Chapter 3

2–increasing aggregation operators

The aim of this chapter is the study of the class of binary aggregation operators (agops, for short) satisfying the 2–increasing property, specifically, by recalling for sake of completeness the definitions already given, we are interested in the functions $A : [0,1]^2 \to [0,1]$ such that

- $A(0,0) = 0$ and $A(1,1) = 1$;
- $A(x,y) \le A(x',y')$ for $x \le x'$ and $y \le y'$;
- $V_A(R) \ge 0$ for every rectangle $R \subseteq [0,1]^2$.

One of the main reasons to study the class $\mathcal{A}_2$ of 2–increasing agops is that it contains, as a distinguished subclass, the restrictions to $[0,1]^2$ of all the bivariate distribution functions F such that $F(0,0) = 0$ and $F(1,1) = 1$; in particular copulas are in this class. On other hand, the 2–increasing property has a relevant connection with the theory of fuzzy measures, where it is also known as “supermodularity” (see [30]).

Notice that, we may limit ourselves to considering only 2–increasing agops because, if A is a 2–increasing agop, it is immediately seen that its dual A^d is 2–decreasing, and conversely. Therefore, analogous results for the 2–decreasing ones can be obtained by duality.

In section 3.1, we characterize some subclasses of 2–increasing agops and some construction methods are presented in section 3.2. Instead, section 3.3 presents the lattice structure of several subsets of $\mathcal{A}_2$. A method for generating a copula using 2–increasing agops is presented in section 3.4.

The results of this chapter are also contained in [38]

3.1 Characterizations of 2–increasing agops

In this section, some subclasses of agops satistying the 2–increasing property are characterized.

Proposition 3.1.1. *Let A be a 2–increasing agop. The following statements hold:*

(a) *the neutral element $e \in [0,1]$ of A, if it exists, is equal to 1;*

(b) *the annihilator $a \in [0,1]$ of A, if it exists, is equal to 0;*

(c) *if A is continuous on the border of $[0,1]^2$, then A is continuous on $[0,1]^2$.*

Proof. Let A be a 2–increasing agop.

If A has neutral element $e \in [0,1[$, then

$$A(1,1) + A(e,e) = 1 + A(e,e) \geq A(e,1) + A(1,e) = 1 + 1,$$

a contradiction. Therefore $e = 1$ (and, as a consequence, A is a copula).

If A has an annihilator $a \in [0,1]$, we assume, if possible, that $a > 0$. We have

$$A(a,a) - A(a,0) - A(0,a) + A(0,0) = -a \geq 0,$$

a contradiction; as a consequence, $a = 0$.

Let A be continuous on the border of $[0,1]^2$ and let (x_0, y_0) be a point in $]0,1[^2$ such that A is not continuous in (x_0, y_0). Suppose, without loss of generality, that there exists a sequence $\{x_n\}_{n\in\mathbb{N}}$ in $[0,1]$, $x_n \leq x_0$ for every $n \in \mathbb{N}$, which tends to x_0, and we have

$$\lim_{n\to+\infty} A(x_n, y_0) < A(x_0, y_0).$$

Therefore, there exists $\epsilon > 0$ and $n_0 \in \mathbb{N}$ such that $A(x_0,y_0) - A(x_n,y_n) > \epsilon$ for every $n \geq n_0$. But, because A is continuous on the border of the unit square, there exists $\overline{n} > n_0$ such that $A(x_0,1) - A(x_{\overline{n}},1) < \epsilon$. But this violates the 2–increasing property, because, in this case,

$$V\left([x_{\overline{n}}, x_0] \times [y_0, 1]\right) < 0.$$

Thus the only possibility is that A is continuous on $[0,1]$. □

Remark 3.1.1. Note that, if $A : [0,1]^2 \to [0,1]$ is 2–increasing and has an annihilator element (which is necessarily equal to 0), then A is increasing in each place. In fact, because of the 2–increasing property, for every x_1, x_2 and y in $[0,1]$, $x_1 \leq x_2$, we have

$$A(x_2,y) - A(x_1,y) \geq A(x_2,0) - A(x_1,0) = 0.$$

But, in general, if $A : [0,1]^2 \to [0,1]$ is 2–increasing, then A need not be increasing in each place. Consider, for example, $A(x,y) = (2x-1)(2y-1)$.

Proposition 3.1.2. *Let M_f be a quasi–arithmetic mean, viz. let a continuous strictly monotone function $f : [0,1] \to \mathbb{R}$ exist such that*

$$M_f(x,y) := f^{-1}\left(\frac{f(x)+f(y)}{2}\right).$$

Then M_f is 2–increasing if, and only if, f^{-1} is convex.

Proof. Let s and t be real numbers and set $a := f^{-1}(s)$ and $b := f^{-1}(t)$. If M_f is 2–increasing, we have, because M_f is also commutative,

$$M_f(a,a) + M_f(b,b) \geq 2\, M_f(a,b),$$

which is equivalent to

$$f^{-1}(s) + f^{-1}(t) \geq 2\, f^{-1}\left(\frac{s+t}{2}\right).$$

This shows that f^{-1} is Jensen–convex and hence convex.

Conversely, let f^{-1} be convex; we have to prove that, whenever $x_1 \leq x_2$ and $y_1 \leq y_2$,

$$M_f(x_1,y_1) + M_f(x_2,y_2) \geq M_f(x_2,y_1) + M_f(x_1,y_2),$$

or, equivalently, that

$$f^{-1}(s_1) + f^{-1}(s_4) \geq f^{-1}(s_2) + f^{-1}(s_3),$$

where

$$s_1 := \frac{f(x_1)+f(y_1)}{2}, \qquad s_4 := \frac{f(x_2)+f(y_2)}{2},$$
$$s_2 := \frac{f(x_2)+f(y_1)}{2}, \qquad s_3 := \frac{f(x_1)+f(y_2)}{2}.$$

Assume now that f is (strictly) increasing; setting

$$\alpha := \frac{s_4 - s_2}{s_4 - s_1},$$

we obtain $\alpha \in [0,1]$ and

$$s_2 = \alpha\, s_1 + (1-\alpha)\, s_4, \qquad s_3 = (1-\alpha)\, s_1 + \alpha\, s_4.$$

Because f^{-1} is convex, we have

$$f^{-1}(s_2) + f^{-1}(s_3) \leq f^{-1}(s_1) + f^{-1}(s_4),$$

namely the assertion.

If, on the other hand, f is (strictly) decreasing, then we set

$$\alpha := \frac{s_1 - s_2}{s_1 - s_4}$$

in order to reach the same conclusion. □

Corollary 3.1.1. *If M_f is a 2–increasing quasi–arithmetic mean generated by f, then*

$$M_f(x, y) \le \frac{x + y}{2} \quad \text{for every } (x, y) \in [0, 1]^2.$$

Proof. In view of Proposition 3.1.2, M_f is 2–increasing if, and only if, f^{-1} is convex. But, if f is increasing, so is f^{-1}, and $M_f(x, y) \le \frac{x+y}{2}$ is equivalent to the fact that f is Jensen–concave and, thus, f^{-1} convex. Instead, if f is decreasing, so is f^{-1}, and $M_f(x, y) \le \frac{x+y}{2}$ is equivalent to the fact that f is Jensen–convex and, thus, f^{-1} convex. □

Proposition 3.1.3. *The Choquet integral–based agop, defined for a and b in $[0, 1]$ by*

$$A_{Ch}(x, y) = \begin{cases} (1 - b)x + by, & \text{if } x \le y, \\ ax + (1 - a)y, & \text{if } x > y, \end{cases}$$

is 2–increasing if, and only if, $a + b \le 1$.

Proof. It is easily proved that A_{Ch} is 2–increasing on every rectangle contained either in Δ_+ or in Δ_-. Now, let $R := [s, t]^2$. Then, for all s and t such that $0 \le s < t \le 1$,

$$V_{A_{Ch}}([s, t]^2) = s + t - [(1 - b)s + bt] - [at + (1 - a)s] \ge 0$$

if, and only if, $a+b \le 1$. Now, the assertion follows directly from Proposition 1.6.1. □

Notice that, if $a+b = 1$, A_{Ch} is the weighted arithmetic mean; and, if $a = b \le 1/2$, we have an OWA operator, $A_{Ch}(x, y) = (1 - a) \min\{x, y\} + a \max\{x, y\}$ (see [159]).

Remark 3.1.2. The above proposition can be also proved by using some known results on fuzzy measures. In fact, following [30], it is known that a Choquet integral operator based on a fuzzy measure m is supermodular if, and only if, the fuzzy measure m is supermodular. But, in the case of 2 inputs, say $\mathbb{X}_2 := \{1, 2\}$, we can define a fuzzy measure m on $2^{\mathbb{X}_2}$ by giving the values $m(\{1\}) = a$ and $m(\{2\}) = b$, where a and b are in $[0, 1]$. Moreover, it is also known that m is supermodular if, and only if, $a + b \le 1$.

A special subclass of 2–increasing agops is that formed by modular agops, i.e. those A for which $V_A(R) = 0$ for every rectangle $R \subseteq [0, 1]^2$. For these operators the following characterization holds.

Proposition 3.1.4. *For an agop A the following statements are equivalent:*

(a) *A is modular;*

(b) *increasing functions f and g from $[0, 1]$ into $[0, 1]$ exist such that $f(0) = g(0) = 0$, $f(1) + g(1) = 1$, and*

$$A(x, y) = f(x) + g(y). \tag{3.1}$$

Proof. If A is modular, set $f(x) := A(x, 0)$ and $g(y) := A(0, y)$. From the modularity of A

$$0 = V_A([0, x] \times [0, y]) = A(x, y) - f(x) - g(y) + A(0, 0),$$

which implies (b). Viceversa, it is clear that every function of type (3.1) is modular. □

3.2 Construction of 2–increasing agops

In the literature, there are a variety of construction methods for agops (see [10] and the references therein). In this section, some of these methods are used to obtain an agop satisfying the 2–increasing property.

Proposition 3.2.1. *Let f and g be increasing functions from $[0, 1]$ into $[0, 1]$ such that $f(0) = g(0) = 0$ and $f(1) = g(1) = 1$. Let A be a 2–increasing agop. Then, the function defined by*

$$A_{f,g}(x, y) := A(f(x), g(y)) \tag{3.2}$$

is a 2–increasing agop.

Proof. It is obvious that $A_{f,g}(0, 0) = 0$, $A_{f,g}(1, 1) = 1$ and $A_{f,g}$ is increasing in each place, since it is the composition of increasing functions. Moreover, given a rectangle $R = [x_1, x_2] \times [y_1, y_2]$, we obtain

$$V_{A_{f,g}}(R) = V_A([f(x_1), f(x_2)] \times [g(y_1), g(y_2)]) \geq 0,$$

which is the desired assertion. □

Example 3.2.1. Let f and g be increasing functions from $[0, 1]$ into $[0, 1]$ with $f(0) = g(0) = 0$ and $f(1) = g(1) = 1$. Then

$$A_{f,g}(x, y) := f(x) \wedge g(y), \qquad B_{f,g}(x, y) := f(x) \cdot g(y),$$
$$C_{f,g}(x, y) := \max\{f(x) + g(y) - 1, 0\}.$$

are 2–increasing agops as a consequence of the previous proposition by taking, respectively, $A = M$, $B = \Pi$ and $C = W$.

Corollary 3.2.1. *The following statements are equivalent:*

(a) *H is the restriction to the unit square $[0, 1]^2$ of a bivariate d.f. on $[0, 1]^2$ with $H(0, 0) = 0$ and $H(1, 1) = 1$;*

(b) *there exist a copula C and increasing and left continuous functions f and g from $[0, 1]$ into $[0, 1]$, $f(0) = g(0) = 0$ and $f(1) = g(1) = 1$, such that $H(x, y) := C(f(x), g(y))$.*

Proof. It is a direct consequence of Sklar's Theorem 1.6.1. □

Corollary 3.2.2. *If A is a 2–increasing and continuous agop with annihilator element 0, then there exist two increasing functions f and g from $[0,1]$ into $[0,1]$, $f(0) = g(0) = 0$ and $f(1) = g(1) = 1$, such that $A_{f,g}$ defined by (3.2) is a copula.*

Proof. Let f and g be the functions given by

$$f(x) := \sup\{t \in [0,1] : A(t,1) = x\},$$
$$g(y) := \sup\{t \in [0,1] : A(1,t) = y\}.$$

Then f and g satisfy the assumptions of Proposition 3.2.1 and, hence, $A_{f,g}$ is 2–increasing. Moreover, it is easily proved that 1 is the neutral element of $A_{f,g}$ and, thus, $A_{f,g}$ is a copula. □

Example 3.2.2. Let B and C be copulas and consider the function $A(x,y) = B(x,y) \cdot C(x,y)$. As we will show in the sequel (see chapter 8), A is a continuous 2–increasing agop with annihilator 0. Moreover, we have

$$f(x) = g(x) = \sup\{t \in [0,1] : A(t,1) = x\} = \sqrt{x}.$$

Therefore, in view of Corollary 3.2.2 the function

$$A_{f,g}(x,y) = A(f(x), g(y)) = B(\sqrt{x}, \sqrt{y}) \cdot C(\sqrt{x}, \sqrt{y})$$

is a copula.

Proposition 3.2.2. *Let f be an increasing and convex function from $[0,2]$ into $[0,1]$ such that $f(0) = 0$ and $f(2) = 1$. Then the function*

$$A_f(x,y) := f(x+y) \tag{3.3}$$

is a 2–increasing agop.

Proof. It is obvious that $A_f(0,0) = 0$, $A_f(1,1) = 1$ and A_f is increasing in each place. Moreover, given a rectangle $R = [x_1,x_2] \times [y_1,y_2]$, we obtain

$$V_{A_f}(R) = f(x_2+y_2) + f(x_1+y_1) - f(x_2+y_1) - f(x_1+y_2).$$

By using an argument similar to the proof of Proposition 3.1.2, the convexity of f implies that $V_{A_f}(R) \geq 0$. □

Notice that the agop A_f given in (3.3) is Schur–constant.

Example 3.2.3. Consider the function $f : [0,2] \to [0,1]$, given for every $t \in [0,2]$ by $f(t) := \max\{t-1, 0\}$. Then the function A_f defined by (3.3) is W.

Sometimes, it is useful to construct an agop with specified values on its diagonal, horizontal or vertical section (see, for example, [91, 81]). Specifically, given a suitable function f, the problem is whether there is a 2–increasing agop with (diagonal, horizontal or vertical) section equal to f.

Proposition 3.2.3. *Let h, v and δ be increasing functions from $[0,1]$ into $[0,1]$, $\delta(0) = 0$ and $\delta(1) = 1$. The following statements hold:*

- *$A_\delta(x,y) = \delta(x)$ is a 2–increasing agop with diagonal section is δ;*
- *a 2–increasing agop with horizontal section at $b \in\,]0,1[$ equal to h is given by*

$$A_h(x,y) = \begin{cases} 1, & \textit{if } y = 1; \\ 0, & \textit{if } y = 0; \\ h(x), & \textit{otherwise}; \end{cases}$$

- *a 2–increasing agop with vertical section at $a \in\,]0,1[$ equal to v is given by*

$$A_v(x,y) = \begin{cases} 1, & \textit{if } x = 1; \\ 0, & \textit{if } x = 0; \\ v(y), & \textit{otherwise.} \end{cases}$$

Proof. The proof is a consequence of Proposition 3.1.4 because A_δ, A_h and A_v are all modular agops. □

In [107] (see also [10]), an ordinal sum construction for agops is given. Here, we modify that method in order to ensure that an ordinal sum of 2–increasing agops is again 2–increasing.

Consider a partition of the unit interval $[0,1]$ by the points $0 = a_0 < a_1 < \cdots < a_n = 1$ and let A_1, A_2,...,A_n be 2–increasing agops. For every $i \in \{1, 2, \dots, n\}$, consider the function $\widetilde{A}_i$ defined on the square $[a_i, a_{i+1}]^2$ by

$$\widetilde{A}_i(x,y) = a_i + (a_{i+1} - a_i) A_i \left(\frac{x - a_i}{a_{i+1} - a_i}, \frac{y - a_i}{a_{i+1} - a_i} \right).$$

Then we can easily prove that $\widetilde{A}_i$ is 2–increasing on $[a_i, a_{i+1}]^2$. Now, define, for every point (x, y) such that $a_i \le \min\{x, y\} < a_{i+1}$,

$$A_{1,n}(x,y) := \widetilde{A}_i \left(\min\{x, a_{i+1}\}, \min\{y, a_{i+1}\}\right) \tag{3.4}$$

(and $A_{1,n}(1,1) = 1$ by definition). Therefore, it is not difficult to prove that $A_{1,n}$ is also a 2–increasing agop, called the *ordinal sum* of the agops $\{A_i\}_{i=1,2,\dots,n}$; we write

$$A_{1,n} = (\langle a_i, A_i \rangle)_{i=1,2,\dots,n}\,.$$

Example 3.2.4. Consider a partition of $[0,1]$ by means of the points $0 = a_0 < a_1 < \cdots < a_n = 1$. Let $A_1, A_2, \ldots, A_n$ be 2–increasing agops such that, for every index i, $A_i = A_S$, the smallest agop. Let $A_{1,n}$ be the ordinal sum $(\langle a_i, a_{i+1}, A_i\rangle)_{i=1,2,\ldots,n}$. For every point (x,y) such that $a_i \le \min\{x,y\} < a_{i+1}$, $A_{1,n}(x,y) = a_i$. Note that $A_{1,n}$ is the smallest agop with idempotent elements $a_0, a_1, \ldots, a_n$.

3.3 Bounds on sets of 2–increasing agops

Given a (2–increasing) agop A, it is obvious that

$$A_S(x,y) \le A(x,y) \quad \text{for every } (x,y) \text{ in } [0,1],$$

where A_S is the smallest agop defined in section 1.11. Because A_S is 2–increasing, it is also the best–possible lower bound in the set $\mathcal{A}_2$, because it is 2–increasing.

The best–possible upper bound in $\mathcal{A}_2$ is the greatest agop A_G. Notice that A_G is not 2–increasing, e.g. $V_{A_G}([0,1]^2) = -1$, but it is the pointwise limit of the sequence $\{A_n\}_{n\in\mathbb{N}}$ of 2–increasing agops, defined by

$$A_n(x,y) = \begin{cases} 1, & \text{if } (x,y) \in [1/n, 1]^2; \\ 0, & \text{otherwise.} \end{cases}$$

In particular, $(\mathcal{A}, \le)$ is not a complete lattice. But, the following result holds.

Proposition 3.3.1. *Every agop is the supremum of a suitable subset of* $\mathcal{A}_2$.

Proof. Let A be an agop; we may (and, in fact do) suppose that $A \ne A_G$, since this case has already been considered, and that A is not 2–increasing, this case being trivial. For every (x_0, y_0) in $[0,1]$, let $z_0 = A(x_0, y_0)$ and consider the following 2–increasing agop

$$\widehat{A}_{x_0,y_0} := \begin{cases} 1, & \text{if } (x,y) = (1,1); \\ z_0, & \text{if } (x,y) \in [x_0,1] \times [y_0,1] \setminus \{(1,1)\}; \\ 0, & \text{otherwise.} \end{cases}$$

Then we have

$$A(x,y) = \sup\{\widehat{A}_{x_0,y_0} : (x_0,y_0) \in [0,1]^2\}.$$

□

The lattice structure of the class of copulas was considered in [123]. Here, other cases will be considered. The following result, for instance, gives the bounds on the subsets of 2–increasing agops with the same margins.

Proposition 3.3.2. *Let A be a 2–increasing agop with margins h_0, h_1, v_0 and v_1. Let*

$$A_*(x,y) := \max\{h_0(x)+v_0(y), h_1(x)+v_1(y)-1\} \tag{3.5}$$

and

$$A^*(x,y) := \min\{h_1(x)+v_0(y)-A(0,1), h_0(x)+v_1(y)-A(1,0)\}. \tag{3.6}$$

Then, for every (x,y) in $[0,1]$,

$$A_*(x,y) \le A(x,y) \le A^*(x,y). \tag{3.7}$$

Proof. Let A be a 2–increasing agop. Let (x,y) be a point in $]0,1[^2$. In view of the 2–increasing property, we have

$$\begin{aligned} A(x,y) &\ge A(x,0)+A(0,y) = h_0(x)+v_0(y), \\ A(x,y) &\ge A(x,1)+A(1,y)-1 = h_1(x)+v_1(y)-1, \end{aligned}$$

which together yield the first of the inequalities (3.7). Analogously,

$$\begin{aligned} A(x,y) &\le A(0,y)+A(x,1)-A(0,1) = h_1(x)+v_0(y)-A(0,1), \\ A(x,y) &\le A(x,0)+A(1,y)-A(1,0) = h_0(x)+v_1(y)-A(1,0), \end{aligned}$$

namely the second of the inequalities (3.7). □

It should be noticed that, in the special case of copulas, the bounds of (3.7) coincide with the usual Fréchet–Hoeffding bounds (1.13).

The subclasses of 2–increasing agops with prescribed margins have the smallest and the greatest element (in the pointwise ordering), as stated here.

Theorem 3.3.1. *For every 2–increasing agop A, the bounds A_* and A^* defined by (3.5) and (3.6) are 2–increasing agops.*

Proof. The functions A_* and A^* defined by (3.5) and (3.6), respectively, are obviously agops. Below we shall prove that they are also 2–increasing. To this end, let $R = [x,x'] \times [y,y']$ be any rectangle contained in the unit square.

Consider, first, the case of A^*. Then

$$\begin{aligned} A^*(x',y') &:= \min\{h_1(x')+v_0(y')-A(0,1), h_0(x')+v_1(y')-A(1,0)\}, \\ A^*(x,y) &:= \min\{h_1(x)+v_0(y)-A(0,1), h_0(x)+v_1(y)-A(1,0)\}, \\ A^*(x',y) &:= \min\{h_1(x')+v_0(y)-A(0,1), h_0(x')+v_1(y)-A(1,0)\}, \\ A^*(x,y') &:= \min\{h_1(x)+v_0(y')-A(0,1), h_0(x)+v_1(y')-A(1,0)\}. \end{aligned}$$

There are four cases to be considered.

Case 1. If

$$A^*(x',y') = h_1(x')+v_0(y')-A(0,1),\ A^*(x,y) = h_1(x)+v_0(y)-A(0,1),$$

then

$$A^*(x', y') + A^*(x, y) = h_1(x') + v_0(y) - A(0, 1) \\ + h_1(x) + v_0(y') - A(0, 1) \geq A^*(x', y) + A^*(x, y').$$

Case 2. If

$$A^*(x', y') = h_0(x') + v_1(y') - A(1, 0), \; A^*(x, y) = h_0(x) + v_1(y) - A(1, 0),$$

then

$$A^*(x', y') + A^*(x', y') = h_0(x') + v_1(y) - A(1, 0) \\ + h_0(x) + v_1(y') - A(1, 0) \geq A^*(x', y) + A^*(x, y').$$

Case 3. If

$$A^*(x', y') = h_1(x') + v_0(y') - A(0, 1), \; A^*(x, y) = h_0(x) + v_1(y) - A(1, 0),$$

then, since A is 2–increasing, we have $h_1(x') + h_0(x) \geq h_1(x) + h_0(x')$, so that

$$\begin{aligned} & A^*(x', y') + A^*(x', y') \\ & \qquad = h_1(x') + h_0(x) - A(0, 1) + v_0(y') + v_1(y) - A(1, 0) \\ & \qquad \geq h_1(x) + v_0(y') - A(0, 1) + h_0(x') + v_1(y) - A(0, 1) \\ & \qquad \geq A^*(x', y) + A^*(x, y'). \end{aligned}$$

Case 4. If

$$A^*(x', y') = h_0(x') + v_1(y') - A(1, 0), \; A^*(x, y) = h_1(x) + v_0(y) - A(0, 1),$$

then, since A is 2–increasing, we have $v_1(y') + v_0(y) \geq v_1(y) + v_0(y')$, so that

$$\begin{aligned} & A^*(x', y') + A^*(x', y') \\ & \qquad = h_0(x') + v_1(y') - A(1, 0) + h_1(x) + v_0(y) - A(0, 1) \\ & \qquad \geq h_0(x') + v_1(y) - A(1, 0) + h_1(x) + v_0(y') - A(0, 1) \\ & \qquad \geq A^*(x', y) + A^*(x, y'). \end{aligned}$$

This proves that A^* is 2–increasing.

A similar proof holds for A_*. Given a rectangle $R = [x, x'] \times [y, y']$ in the unit square, we have

$$\begin{aligned} A_*(x', y') &:= \max\{h_0(x') + v_0(y'), h_1(x') + v_1(y') - 1\}, \\ A_*(x, y) &:= \max\{h_0(x) + v_0(y), h_1(x) + v_1(y) - 1\}, \\ A_*(x', y) &:= \max\{h_0(x') + v_0(y), h_1(x') + v_1(y) - 1\}, \\ A_*(x, y') &:= \max\{h_0(x) + v_0(y'), h_1(x) + v_1(y') - 1\}. \end{aligned}$$

Here, again, four cases will be considered.
Case 1. If

$$A_*(x', y) = h_0(x') + v_0(y), \quad A_*(x, y') = h_0(x) + v_0(y'),$$

then

$$\begin{aligned} A_*(x', y) + A_*(x, y') &= h_0(x) + v_0(y) + h_0(x') + v_0(y') \\ &\leq A_*(x', y') + A_*(x, y). \end{aligned}$$

Case 2. If

$$A_*(x', y) = h_0(x') + v_0(y), \quad A_*(x, y') = h_1(x) + v_1(y') - 1,$$

then, since A is 2–increasing, we have $h_0(x') + h_1(x) \leq h_1(x') + h_0(x)$ so that

$$\begin{aligned} A_*(x', y) + A_*(x, y') &= h_0(x') + v_0(y) + h_1(x) + v_1(y') - 1 \\ &\leq h_1(x') + v_1(y') - 1 + h_0(x) + v_0(y) \\ &\leq A_*(x', y') + A_*(x, y). \end{aligned}$$

Case 3. If

$$A_*(x', y) = h_1(x') + v_1(y) - 1, \quad A_*(x, y') = h_0(x) + v_0(y'),$$

then, since A is 2–increasing, we have $v_1(y) + v_0(y') \leq v_1(y') + v_0(y)$, so that

$$\begin{aligned} A_*(x', y) + A_*(x, y') &= h_1(x') + v_1(y) - 1 + h_0(x) + v_0(y') \\ &\leq h_1(x') + v_1(y') - 1 + h_0(x) + v_0(y) \\ &\leq A_*(x', y') + A_*(x, y). \end{aligned}$$

Case 4. If

$$A_*(x', y) = h_1(x') + v_1(y) - 1, \; A_*(x, y') = h_1(x) + v_1(y') - 1,$$

then

$$\begin{aligned} A_*(x', y) + A_*(x, y') &= h_1(x') + v_1(y') - 1 + h_1(x) + v_1(y) - 1 \\ &\leq A_*(x', y') + A_*(x, y). \end{aligned}$$

□

The following result gives a necessary and sufficient condition that ensures $A_* = A^*$ in the case of a symmetric agop A.

Proposition 3.3.3. *For a symmetric and 2–increasing agop A, the following statements are equivalent:*

(a) $A_* = A^*$;

(b) *there exists an interval* $I \subseteq [0,1]$, $0 \in I$, *and* $a \in [0,1]$ *such that*

$$h_1(t) = \begin{cases} h_0(t) + a, & \text{if } t \in I, \\ h_0(t) + (1-a), & \text{if } t \in [0,1] \setminus I. \end{cases} \tag{3.8}$$

Proof. If A is a symmetric agop, then $h_0 = v_0$ and $h_1 = v_1$. Set $a := A(0,1) = A(1,0)$, $a \leq 1/2$. Therefore

$$A_*(x,y) := \max\{h_0(x) + h_0(y), h_1(x) + h_1(y) - 1\}$$

and

$$A^*(x,y) := \min\{h_1(x) + h_0(y) - a, h_0(x) + h_1(y) - a\}.$$

If $A = A^*$, then $A(x,x) = h_1(x) + h_0(x) - a$. Now, from $A = A_*$, we obtain that either $A(x,x) = 2h_0(x)$ or $A(x,x) = 2h_1(x) - 1$. Therefore, either

$$h_1(x) - h_0(x) = a, \tag{3.9}$$

or

$$h_1(x) - h_0(x) = 1 - a. \tag{3.10}$$

If $a = 1/2$, then $h_1(x) = h_0(x) + a$ on $[0,1]$. Otherwise, note that (3.9) holds at the point $x = 0$ and (3.10) holds at the point $x = 1$. Moreover, if (3.9) does not hold at a point x_1, then (3.9) does not hold also for every $x_2 > x_1$. In fact, for the 2–increasing property, we obtain

$$h_1(x_2) - h_0(x_2) \geq h_1(x_1) - h_0(x_1) = 1 - a > 1/2.$$

Thus h_1 has the form (3.8), where I is an interval. The converse is just a matter of straightforward verification. □

Note that if $A = A^* = A_*$, then $A = 2aB + (1-2a)C$, where B is a symmetric and modular agop, and $C = 1_{I^2}$ is the indicator function of the set I^2.

Example 3.3.1. Consider the arithmetic mean $A(x,y) := (x+y)/2$, which is obviously 2–increasing. Then, we easily evaluate $A_* = A^* = A$.

Consider the 2–increasing agop given by the geometric mean $G(x,y) := \sqrt{xy}$. We have

$$G_*(x,y) = \max\{0, \sqrt{x} + \sqrt{y} - 1\} \qquad \text{and} \qquad G^*(x,y) = \min\{\sqrt{x}, \sqrt{y}\},$$

both of which are 2–increasing.

Remark 3.3.1. In the general case of a 2–increasing agop A such that $A = A_* = A^*$, as above it can be proved that one among the following four equalities holds:

- $h_1(x) - h_0(x) = A(0,1)$;

- $h_1(x) - h_0(x) = 1 - A(1,0)$;
- $v_1(y) - v_0(y) = 1 - A(0,1)$;
- $v_1(y) - v_0(y) = A(1,0)$.

However, one need not have explicit conditions as in the symmetric case for $h_1(x) - h_0(x)$ and $v_1(y) - v_0(y)$.

Let h, v and δ be increasing functions from $[0,1]$ into $[0,1]$, $\delta(0) = 0$ and $\delta(1) = 1$. Denote by $\mathcal{A}_h$, $\mathcal{A}_v$ and $\mathcal{A}_\delta$, respectively, the subclasses of 2–increasing agops with horizontal section at $b \in]0,1[$ equal to h, vertical section at $a \in]0,1[$ equal to v, diagonal section equal δ, respectively. Notice that the sets $\mathcal{A}_h$, $\mathcal{A}_v$ and $\mathcal{A}_\delta$ are not empty, in view of Proposition 3.2.3. The following results give the best–possible bounds in these subclasses.

Proposition 3.3.4. *Let $h : [0,1] \to [0,1]$ be an increasing function. For every A in $\mathcal{A}_h$ we obtain*

$$(A_h)_* \le A(x,y) \le (A_h)^*, \tag{3.11}$$

where

$$(A_h)_*(x,y) := \begin{cases} 1, & \text{if } (x,y) = (1,1); \\ 0, & \text{if } 0 \le y < b; \\ h(x), & \text{otherwise}; \end{cases}$$

$$(A_h)^*(x,y) := \begin{cases} 0, & \text{if } (x,y) = (0,0); \\ 1, & \text{if } b < y \le 1; \\ h(x), & \text{otherwise}. \end{cases}$$

Moreover,

$$(A_h)_*(x,y) = \bigwedge_{A \in \mathcal{A}_h} A(x,y) \quad \text{and} \quad (A_h)^*(x,y) = \bigvee_{A \in \mathcal{A}_h} A(x,y),$$

where $(A_h)_*$ is a 2–increasing agop and $(A_h)^*$, while it is still an agop, is not necessarily 2–increasing.

Proof. For all $(x,y) \in [0,1]^2$ and $A \in \mathcal{A}_h$, $A(x,y) \ge 0$ for every $y \in [0,b[$ and $A(x,y) \ge h(x)$ for every $y \in [b,1]$, viz. $A(x,y) \ge (A_h)_*(x,y)$ on $[0,1]^2$. Analogously, $A(x,y) \le h(x)$ for every $y \in [0,b]$ and $A(x,y) \le 1$ for every $y \in]b,1]$, viz. $A(x,y) \le (A_h)^*(x,y)$ on $[0,1]^2$. Both $(A_h)_*$ and $(A_h)^*$ are agops, as is immediately seen; it is also immediate to check that $(A_h)_*$ is 2–increasing and, therefore, that $(A_h)_* = \bigwedge_{A \in \mathcal{A}_h} A$. Now, suppose that B is any agop greater than, or at least equal to,

$\bigvee_{A\in\mathcal{A}_h} A$. Then $B(x,y) \geq A_1(x,y)$, where A_1 is the 2–increasing agop given by

$$A_1(x,y) := \begin{cases} 0, & \text{if } y = 0; \\ h(x), & \text{if } 0 < y \leq b; \\ 1, & \text{if } b < y \leq 1; \end{cases}$$

and $B(x,y) \geq A_2(x,y)$, where A_2 is the 2–increasing agop given by

$$A_2(x,y) := \begin{cases} 0, & \text{if } x = 0; \\ h(x), & \text{if } x \neq 0 \text{ and } 0 < y \leq b; \\ 1, & \text{if } x \neq 0 \text{ and } b < y \leq 1; \end{cases}$$

therefore $B(x,y) \geq \max\{A_1(x,y), A_2(x,y)\} = (A_h)^*(x,y)$ on $[0,1]^2$ and we obtain $(A_h)^* = \bigvee_{A\in\mathcal{A}_h} A$. However $(A_h)^*$ need not be 2–increasing; in fact,

$$V_{(A_h)^*}([0,1]\times[b,1]) = h(0) - h(1),$$

and thus $(A_h)^*$ is 2–increasing if, and only if, $h = 0$. □

Analogously, we prove the following result for the class $\mathcal{A}_v$.

Proposition 3.3.5. *Let $v : [0,1] \to [0,1]$ be an increasing function. For every A in $\mathcal{A}_v$ we obtain*

$$(A_v)_* \leq A(x,y) \leq (A_v)^*, \tag{3.12}$$

where

$$(A_v)_*(x,y) := \begin{cases} 1, & \textit{if } (x,y) = (1,1); \\ 0, & \textit{if } 0 \leq x < a; \\ v(y), & \textit{otherwise}; \end{cases}$$

$$(A_v)^*(x,y) := \begin{cases} 0, & \textit{if } (x,y) = (0,0); \\ 1, & \textit{if } a < x \leq 1; \\ v(y), & \textit{otherwise}. \end{cases}$$

Moreover,

$$(A_v)_*(x,y) = \bigwedge_{A\in\mathcal{A}_v} A(x,y) \quad \textit{and} \quad (A_v)^*(x,y) = \bigvee_{A\in\mathcal{A}_v} A(x,y),$$

where $(A_v)_*$ is a 2–increasing agop and $(A_v)^*$, while it is still an agop, is not necessarily 2–increasing.

Proposition 3.3.6. *Let δ be an increasing function with $\delta(0) = 0$ and $\delta(1) = 1$. For every A in $\mathcal{A}_\delta$, we obtain*

$$(A_\delta)_* := \min\{\delta(x), \delta(y)\} \leq A(x,y) \leq (A_\delta)^* := \max\{\delta(x), \delta(y)\}. \tag{3.13}$$

Moreover, $(A_\delta)_*$ and $(A_\delta)^*$ are the best–possible bounds, in the sense that

$$(A_\delta)_*(x,y) = \bigwedge_{A\in\mathcal{A}_\delta} A(x,y) \quad \text{and} \quad (A_\delta)^*(x,y) = \bigvee_{A\in\mathcal{A}_\delta} A(x,y),$$

where $(A_\delta)_*$ is a 2–increasing agop and $(A_\delta)^*$, while it is still an agop, is never 2–increasing.

Proof. For all $(x,y) \in [0,1]^2$ and $A \in \mathcal{A}_\delta$,

$$A(x,y) \geq A(x\wedge y, x\wedge y) = \min\{\delta(x), \delta(y)\}$$

and

$$A(x,y) \leq A(x\vee y, x\vee y) = \max\{\delta(x), \delta(y)\}.$$

This proves (3.13). Both $(A_\delta)_*$ and $(A_\delta)^*$ are agops, as is immediately seen; it is also immediate to check that $(A_\delta)_*$ is 2–increasing (because of Proposition 3.2.1) and, therefore, that $(A_\delta)_* = \bigwedge_{A\in\mathcal{A}_\delta} A$. Now, suppose that B is any agop greater than, or at least equal to, $\bigvee_{A\in\mathcal{A}_\delta} A$. Then $B(x,y) \geq A_1(x,y) := \delta(x)$ and $B(x,y) \geq A_2(x,y) := \delta(y)$, where A_1 and A_2 are 2–increasing agops. Thus, $B(x,y) \geq (A_\delta)^*$ so that $(A_\delta)^* = \bigvee_{A\in\mathcal{A}_\delta} A$. This proves that $(A_\delta)^*$ is the best possible upper bound for the set $\mathcal{A}_\delta$. However $(A_\delta)^*$ is never 2–increasing, in fact

$$V_{(A_\delta)^*}\left([0,1]^2\right) = \delta(0) - \delta(1) = -1 < 0. \qquad \square$$

Corollary 3.3.1. *Let δ be an increasing function with $\delta(0) = 0$ and $\delta(1) = 1$. For every symmetric agop A in $\mathcal{A}_\delta$, we obtain*

$$(A_\delta)_* := \min\{\delta(x), \delta(y)\} \leq A(x,y) \leq \frac{\delta(x)+\delta(y)}{2},$$

where $(\delta(x)+\delta(y))/2$ is the maximal element in the subclass of the symmetric agops in $\mathcal{A}_2$.

Proof. If A is symmetric and 2–increasing, we have, for every x, y in $[0,1]$,

$$\delta(x) + \delta(y) = A(x,x) + A(y,y) \geq 2\ A(x,y). \qquad \square$$

3.4 A construction method for copulas

The main result of this section is to give a simple method of constructing a copula from a 2–increasing and 1–Lipschitz agop.

Theorem 3.4.1. *For every 2–increasing and 1–Lipschitz agop A, the function*

$$C(x,y) := \min\{x, y, A(x,y)\}$$

is a copula.

Proof. First, in order to prove that C is a copula, we note that C has neutral element 1 and annihilator 0; in fact, for every $x \in [0,1]$, we have

$$|A(1,1) - A(x,1)| \leq 1 - x$$

and thus $A(x,1) \geq x$. Consequently, we have

$$C(x,1) = \min\{A(x,1), x\} = x, \quad C(x,0) = \min\{A(x,0), 0\} = 0,$$

and, similarly, $C(1,x) = x$ and $C(0,x) = 0$. Then, we prove that C is 2–increasing by using Proposition 1.6.1.

For every rectangle $R := [s,t] \times [s,t]$ on $[0,1]^2$, set

$$V_C(R) = \min\{A(s,s), s\} + \min\{A(t,t), t\} - \min\{A(s,t), s\} - \min\{A(t,s), s\}.$$

We have to prove that $V_C(R) \geq 0$ and several cases are considered.

If $A(s,s) \geq s$, then also $A(s,t)$, $A(t,s)$ and $A(t,t)$ are greater than s, because A is increasing in each variable, and thus

$$V_C(R) = \min\{A(t,t), t\} - s \geq 0.$$

If $A(s,s) < s$, then we distinguish:

- if $A(t,t) < t$, since A is 2–increasing, we have

  $$A(s,s) + A(t,t) \geq A(s,t) + A(t,s) \geq \min\{A(s,t), s\} + \min\{A(t,s), s\},$$

  viz. $V_C(R) \geq 0$;

- if $A(t,t) \geq t$, since A is 1–Lipschitz, we have

  $$\min\{A(t,s), s\} - \min\{A(s,s), s\} \leq t - s \leq t - \min\{A(t,s), s\},$$

  and thus $V_C(R) \geq 0$.

Now, let $R = [x_1, x_2] \times [y_1, y_2]$ be a rectangle contained in Δ_+. Then $V_C(R)$ is given by

$$\begin{aligned} V_C(R) = &\min\{A(x_1,y_1), y_1\} + \min\{A(x_2,y_2), y_2\} \\ &- \min\{A(x_2,y_1), y_1\} - \min\{A(x_1,y_2), y_2\}. \end{aligned}$$

If $A(x_1,y_1) \geq y_1$, then also $A(x_2,y_1)$, $A(x_1,y_2)$ and $A(x_2,y_2)$ are greater than y_1, because A is increasing in each variable, and thus

$$V_C(R) = \min\{A(x_2,y_2), y_2\} - y_1 \geq 0.$$

If $A(x_1,y_1) < y_1$, then we distinguish:

- if $A(x_2, y_2) < y_2$, since A is 2–increasing, we have

$$\begin{aligned} A(x_2, y_2) + A(x_1, y_1) &\geq A(x_2, y_1) + A(x_1, y_2) \\ &\geq \min\{A(x_2, y_1), y_1\} + \min\{A(x_1, y_2), y_2\}, \end{aligned}$$

  viz. $V_C(R) \geq 0$;

- if $A(x_2, y_2) \geq y_2$, we have

$$V_C(R) = A(x_1, y_1) + y_2 - A(x_1, y_2) - \min\{A(x_2, y_1), y_1\},$$

  and, since A is 1–Lipschitz,

$$A(x_1, y_2) \leq y_2 - y_1 + A(x_1, y_1) \leq y_2,$$

  moreover, from the fact that

$$A(x_1, y_2) - A(x_1, y_1) \leq y_2 - y_1 \leq y_2 - \min\{A(x_2, y_1), y_1\},$$

  it follows that $V_C(R) \geq 0$.

Finally, let $R = [x_1, x_2] \times [y_1, y_2]$ be a rectangle contained in Δ_-. Then $V_C(R)$ is given by

$$\begin{aligned} V_C(R) = &\min\{A(x_1, y_1), x_1\} + \min\{A(x_2, y_2), x_2\} \\ &- \min\{A(x_2, y_1), x_2\} - \min\{A(x_1, y_2), x_1\}. \end{aligned}$$

If $A(x_1, y_1) \geq x_1$, then, because A is increasing in each variable,

$$V_C(R) = \min\{A(x_2, y_2), x_2\} - x_1 \geq 0.$$

If $A(x_1, y_1) < x_1$, then we distinguish:

- if $A(x_2, y_2) < x_2$, since A is 2–increasing, we have

$$\begin{aligned} A(x_2, y_2) + A(x_1, y_1) &\geq A(x_2, y_1) + A(x_1, y_2) \\ &\geq \min\{A(x_2, y_1), x_1\} + \min\{A(x_1, y_2), x_2\}, \end{aligned}$$

  viz. $V_C(R) \geq 0$;

- if $A(x_2, y_2) \geq x_2$, we have

$$V_C(R) = A(x_1, y_1) + x_2 - \min\{A(x_1, y_2), x_1\} - A(x_2, y_1),$$

  and, since A is 1–Lipschitz

$$A(x_2, y_1) \leq x_2 - x_1 + A(x_1, y_1) \leq x_2;$$

  moreover, from the inequality

$$A(x_2, y_1) - A(x_1, y_1) \leq x_2 - x_1 \leq x_2 - \min\{A(x_1, y_2), x_1\},$$

  it follows that $V_C(R) \geq 0$. □

Notice that agops satisfying the assumptions of Theorem 3.4.1 are stable under convex combinations. Thus, many examples can be provided by using, for examples, copulas, quasi–arithmetic means bounded from above by the arithmetic mean, and their convex combinations.

Example 3.4.1. Let A be the modular agop $A(x,y) = (\delta(x)+\delta(y))/2$, where $\delta : [0,1] \to [0,1]$ is an increasing and 2–Lipschitz function with $\delta(0) = 0$ and $\delta(1) = 1$. Then A satisfies the assumptions of Theorem 3.4.1 and it generates the following copula

$$C_\delta(x,y) = \min\left\{x, y, \frac{\delta(x)+\delta(y)}{2}\right\}.$$

Copulas of this type were introduced in [56] and are called *diagonal copulas*.

Example 3.4.2. Let consider the following 2–increasing and 1–Lipschitz agop

$$A(x,y) = \lambda B(x,y) + (1-\lambda)\frac{x+y}{2},$$

defined for every $\lambda \in [0,1]$ and for every copula B. This A satisfies the assumptions of Theorem 3.4.1 and, therefore, the following class of copulas is obtained

$$C_\lambda(x,y) := \min\left\{x, y, \lambda B(x,y) + (1-\lambda)\frac{x+y}{2}\right\}.$$

Example 3.4.3. Let A be a 2–increasing agop of the form $A(x,y) = f(x)\cdot g(y)$. If A is 1–Lipschitz, then A satisfies the assumptions of Theorem 3.4.1. Consider, for instance, either $f(x) = x$ and $g(y) = (y+1)/2$, or $f(x) = (x+1)/2$ and $g(y) = y$, which yield, respectively, the following copulas

$$C_1(x,y) = \min\left\{y, \frac{x(y+1)}{2}\right\}, \quad C_2(x,y) = \min\left\{x, \frac{y(x+1)}{2}\right\}.$$