic to* Π.

(ii) *T is a nilpotent t–norm if, and only if, T is isomorphic to W.*

9.2 Transformation of copulas

Given a copula C and a function $h \in \Theta$, let C_h be the h–transformation of C,

$$C_h(x, y) := h^{[-1]}\left(C(h(x), h(y))\right). \tag{9.2}$$

From Theorem 9.1.1, it follows that C_h is a semicopula for all $h \in \Theta$ and for every copula $C \in \mathcal{C}$. However, it is easily checked that C_h need not be a copula, as the following example shows.

Example 9.2.1. Let h be in Θ defined by $h(t) := t^2$. Then

$$W_h(x, y) = h^{-1}\left(W(h(x), h(y))\right) = \sqrt{\max\{x^2 + y^2 - 1, 0\}},$$

namely

$$W_h(x, y) = \begin{cases} 0, & \text{if } x^2 + y^2 \leq 1, \\ \sqrt{x^2 + y^2 - 1}, & \text{otherwise.} \end{cases}$$

And we have

$$W_h\left(1, \frac{6}{10}\right) - W_h\left(\frac{6}{10}, \frac{6}{10}\right) = \frac{6}{10} > \frac{4}{10}.$$

Thus W_h is not 1–Lipschitz, therefore neither the class of copulas nor the class of quasi–copulas are stable under Ψ.

In the following result, we characterize the transformations of copulas.

Theorem 9.2.1. *For each $h \in \Theta$, the following statements are equivalent:*

(a) *h is concave;*

(b) *for every copula C, the transform* (9.2) *is a copula.*

Proof. (a) $\Longrightarrow$ (b) In view of Theorem 9.1.1, it suffices to show that C_h satisfies the rectangular inequality (C2). To this end, let x_1, y_1, x_2, y_2 be points of $[0, 1]$ such that $x_1 \leq x_2$ and $y_1 \leq y_2$. Then the points s_i $(i = 1, 2, 3, 4)$, defined by

$$\begin{aligned} s_1 &= C(h(x_1), h(y_1)), \quad & s_2 &= C(h(x_1), h(y_2)), \\ s_3 &= C(h(x_2), h(y_1)), \quad & s_4 &= C(h(x_2), h(y_2)), \end{aligned}$$

satisfy

$$s_1 \leq s_2 \wedge s_3 \leq s_2 \vee s_3 \leq s_4 \quad \text{and} \quad s_1 + s_4 \geq s_2 + s_3, \tag{9.3}$$

viz. $(s_3, s_2) \prec_w (s_4, s_1)$. Because $h^{[-1]}$ is convex, continuous and increasing, it follows from Tomic's theorem 1.2.3 that

$$h^{[-1]}(s_3) + h^{[-1]}(s_2) \leq h^{[-1]}(s_4) + h^{[-1]}(s_1).$$

Therefore we have

$$h^{[-1]}(C(h(x_2),h(y_1))) + h^{[-1]}(C(h(x_1),h(y_2))) \\ \leq h^{[-1]}(C(h(x_2),h(y_2))) + h^{[-1]}(C(h(x_1),h(y_1))),$$

namely C_h satisfies (C2).

(b) $\Longrightarrow$ (a) It suffices to show that $h^{[-1]}$ is mid–convex, that is

$$\forall s,t \in [0,1] \qquad h^{[-1]}\left(\frac{s+t}{2}\right) \leq \frac{h^{[-1]}(s)+h^{[-1]}(t)}{2}, \tag{9.4}$$

because, then, $h^{[-1]}$ is convex and, hence, h is concave.

Without loss of generality consider the copula W and s and t in $[0,1]$ with $s \leq t$. If $(s+t)/2$ is in $[0,h(0)]$, then (9.4) is immediate. If $(s+t)/2$ is in $]h(0),1]$, then we have

$$W\left(\frac{s+1}{2},\frac{s+1}{2}\right) = s, \quad W\left(\frac{t+1}{2},\frac{t+1}{2}\right) = t$$
$$W\left(\frac{s+1}{2},\frac{t+1}{2}\right) = \frac{s+t}{2} = W\left(\frac{t+1}{2},\frac{s+1}{2}\right).$$

There are points x_1 and x_2 in $[0,1]$ such that

$$h(x_1) = \frac{1+s}{2} \qquad \text{and} \qquad h(x_2) = \frac{1+t}{2}.$$

Since W_h is a copula, we have

$$W_h(x_1,x_1) - W_h(x_1,x_2) - W_h(x_2,x_1) + W_h(x_2,x_2) \geq 0;$$

and, as a consequence

$$h^{[-1]}(s) - h^{[-1]}\left(\frac{s+t}{2}\right) - h^{[-1]}\left(\frac{s+t}{2}\right) + h^{[-1]}(t) \geq 0,$$

which is the desired conclusion. □

Remark 9.2.1. In a special case, an interesting probabilistic interpretation of formula (9.2) is presented in [59, Theorem 5.2.3]: if $h(t) = t^{1/n}$ for some $n \geq 1$, then C_h is the copula associated with componentwise maxima, $X = \max\{X_1,\dots,X_n\}$ and $Y = \max\{Y_1,\dots,Y_n\}$, of a random sample (X_1,Y_1), (X_2,Y_2), $\dots$, (X_n,Y_n) of i.i.d. random vectors with the same copula C. Power transformations of copulas are useful in the theory of extreme value distributions ([104, 14, 20, 87]).

Remark 9.2.2. Let H be a bivariate distribution function with marginals F and G and let h be a concave and strictly increasing function. From the proof of Theorem 9.2.1, it is easily proved that the function $\widetilde{H}$ given, for every $(x,y) \in \overline{\mathbb{R}}^2$, by

$$\widetilde{H}(x,y) = h(H(x,y)) \tag{9.5}$$

is a bivariate distribution function with margins $h(F)$ and $h(G)$. Moreover, if the margins are continuous, the copula of $\widetilde{H}$ is $C_{h^{-1}}$. Transformations of type (9.5) were used in the field of insurance pricing ([58, 156]) and they are also called *distorted probability measure* in the context of non–additive probabilities ([30]).

9.3 Properties of the transformed copula

We denote by Θ_C the set of concave functions in Θ. These properties can be easily proved:

Proposition 9.3.1. *Let h and g be two functions in Θ_C. Then*

(a) $\lambda h + (1-\lambda)g$ *is in* Θ_C *for every* $\alpha \in [0,1]$*;*

(b) $h \circ g$ *is in* Θ_C*;*

(c) $h(t^\alpha)$ *and* $(h(t))^\alpha$ *are in* Θ_C *for all* $\alpha \in]0,1[$.

$h(x)$	$h^{[-1]}(x)$	Parameter
$x^{1/\alpha}$	x^α	$\alpha \geq 1$
$\frac{1-e^{-\alpha x}}{1-e^{-\alpha}}$	$-\frac{1}{\alpha}\log\left(1 - x(1-e^{-\alpha})\right)$	$\alpha > 0$
$\frac{bx}{bx+a(1-x)}$	$\frac{ax}{ax-bx+b}$	$0 < a < b$
$\sin(\pi x/2)$	$(2/\pi)\arcsin x$	
$(4/\pi)\arctan x$	$\tan(\pi x/4)$	

Table 9.1: Examples of functions in Θ_C

Example 9.3.1. Let C be a copula and let r be a function defined on $[0,1]$ by $r(t) = at + b$, with $a, b \in]0,1[$, $a + b = 1$. Then $r^{[-1]}(t) = \max\{0, (t-b)/a\}$ and we have

$$C_r(x,y) = \begin{cases} \frac{1}{a}\left[C(ax+b, ay+b) - b\right], & \text{if } C(ax+b, ay+b) \geq b; \\ 0, & \text{otherwise.} \end{cases}$$

The copula C_r is said to be *linear transformation of* C.

In particular, given $r(t) = (t+1)/2$, let C' be an ordinal sum of type $(\langle 0, 1/2, C\rangle)$. Then $C_r = M$.

Remark 9.3.1. Let h and g be in Θ_C. Given a copula C, the transformations C_h and C_g may be equal, $C_h = C_g$, even though the functions h and g are not equal,

$h \neq g$. For instance, we consider the copula W and let h be the function defined on $[0, 1]$ by $h(t) = (t+1)/2$. Then $W_h = W$ and $W_{\text{id}} = W$, but $\text{id} \neq h$.

Conversely, Let C and D be copulas. Given $h \in \Theta_C$, we may have $C_h = D_h$ even though $C \neq D$. In fact, $C_h(x, y) = D_h(x, y)$ if, and only if,

$$\max\{h(0), C(h(x), h(y))\} = \max\{h(0), D(h(x), h(y))\},$$

viz. it suffices $C = D$ on $[h(0), 1]^2$.

Theorem 9.2.1 introduces, for all $h \in \Theta_C$, a mapping

$$\Psi_h : \mathcal{C} \to \mathcal{C}, \qquad C \mapsto \Psi_h C := C_h,$$

which verifies the properties given in the proposition below.

Proposition 9.3.2. *For every h and g in Θ_C, we have*

(a) $\Psi_h \circ \Psi_g = \Psi_{g \circ h}$*;*

(b) *if $\{C^n\}$ is a sequence of copulas that converges pointwise to the copula C, then $\{\Psi_h C^n\}$ converges pointwise to $\Psi_h C$;*

(c) *Ψ_h is continuous, in the sense that, for every $\epsilon > 0$ there exists $\delta > 0$ such that*

$$\forall A, B \in \mathcal{C} \quad \|A - B\|_\infty < \delta \quad \Longrightarrow \quad \|\Psi_h A - \Psi_h B\|_\infty < \epsilon.$$

(d) *Ψ_h is convex, in the sense that, for every $A, B \in \mathcal{C}$ and $\lambda \in [0, 1]$*

$$\Psi_h(\lambda A + (1-\lambda)B) \leq \lambda \Psi_h A + (1-\lambda)\Psi_h B.$$

Proof. Let h and g be in Θ_C.
(a) For every copula C, we have

$$\begin{aligned}\Psi_h \circ \Psi_g(C) &= \Psi_h\left(g^{[-1]}\left(C(g(x), g(y))\right)\right)\\ &= h^{[-1]}\left(g^{[-1]}\left(C(g(h(x)), g(h(y)))\right)\right) = \Psi_{g\circ h} C,\end{aligned}$$

and, from Proposition 9.3.1, $g \circ h$ is in Θ_C.

(b) For every (x, y) in $[0, 1]^2$, we have

$$C_n(x, y) \xrightarrow{n \to +\infty} C(x, y);$$

and, in particular,

$$C_n(h(x), h(y)) \xrightarrow{n \to +\infty} C(h(x), h(y)).$$

Now, the assertion follows from the continuity of $h^{[-1]}$.

(c) Given two copulas A and B, since $h^{[-1]}$ is convex, we obtain

$$\begin{aligned}&\Psi_h\left(\lambda A(x,y)+(1-\lambda)B(x,y)\right)\\&\quad=h^{[-1]}\left(\lambda A(h(x),h(y))+(1-\lambda)B(h(x),h(y))\right)\\&\quad\le\lambda h^{[-1]}\left(A(h(x),h(y))\right)+(1-\lambda)h^{[-1]}\left(B(h(x),h(y))\right)\\&\quad=\lambda\Psi_hA(x,y)+(1-\lambda)\Psi_hB(x,y),\end{aligned}$$

which concludes the proof. □

As in section 9.1, a subset $\mathcal{B}$ of $\mathcal{C}$ is said to be *stable* with respect to Ψ if the image of $\mathcal{B}\times\Theta_C$ under Ψ is contained in $\mathcal{B}$, $\Psi(\mathcal{B}\times\Theta_C)\subseteq\mathcal{B}$.

Proposition 9.3.3. *The following class of copulas are stable with respect to Ψ:*

(a) *the Archimedean family;*

(b) *the class of associative copulas;*

(c) *the Archimax family.*

Proof. (a) Let C be an Archimedean copula additively generated by φ. For every $h\in\Theta_C$, the h–transformation of C is given by

$$C_h(x,y)=h^{[-1]}\left(\varphi^{[-1]}\left(\varphi(h(x))+\varphi(h(y))\right)\right),$$

viz. C_h is the Archimedean copula generated by $\varphi\circ h$.
Part (b) is a direct consequence of Proposition 9.1.1.
(c) Let C be an Archimax copula defined by the dependence function A and the Archimedean generator φ (see Example 1.6.9). As in part (a), we can prove that the h–transformation of C, C_h, is also an Archimax copula defined by the dependence function A and the Archimedean generator $\varphi\circ h$. □

In [7] some results are presented about the preservation of some dependence properties of a copula C that is transformed via a concave bijection (see Propositions 6.6 and 6.7). Here, we present only a result about the concordance order.

Proposition 9.3.4. *Given C and C' in $\mathcal{C}$, and h in Θ_C, we have*

(a) *the operation Ψ_h is order–preserving in the first place, i.e., $C\le C'$ implies $\Psi_hC\le\Psi_hC'$;*

(b) *if $\Psi_hC\le\Psi_hC'$, then $C(x,y)\le C'(x,y)$ for all $(x,y)\in[h(0),1]^2$.*

Proof. Part (a) is a consequence of the fact that h and $h^{[-1]}$ are both increasing. Part (b) follows by considering that the restriction of h on $[h(0),1]$ is a bijection. □

Notice that, in general, C and its transformation C_h are not ordered in concordance order. It suffices to take, for $\alpha \in]0,1[$, the copula

$$C_\alpha(x,y) := \frac{xy}{[1+(1-x^\alpha)(1-y^\alpha)]^{1/\alpha}},$$

and $h(t) = t^{1/2}$ a function in Θ_C. Then $\Psi_h C_\alpha = C_{\alpha/2}$ and $C_{\alpha/2} \leq C_\alpha$ if, and only if, $x^{\alpha/2} + y^{\alpha/2} \leq 1$ (see also [114, Example 4.15]).