Introduction

In the framework of the theory of Continuum Mechanics, exact solutions play a fundamental role for several reasons. They allow to investigate in a direct way the physics of various constitutive models (for example, in suggesting specific experimental tests); to understand in depth the qualitative characteristics of the differential equations under investigation (for example, giving explicit appreciation on the well-posedness of these equations); and they provide benchmark solutions of complex problems.

The Mathematical method used to determine these solutions is usually called the semi-inverse method. This is essentially a heuristic method that consists in formulating a priori a special ansatz on the geometric and/or kinematical fields of interest, and then introducing this ansatz into the field equations. Luck permitting, these field equations reduce to a simple set of equations and then some special boundary value problems may be solved.

Although the semi-inverse method has been used in a systematic way during the whole history of Continuum Mechanics (for example the celebrated Saint Venant solutions in linear elasticity have been found by this method), it is still not known how to generate meaningful ansatzes to determine exact solutions for sure. In this direction, the only step forward has been a partial confirmation of the conjecture by Ericksen [36] on the connection between group analysis and semi-inverse methods [96].

Another important aspect in the use of the semi-inverse method is associated in fluid dynamics with the emergence of secondary flows and in solid mechanics with latent deformations. It is clear that "Navier-Stokes fluid" and an "isotropic incompressible hyperelastic material" are intellectual constructions. No real fluid is exactly a Navier-Stokes fluid and no-real world elastomer can be characterized from a specific elastic potential, such as for example the "neo-Hookean" or "Mooney-Rivlin" models. The experimental data associated with the extension of a rubber band can be approximated by several different models, but we still do not know of a fully satisfying mathematical model. This observation is fundamental in order to understand that the results obtained by a semi-inverse method could be dangerous and misleading.

We know that a Navier-Stokes fluid can move by parallels flows in a cylindrical tube of arbitrary section. We obtain that solution by considering that the kinematic field is a function of the section variables only. In this way, the Navier-Stokes equations are reduced to linear parabolic equations which we solve by considering the usual no-slip boundary conditions. This picture is peculiar to Navier-Stokes fluids. In fact, if the relation between the stress and the stretching is not linear, a fluid can flow in a tube by parallel flows if and only if the tube possesses cylindrical symmetry (see [40]). If the tube is not perfectly cylindrical, then what is going on? Clearly any real fluid may flow in a tube, whether or not it is a Navier-Stokes fluid. In the real world, what is different from what it is predicted by the Navier-Stokes theory is the presence of secondary flows, i.e. flows in the section of the cylinder. This means that a pure parallel flow in a tube is a strong idealization of reality. A classic example illustrating such an approach in solid mechanics is obtained by considering deformations of anti-plane shear type. Knowles [72] shows that a non-trivial (non-homogeneous) equilibrium state of anti-plane shear is not always (universally) admissible, not only for compressible solids (as expected from Ericksen's result [34]) but also for incompressible solids. Only for a special class of incompressible materials (inclusive of the so-called "generalized neo-Hookean materials") is an anti-plane shear deformation controllable. Let us consider, for example, the case of an elastic material filling the annular region between two coaxial cylinders, with the following boundary-value problem: hold fixed the outer cylinder and pull the inner cylinder by applying a tension in the axial direction. It is known that the deformation field of pure axial shear is a solution to this problem valid for every incompressible isotropic elastic solid. In the assumption of non-coaxial cylinders, thereby losing the axial symmetry, we cannot expect the material to deform as prescribed by a pure axial shear deformation. Knowles's result [72] tells us that now the boundary-value problem can be solved with a general anti-plane deformation (not axially symmetric) only for a certain subclass of incompressible isotropic elastic materials. Of course, this restriction does not mean that, for a generic material, it is not possible to deform the annular material as prescribed by our boundary conditions, but rather that, in general, these lead to a deformation field that is more complex than an anti-plane shear.

Hence, we also expect secondary in-plane deformations. The true problem is therefore to understand when these secondary fields can be or cannot be neglected; it is not to determine the special theory for which secondary flows disappears in our mathematical world. These issues are relevant to many stability issues.

The present Thesis originates from the desire to understand in greater detail the analogy between secondary flows and latent deformations (i.e. deformations that are awoken from particular boundary conditions) in solid mechanics. We would also like to question those boundary conditions that allow a semi-inverse simple solution for special materials, but pose very difficult problem for general materials. In some sense we are criticizing all studies that characterize the special strain energy functions for which particular classes of deformations turn out to be possible (or using a standard terminology, turn out to be controllable).

We wish to point out that our criticism is not directed at the mathematical results obtained by these studies. Those results can and do lead to useful exact solutions if the correct subclass of materials is picked. However, with regard to the whole class of materials that are identified in the literature, one has to exercise a great deal of caution, because models that are obtained on the basis of purely mathematical arguments may exhibit highly questionable physical behavior. For example, some authors have determined which elastic compressible isotropic materials support simple isochoric torsion. In fact, it is not of any utility to understand which materials possess this property, because these materials do not exist. It is far more important to understand which complex geometrical deformation accompanies the action of a moment twisting a cylinder. That is why universal solutions are so precious (see [113]). These results may also have important repercussions in biomechanics. In the study of the hemo-dynamics, the hypothesis that the arterial wall deforms according to simple geometric fields does not account for several fundamental factors. A specific example of a missing factor is the effect of torsion on microvenous anastomic patency and early thrombolytic phenomenon (see for example [116]). Nonetheless, we do acknowledge the value of simple exact solutions obtained by inverse or semi-inverse investigations for understanding directly the nonlinear behavior of solids.

The plan of the Thesis is the following: in the first two chapters, we develop an introduction to nonlinear elasticity, essential to the subsequent chapters. The third chapter is entirely devoted to the inverse procedures of Continuum Mechanics and we illustrate some of the most important results obtained by their use, including the "universal solutions". While the inverse procedures have been truly important to obtain exact solutions, on the other hand some of them may misguide and miss real and interesting real phenomena. Here we also begin to expose our criticism of some uses of the semi-inverse method and we describe in detail the "anti-plane shear problem". The core of these considerations is presented in the fourth chapter (see also [28]). Here we illustrate some possible dangers inherent to the use of special solutions to determine classes of constitutive equations. We consider some specific solutions obtained for isochoric deformations but for compressible nonlinear elastic materials: "pure torsion" deformation, "pure axial shear" deformation and the "propagation of transverse waves". We use a perturbation tecnique to predict some risks that they may lead to when they are considered. Mathematical arguments are therefore important when they determine general constitutive arguments, not very special strain energies as the compressible potential that admits isochoric deformations. In the fifth chapter (see also [27]), we give an elegant and analytic example of secondary (or latent) deformations in the framework of nonlinear elasticity. We consider a complex deformation field for an isotropic incompressible nonlinear elastic cylinder and we show that this deformation field provides an insight into the possible appearance of secondary deformation fields for special classes of materials. We also find that these latent deformation fields are woken up by normal stress differences. Then we present some more general and universal results in the sixth chapter, where we use incremental solutions of nonlinear elasticity and we provide an exact solution for buckling instability of a nonlinear elastic cylinder and an explicit derivation for the first nonlinear correction of Euler's celebrated buckling formula (see also [26]).