Chapter 4

A new family of PQD copulas

In this chapter we introduce a new class of bivariate copulas, depending on a univariate function, that includes some already known families. This class is characterized in section 4.1, where a probabilistic interpretation is given, and its properties (dependence, measures of association, symmetries, associativity, absolute continuity) are studied in detail in section 4.2. Section 4.3 is devoted to the introduction of a similar class in the set of quasi–copulas.

The contents of this chapter can be also found in [36, 42, 43].

4.1 Characterization of the new class

Let \( f \) be a mapping from \([0, 1]\) into \([0, 1]\). Consider the function \( C_f \) given, for every \( x, y \in [0, 1] \), by

\[
C_f(x, y) := (x \land y) f(x \lor y).
\]

(4.1)

It is obvious that every \( C_f \) is symmetric and the copulas \( \Pi \) and \( M \) are of this type: it suffices to take, respectively, \( f(t) = t \) and \( f(t) = 1 \) for all \( t \in [0, 1] \). Our aim is to study under which conditions on \( f \), \( C_f \) is a copula. Notice that, in view of the properties (1.9) and (1.10) of a copula, it is quite natural to require that \( f \) is increasing and continuous and, then, simple considerations of real analysis imply that \( f \) is differentiable almost everywhere on \([0, 1]\) and the left and right derivatives of \( f \) exist for every \( x \in [0, 1] \) and assume finite values. We aim to characterize the copulas of type (4.1).

Lemma 4.1.1. Let \( f : [0, 1] \to [0, 1] \) be a continuous and increasing function, differentiable except at finitely many points. The following statements are equivalent:

(a) for every \( s, t \in [0, 1] \), with \( s \leq t \), \( sf(s) + tf(t) - 2sf(t) \geq 0 \);

(b) the function \( t \mapsto f(t)/t \) is decreasing on \([0, 1]\).
Proof. \((a) \Rightarrow (b)\): Let \(s_i \ (i = 1,2,\ldots,n)\) be the points in \([0,1]\) such that \(f'(s^-_i) \neq f'(s^+_i)\). Set \(s_0 := 0\) and \(s_{n+1} := 1\). For every \(i \in \{0,1,\ldots,n\}\), let \(s\) and \(t\) be in \([s_i,s_{i+1}[, s < t\). The inequality

\[
sf(s) + tf(t) - 2sf(t) \geq 0
\]

is equivalent to

\[
\frac{f(t)}{s} \geq \frac{f(t) - f(s)}{t - s}.
\]

In the limit \(t \downarrow s\), we have \(f(s) \geq sf'(s)\). It follows that

\[
\left( \frac{f(s)}{s} \right)' = \frac{sf'(s) - f(s)}{s^2} \leq 0,
\]

viz. \(t \mapsto f(t)/t\) is decreasing in each interval \([s_i,s_{i+1}[, (i = 0,1,\ldots,n)\). But \(f(t)/t\) is continuous and, therefore, it is decreasing on the whole \([0,1]\).

\((b) \Rightarrow (a)\): Let \(s,t\) be in \([0,1]\), with \(s < t\). Then

\[
\frac{f(s)}{s} \geq \frac{f(t)}{t}
\]

is equivalent to

\[
\frac{f(s)}{s} \geq \frac{f(t) - f(s)}{t - s},
\]

and, because \(f\) is increasing,

\[
\frac{f(t)}{s} \geq \frac{f(t) - f(s)}{t - s},
\]

viz. condition (a).

\[\square\]

**Theorem 4.1.1.** Let \(f: [0,1] \to [0,1]\) be a differentiable function (except at finitely many points). Let \(C_f\) be the function defined by (4.1). Then \(C_f\) is a copula if, and only if, the following statements hold:

(i) \(f(1) = 1\);

(ii) \(f\) is increasing;

(iii) the function \(t \mapsto f(t)/t\) is decreasing on \([0,1]\).

**Proof.** It is immediate that \(C_f\) satisfies the boundary conditions (C1) if, and only if, \(f(1) = 1\). We now prove that \(C_f\) is 2–increasing if, and only if, (ii) and (iii) hold. Let \(x,x',y,y'\) be in \([0,1]\) with \(x \leq x'\) and \(y \leq y'\). First, we suppose that the rectangle \([x,x'] \times [y,y']\) is a subset of \(\Delta_+\) (see notations (1.12)). Then

\[
V_C([x,x'] \times [y,y']) = (y' - y) (f(x') - f(x)) \geq 0
\]

if, and only if, \(f\) is increasing. Analogously, the 2–increasing property is equivalent to (ii) for rectangles contained in \(\Delta_-\). If, instead, the diagonal of \([x,x'] \times [y,y']\) lies
on the diagonal \( \{(x, y) \in [0,1]^2 : y = x\} \) of the unit square, then \( x = y \) and \( x' = y' \) and, in view of Lemma 4.1.1,

\[
V_C([x, x'] \times [x, x']) = xf(x) + x'f(x') - 2xf(x') \geq 0
\]

if, and only if, (iii) holds. Now, the assertion follows from Proposition 1.6.1.

A function \( f \) that satisfies the assumptions of Theorem 4.1.1 is called *generator* of a copula of type (4.1). In particular, the class of generators is convex and, because of condition (iii), it has minimal element \( \text{id}_{[0,1]} \) and maximal element the constant function equal to 1. Note that \( f : [0,1] \to [0,1] \) satisfies condition (iii) of Theorem 4.1.1 if, and only if, \( f \) is *star–shaped*, i.e., \( f(\alpha x) \geq \alpha f(x) \) for all \( \alpha \in [0,1] \). Moreover, every concave function satisfies (iii) (these results can also be found in [103, Chap. 16]). Now, we give a probabilistic interpretation of the generators.

**Proposition 4.1.1.** Let \( U \) and \( V \) be r.v.’s uniformly distributed on \([0,1]\) with copula \( C_f \) of type (4.1). Then

\[
f(t) = P(\max\{U, V\} \leq t | U \leq t).
\]

*Proof.* For every \( t \) in \([0,1]\), we have

\[
C(t,t) = tf(t) = P(U \leq t, V \leq t),
\]

and

\[
P(\max\{U, V\} \leq t | U \leq t) = \frac{P(U \leq t, V \leq t)}{P(U \leq t)} = f(t),
\]

namely the assertion. \( \Box \)

In the sequel we give some sub–classes of copulas \( \{C_\alpha\} \) of type (4.1) generated by a one–parameter family \( \{f_\alpha\} \).

**Example 4.1.1** (Fréchet copulas). Given \( f_\alpha(t) := \alpha t + (1 - \alpha) (\alpha \in [0,1]) \), we obtain \( C_\alpha = \alpha \Pi + (1 - \alpha)\mathcal{M} \), which is a convex sum of \( \Pi \) and \( \mathcal{M} \) and, therefore, is a member of the Fréchet family of copulas (see Example 1.6.2) (see, also, family (B11) in [74]). Notice that \( C_0 = \mathcal{M} \) and \( C_1 = \Pi \).

**Example 4.1.2** (Cuadras–Augé copulas). Given \( f_\alpha(t) := t^\alpha (\alpha \in [0,1]) \), \( C_\alpha \) is defined by

\[
C_\alpha(x, y) = (x \wedge y)(x \vee y)^\alpha = \begin{cases} 
xy^\alpha, & \text{if } x \leq y; \\
x^\alpha y, & \text{if } x > y.
\end{cases}
\]

Then \( C_\alpha \) describes the Cuadras–Augé family of copulas (see Example 1.6.4). Notice that \( C_0 = \mathcal{M} \) and \( C_1 = \Pi \).
Example 4.1.3. Given \( f_\alpha(t) := \min(\alpha t, 1) \) \((\alpha \geq 1)\), \( C_\alpha \) is defined by \( C_\alpha(x, y) = (x \wedge y) \min\{\alpha(x \vee y), 1\} \) \( = \begin{cases} \alpha xy, & \text{if } (x, y) \in \left[0, \frac{1}{\alpha}\right]^2; \\ x \wedge y, & \text{otherwise;} \end{cases} \) viz. \( C_\alpha \) is the ordinal sum \( \langle 0, \frac{1}{\alpha}, \Pi \rangle \). Notice that \( C_1 = \Pi \) and \( C_\infty = M \), where, if \( g(x) = \lim f_\alpha(x) \) as \( \alpha \to +\infty \) and \( x \in [0, 1] \), \( C_\infty := C_g \).

Example 4.1.4. Given the function \( f_\alpha(t) := c \exp(t^\alpha) \), where \( \alpha > 0 \) and \( c = \exp(-1/\alpha) \), we obtain the following family \( C_\alpha(x, y) = \begin{cases} cx \exp(y^\alpha/\alpha), & \text{if } x \leq y; \\ cy \exp(x^\alpha/\alpha), & \text{if } x > y. \end{cases} \)

Example 4.1.5. The function \( f_\alpha(t) := \frac{1}{\sin \alpha} \sin(\alpha t) \) \((\alpha \in [0, \pi/2])\) is increasing with \( f_\alpha(t)/t \) decreasing on \([0, 1]\), as is easily proved. Therefore, Theorem 4.1.1 ensures that \( C_\alpha(x, y) = \begin{cases} \frac{x}{\sin \alpha} \sin(\alpha y), & \text{if } x \leq y; \\ \frac{y}{\sin \alpha} \sin(\alpha x), & \text{if } x > y. \end{cases} \)

is a copula.

For a copula \( C_f \) of type (4.1) the following result holds (see [100] for details).

**Theorem 4.1.2.** If \( C_f \) is the copula given by (4.1) and \( H(x, y) = C_f(F_1(x), F_2(y)) \) for univariate d.f.’s \( F_1 \) and \( F_2 \), then the following statements are equivalent:

(a) random variables \( X \) and \( Y \) with joint d.f. \( H \) have a representation of the form

\[ X = \max\{R, W\} \quad \text{and} \quad Y = \max\{S, W\} \]

where \( R, S \) and \( W \) are independent r.v.’s;

(b) \( H \) has the form \( H(x, y) = F_R(x)F_S(y)F_W(x \wedge y) \), where \( F_R, F_S \) and \( F_W \) are univariate d.f.’s.

### 4.2 Properties of the new class

In this section we give the most important properties of a copula \( C_f \) of type (4.1).

#### 4.2.1 Concordance order

**Proposition 4.2.1.** Let \( C_f \) and \( C_g \) be two copulas of type (4.1). Then \( C_f \leq C_g \) if, and only if, \( f(t) \leq g(t) \) for all \( t \in [0, 1] \).
In particular, for every copula $C_f$, $\Pi \leq C_f \leq M$ and, therefore, every $C_f$ is positively quadrant dependent.

**Example 4.2.1.** Consider the family $\{f_\alpha\}$ $(\alpha \geq 1)$, given by $f_\alpha(t) := 1 - (1 - t)^\alpha$. It is easily proved by differentiation that every $f_\alpha$ is increasing with $f_\alpha(t)/t$ decreasing on $[0,1]$. Therefore, this family generates a family of copulas $C_\alpha$, that is positively ordered, with $C_1 = \Pi$ and $C_\infty = M$.

**Example 4.2.2.** Consider the family of copulas generated by the function $f_\alpha(t) := (1 + \alpha) t / \alpha t + 1$ for every $\alpha \geq 0$. This family is positively ordered with $C_0 = \Pi$ and $C_\infty = M$.

### 4.2.2 Dependence concepts

**Theorem 4.2.1.** Let $(X,Y)$ be a continuous random pair with copula $C_f$. Then

(a) $Y$ is left tail decreasing in $X$;

(b) $Y$ is stochastically increasing in $X$ if, and only if, $f'$ is decreasing a.e. on $[0,1]$;

(c) $X$ and $Y$ are left corner set decreasing.

**Proof.** In order to prove $\text{LTD}(Y|X)$, according to Proposition 1.7.2 it suffices to notice that, for every $(x,y) \in [0,1]^2$,

$$\frac{C_f(x,y)}{x} = \begin{cases} f(y), & \text{if } x \leq y; \\ y f(x), & \text{if } x > y; \end{cases}$$

is decreasing in $x$.

Property $\text{SI}(Y|X)$ follows from Proposition 1.7.3, observing that $\partial_x C_f$ is decreasing in the first place if, and only if, $f'$ is decreasing a.e. on $[0,1]$.

In order to prove (c), because of Proposition 1.7.4, it suffices to prove that, for all $x, x', y, y' \in [0,1]$, with $x \leq x'$ and $y \leq y'$,

$$C_f(x,y) C_f(x',y') \geq C_f(x,y') C_f(x',y) \geq 0. \quad (4.2)$$

Because $f(t)/t$ is decreasing and $C_f$ is symmetric, inequality (4.2) follows easily from simple calculations on rectangles $[x, x'] \times [y, y']$ that have 4, 3 or 2 vertices in the set $\Delta_+$. For instance, if $[x, x'] \times [y, y']$ has only two vertices, say $(x, y)$ and $(x', y)$ in $\Delta_+$, then (4.2) holds if, and only if, $x' f(x) \geq x f(x')$, viz. $f(t)/t$ is decreasing.

The following result for the tail dependence holds.

**Proposition 4.2.2.** Let $C_f$ be a copula of type (4.1). Then, the lower tail dependence of $C_f$ is $f(0^+)$ and the upper tail dependence of $C_f$ is $1 - f'(1^-)$. 
Proof. The diagonal section of $C_f$ is $\delta C_f(t) = tf(t)$. Therefore, from Proposition 1.7.5, we have $\lambda_L = \delta'_C(0^+) = f(0^+)$ and $\lambda_U = 2 - \delta'_C(1^-) = 1 - f'(1^-)$.

Remark 4.2.1. As noted, a copula of type (4.1) is PQD and, therefore, it is suitable to describe positive dependence of a random vector $(X, Y)$. However, it is very simple to introduce a copula to describing, for example, the (negative) dependence of the random vector $(X, -Y)$. It suffices to consider the copula $C_{0,1}^f$ given by

$$C_{0,1}^f(x, y) := x - C(x, 1 - y) = \begin{cases} x(1 - f(1 - y)), & \text{if } x + y \leq 1; \\ x - (1 - y)f(x), & \text{otherwise}. \end{cases}$$

4.2.3 Measures of association

Theorem 4.2.2. The values of several measures of association of $C_f$ are, respectively, given by

$$\tau_C = 4 \int_0^1 xf^2(x) \, dx - 1, \quad \rho_C = 12 \int_0^1 x^2 f(x) \, dx - 3,$$

$$\gamma_C = 4 \left( \int_0^{1/2} x [f(x) + f(1 - x)] \, dx + \int_{1/2}^1 f(x) \, dx \right) - 2,$$

$$\beta_C = 2f(1/2) - 1, \quad \varphi_C = 6 \int_0^1 x f(x) \, dx - 2.$$  

Proof. In view of Theorem 1.8.1, the Kendall’s tau of $C_f$ is given by

$$\tau_C = 1 - 4 \int_0^1 \int_0^1 \partial_x C(x, y) \partial_y C(x, y) \, dx \, dy.$$  

Now, we have

$$\int_0^1 \int_0^1 \partial_x C(x, y) \partial_y C(x, y) \, dx \, dy = \int_0^1 dy \int_0^y x f(y)f'(y) \, dx + \int_0^1 dx \int_0^x y f(x)f'(x) \, dy$$

$$= \int_0^1 x^2 f(x)f'(x) \, dx = \frac{1}{2} \int_0^1 x f^2(x) \, dx,$$

where the last equality is obtained through integration by parts. Then

$$\tau_C = 4 \int_0^1 x f^2(x) \, dx - 1.$$  

From Theorem 1.8.2, Spearman’s rho is given by:

$$\rho_C = 12 \int_0^1 \int_0^1 C(x, y) \, dx \, dy - 3$$

$$= 12 \int_0^1 dy \int_0^y x f(y) \, dx + \int_0^1 dx \int_0^x y f(x) \, dy - 3$$

$$= 12 \int_0^1 x^2 f(x) \, dx - 3.$$
Following Theorem 1.8.3, we have

\[
\gamma_C = 4 \left( \int_0^1 C(x, 1-x) \, dx - \int_0^{1/2} (1-x) f(x) \, dx \right)
+ \int_{1/2}^1 (1-x) f(x) \, dx - \int_0^{1/2} [x-x f(x)] \, dx
\]

\[
= 4 \left( \int_0^{1/2} x f(1-x) \, dx - \int_0^{1/2} [x-x f(x)] \, dx \right)
+ \int_{1/2}^1 (1-x) f(x) \, dx - \int_0^{1/2} [x-x f(x)] \, dx
\]

\[
= 4 \left( \int_0^{1/2} x [f(x) + f(1-x)] \, dx + \int_{1/2}^1 f(x) \, dx \right) - 2.
\]

The expressions of \( \beta_C \) and \( \varphi_C \) follow easily from Theorems 1.8.4 and 1.8.5.

As an application of Theorem 4.2.2, the measures of association for the copulas in Examples 1.6.2 and 1.6.4 can be easily given:

- If \( C \) is a copula of the Fréchet family, then

\[
\tau_C = \frac{(\alpha - 1)(\alpha - 3)}{3}, \quad \rho_C = 1 - \alpha = \gamma_C = \varphi_C.
\]

- If \( C \) is a Cuadras–Augé copula, then

\[
\tau_C = \frac{1 - \alpha}{1 + \alpha}, \quad \rho_C = \frac{3 - 3\alpha}{3 + \alpha}, \quad \varphi_C = \frac{2 - 2\alpha}{2 + \alpha}.
\]

### 4.2.4 Symmetry properties

**Theorem 4.2.3.** Let \((X, Y)\) be continuous r.v.‘s with copula \( C_f \).

(a) If \( X \) and \( Y \) are identically distributed, then \( X \) and \( Y \) are exchangeable.

(b) If \( X \) and \( Y \) are symmetric about \( a \) and \( b \), respectively \((a, b \in \mathbb{R})\), then \( (X, Y) \) is radially symmetric about \((a, b)\) if, and only if, \( C_f = \alpha \Pi + (1 - \alpha)M \) for some \( \alpha \in [0, 1] \).

(c) If \( X \) and \( Y \) are symmetric about \( a \) and \( b \), respectively \((a, b \in \mathbb{R})\), then \( (X, Y) \) is jointly symmetric about \((a, b)\) if, and only if, \( C_f = \Pi \).

**Proof.** Statement (a) is a consequence of the symmetry of \( C_f \). From Proposition 1.6.3, statement (b) holds if, and only if, \( C_f \) satisfies the following functional equation:

\[
\forall x, y \in [0, 1] \quad C_f(x, y) = x + y - 1 + C_f(1-x, 1-y). \tag{4.3}
\]

But, equality (4.3) is equivalent to

\[
(x \wedge y)f(x \vee y) = x + y - 1 + [1 - (x \vee y)] f [1 - (x \wedge y)];
\]
in particular, for all \( y \in [x, 1] \), we have
\[
x f(y) = x + y - 1 + (1 - y)f(1 - x)
\]
\[
\implies x (1 - f(y)) + (1 - y)f(1 - x) = 1 - y
\]
\[
\implies x \frac{1 - f(y)}{1 - y} + f(1 - x) = 1 \implies f(1 - x) = 1 - x \cdot \frac{f(y) - 1}{y - 1}.
\]
In the limit \( y \uparrow 1 \), we can derive
\[
\frac{1 - f(y)}{1 - y} \to f'(1^-),
\]
where \( f'(1^-) \) is a real number in \([0, 1]\). Thus \( f(1 - x) = 1 - cx \), i.e. \( f(x) = cx + (1 - c) \), which corresponds to the family \( C_f = c\Pi + (1 - c)M \).

From Proposition 1.6.3, \((X, Y)\) is jointly symmetric about \((a, b)\) if, and only if, for all \((x, y)\in[0,1]^2\)
\[
C_f(x, y) = x - C_f(x, 1 - y) \quad \text{and} \quad C_f(x, y) = y - C_f(1 - x, y). \tag{4.4}
\]
In particular, for \( x = y \), we obtain
\[
\forall x \in [0, 1] \quad x f(x) = x - [x \wedge (1 - x)] f [x \vee (1 - x)],
\]
which implies
\[
\forall x \in [1/2, 1] \quad x f(x) = x - (1 - x)f(x),
\]
\[
\forall x \in [0, 1/2] \quad x f(x) = x - x f(1 - x),
\]
viz. \( f(x) = x \) on \([0, 1]\), which corresponds to \( C_f = \Pi \).

**4.2.5 Associativity**

**Lemma 4.2.1.** Let \( C_f \) be a copula of type (4.1). Then \( C_f \) is Archimedean if, and only if, \( C_f = \Pi \).

**Proof.** If \( C_f \) is an Archimedean copula, then, there exists a convex function \( \varphi : [0, 1] \to [0, +\infty] \), which is continuous and strictly increasing, \( \varphi(1) = 0 \), such that \( C_f(x, y) = \varphi^{-1} (\varphi(x) + \varphi(y)) \). In view of Theorem 1.6.8,
\[
\varphi'(x) \frac{\partial C_f(x, y)}{\partial y} = \varphi'(y) \frac{\partial C_f(x, y)}{\partial x} \quad \text{a.e. on } [0, 1]^2.
\]
In particular, if \( x = y \), we obtain \( \varphi'(x) \cdot x f'(x) = \varphi'(x) \cdot f(x) \), which leads to \( x f'(x) = f(x) \). In the class of the generators of a copula of type (4.1), this differential equation has as unique solution the function \( f(x) = x \), viz. \( C_f = \Pi \).

**Theorem 4.2.4.** Let \( C_f \) be a copula of type (4.1). Then \( C_f \) is associative if, and only if, \( C_f \) is an ordinal sum of type \((0, a, \Pi)\) with \( a \in [0, 1] \).
Proof. First, notice that every ordinal sum of type \((0, a, II)\) is associative and it is generated by the function \(f(t) = \min\{t/a, 1\}\).

Conversely, let \(C_f\) be an associative copula. As asserted in Theorem 1.6.9, the representation of \(C_f\) depends on the set \(ID\) of idempotent elements of \(C_f\), given by \(ID := \{0\} \cup [a, 1]\), where \(a := \inf\{t \in [0, 1] : f(t) = 1\}\). If \(ID = \{0, 1\}\), then \(C_f\) is Archimedean and, therefore, Lemma 4.2.1 ensures that \(C_f = \Pi = ((0, 1, II))\). If \(ID = [0, 1]\), then \(C_f = M = ((0, 0, II))\). Otherwise, \(C_f\) is an ordinal sum of type \((0, a, D)\) for a suitable Archimedean copula \(D\). Therefore, if \(\phi\) is a generator of \(D\), for all \(x, y\) in \([0, a]\),

\[
C_f(x, y) = a \phi^{-1}\left(\frac{x}{a}\right) + \phi\left(\frac{y}{a}\right).
\]

Hence, applying the chain rule to \(\phi(C_f(x, y)/a) = \phi(x/a) + \phi(y/a)\), we obtain

\[
\phi'\left(\frac{C_f(x, y)}{a}\right) \frac{\partial C_f(x, y)}{\partial x} = \phi'\left(\frac{x}{a}\right), \quad \phi'\left(\frac{C_f(x, y)}{a}\right) \frac{\partial C_f(x, y)}{\partial y} = \phi'\left(\frac{y}{a}\right).
\]

Therefore, a.e. on \([0, 1]^2\), we have

\[
\phi'\left(\frac{x}{a}\right) \frac{\partial C_f(x, y)}{\partial y} = \phi'\left(\frac{y}{a}\right) \frac{\partial C_f(x, y)}{\partial x}.
\]

An argument similar to the proof of Lemma 4.2.1 gives \(D = \Pi\), as asserted.

4.2.6 Absolute continuity

Proposition 4.2.3. The only absolutely continuous copula of type (4.1) is \(\Pi\).

Proof. Let \(C_f\) be a copula of type (4.1). If \(C_f\) is absolutely continuous, then

\[
1 = C_f(1, 1) = \int_0^1 \int_0^1 \frac{\partial^2 C}{\partial x \partial y} \, dx \, dy = \int_0^1 \int_0^1 f'(x \vee y) \, dx \, dy.
\]

It follows that

\[
\frac{1}{2} = \int_0^1 ds \int_0^s f'(s) \, dt = \int_0^1 sf'(s) \, ds;
\]

integrating by parts, we have

\[
\int_0^1 f(x) \, dx = \frac{1}{2}.
\]

The function \(f(x) = x\) is a solution of the above equation and, because all functions generating a copula of type (4.1) are greater than \(id_{[0,1]}\), it follows that \(id_{[0,1]}\) is the only solution in this class.

Remark 4.2.2. Let \(C_f\) be a copula of type (4.1), \(C \neq \Pi\). Consider the first derivative of \(C_f\)

\[
\partial_1 C_f(x, y) = \begin{cases} f(y), & \text{if } x < y; \\ x \cdot f'(x), & \text{otherwise.} \end{cases}
\]
For a fixed $y_0$, the mapping $t \mapsto \partial_1 C_f(t, y_0)$ has a jump discontinuity in $y_0$, and, thus, $C_f$ has a singular component along the main diagonal of the unit square. By using [74, Theorem 1.1], the mass of this singular component is given by

$$m = \int_0^1 (f(x) - xf'(x)) \, dx = 2 \cdot \int_0^1 f(x) \, dx - 1.$$

This $m$ has a graphical interpretation if $f$ admits an inverse: in fact, $m$ is the area of the region of the unit square between the graph of $f$ and the graph of $f^{-1}$.

### 4.3 A similar new class of quasi–copulas

Given a function $f : [0, 1] \to [0, 1]$, we are also interested in studying under which conditions on $f$, the following function

$$Q_f(x, y) := (x \land y) f(x \lor y), \quad \text{for all } (x, y) \in [0, 1]^2, \quad (4.5)$$

is a quasi–copula. The following result provides a characterization.

**Theorem 4.3.1.** Let $f : [0, 1] \to [0, 1]$ be a continuous function and let $Q_f$ be defined by (4.5). Then $Q_f$ is a quasi–copula if, and only if, the three following statements hold:

(i) $f(1) = 1$;

(ii) $f$ is increasing;

(iii) $x_1 \cdot \frac{f(x_2) - f(x_1)}{x_2 - x_1} \leq 1$ for every $x_1, x_2 \in [0, 1]$, with $x_1 < x_2$.

**Proof.** First, observe that $Q_f$ satisfies (Q1) if, and only if, $f(1) = 1$ and $Q_f$ satisfies (Q2) if, and only if, (ii) holds. In order to prove that $Q_f$ satisfies (Q3), let $x_1, x_2$ and $y$ be three points in $[0, 1]$ with $x_1 < x_2$. We distinguish three cases. If $x_1 < x_2 \leq y$, then

$$Q_f(x_2, y) - Q_f(x_1, y) = x_2 f(y) - x_1 f(y) \leq x_2 - x_1$$

because $f \leq 1$. If $y \leq x_1 < x_2$, then

$$Q_f(x_2, y) - Q_f(x_1, y) = y \cdot (f(x_2) - f(x_1)) \leq \frac{y}{x_1} \cdot (x_2 - x_1) \leq x_2 - x_1$$

if, and only if, (iii) holds. Finally, if $x_1 \leq y \leq x_2$, in view of the two above cases we obtain

$$Q_f(x_2, y) - Q_f(x_1, y) = (Q_f(x_2, y) - Q_f(y, y)) + (Q_f(y, y) - Q_f(x_1, y)) \leq (x_2 - x_1)$$

if, and only if, (iii) holds. In every case, (iii) is a necessary and sufficient condition that ensures that $Q_f$ satisfies (1.10). \qed
Corollary 4.3.1. Let $f : [0, 1] \rightarrow [0, 1]$ be a differentiable function and let $Q_f$ be defined by (4.5). Then $Q_f$ is a quasi-copula if, and only if, the three following statements hold:

(i) $f(1) = 1$;

(ii) $f$ is increasing;

(iii) $xf'(x) \leq 1$ for every $x \in [0, 1]$.

Notice that if $Q_f$ is a copula, then $t \mapsto f(t)/t$ is decreasing and

$$
\frac{f(x_2) - f(x_1)}{x_2 - x_1} \leq \frac{f(x_1)}{x_1}
$$

for every $x_1, x_2 \in [0, 1]$, with $x_1 < x_2$, from which the condition (iii) of Theorem 4.3.1 follows, viz. $Q_f$ is a quasi-copula. The converse implication need not be true, as the following example shows.

Example 4.3.1. Consider the function $f(t) := t + t^2 - t^3$ on $[0, 1]$. So, $f$ satisfies the assumptions of Theorem 4.3.1, viz. $f'(t) \leq 1/t$ on $[0, 1]$, but $f(t)/t$ is increasing on $[0, 1/2]$. So $Q_f$ is a proper quasi-copula. Another (not everywhere) differentiable function $g$, which leads to a proper quasi-copula, is given by

$$
g(x) = \begin{cases} 
x, & \text{if } x \in [0, 1/4]; \\
2x - 1/4, & \text{if } x \in [1/4, 1/2]; \\
(x + 1)/2, & \text{if } x \in [1/2, 1].
\end{cases}
$$

We have $g'(x) \leq 1/x$ and thus $Q_g$ is a quasi-copula; however, $h(x) := g(x)/x$ is not decreasing (e.g. $h(1/4) = 1$ but $h(1/2) = 3/2$).