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Groups as the union of fusion classes

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Abstract. Fusion class of a group consists of all the elements which are related by the fusion relation, that is; the elements which are conjugate in their automorphism group. In this paper we study the structure of groups in which union of fusion classes with identity element 1 becomes a subgroup.

Keywords: Fusion class, Automorphism, Characteristic subgroup

MSC 2000 classification: 20E45,20E34

1 Introduction

Let G be a finite group. In [3], the relation of fusion has been defined. Two elements a and b of G are said to be fused if there exists an automorphism α of G which sends a to b. It is easy to check that the relation of fusion is an equivalence relation. We will call the equivalence class related to the element a of G as fusion class of a and is denoted by $\overline{cl(a)}$. Let $L(G)$ denote the autocenter of G [2]. $L(G)$ consists of all those elements in G which are fixed by all the automorphisms of G. We have an equation similar to the class equation

$$|G| = |L(G)| + \sum_{a \notin L(G)} |\overline{cl(a)}|.$$

In fact, for $a \notin L(G)$, $|\overline{cl(a)}| = [Aut(G) : C_{Aut(G)}(a)]$,

where $C_{Aut(G)}(a)$ contains all those automorphisms which fix a. In [6, 7, 5] the structure of subgroups which are union of two, three and four conjugacy classes are determined. In [1], the authors gave the structure of groups in which the union of identity element 1 with two, three and four non-trivial conjugacy classes is a subgroup. Instead of taking conjugacy classes, we will find the structure of groups which are union of fusion classes. In the Section 2 of this paper we find the structure of groups in which union of any non trivial fusion class with

1 becomes a subgroup. The Section 3 of this paper aimed to determine the structure of groups in which union of any two non-trivial fusion classes with 1 becomes a subgroup. The Section 4 is devoted to the study the same for three fusion classes. Let $K(G)$ denote the number of fusion classes of a group G.

2 First classification

Proposition 2. Let G be an elementary abelian $p-$ group, then G has only one non-trivial fusion class and therefore $G = 1 \cup \overline{cl(a)}$, for all $a \in G \backslash 1$.

Proof. Suppose $G = < x_1,\ x_2,\ x_3,\ ...,\ x_n | o(x_i) = p, [x_i, x_j] = 1, 1 \leq i, j \leq n >$. We can define automorphisms for each i, $1 \leq i \leq n$, which sends x_1 to x_i and x_i to $x_1$and rest of generators to themselves. Then we have $x_1, x_2, ...x_n \in \overline{cl(x_1)}$. Thus $\overline{cl(x_1)} = \overline{cl(x_2)} = ... = \overline{cl(x_n)}$. Therefore G has only one non-trivial fusion class. QED

Lemma 2. Let G be a finite group and let H be a characteristic subgroup of G which is union of 1 and one non trivial fusion class, that is $H = 1 \cup \overline{cl(a)}$. Then H is an elementary abelian p-group.

Proof. Let p be any prime divisor of $|H|$. So there exist an element in H of order p, since all the non-identity elements in H have same order, therefore all non-identity elements in H are of order p. Therefore H is a p-group. It is easy to check that H is a minimal characteristic subgroup of G. We also have $H' \leq H$ is a characteristic subgroup of G. Hence $H' = 1$. Thus H is abelian. QED

Theorem 1. Let G be a finite group in which the union of any non trivial fusion class with 1 becomes a subgroup. Then G is an elementary abelian group.

Proof. First we claim that the order of each non identity element in G is a prime number. Let $1 \neq a \in G$. Then $H = 1 \cup \overline{cl(a)}$ is a subgroup of G. Let p be a prime number dividing the $|H|$. By Cauchy' s theorem, there exists an element of order p in H. Since order of all elements in $\overline{cl(a)}$ are same and $a \in H$, thus $|a| = p$.
Next we claim G is a p- group. Suppose a and b are two elements of G of order p and q respectively. Let $H = 1 \cup \overline{cl(a)}$ and $K = 1 \cup \overline{cl(b)}$, then H and K are characteristic subgroups of G. Clearly $H \cap K = 1$. Therefore $HK$is a subgroup of G containing an element ab of order pq, a contradiction as order of each non trivial element in G is a prime number. Therefore $|G|$ has only one prime divisor and hence G is a p-group. Suppose G has more than one non trivial fusion classes. Let a and b be elements of G of order p such that $b \notin \overline{cl(a)}$. Let

$H = 1 \cup \overline{cl(a)}$ and $K = 1 \cup \overline{cl(b)}$. Then H and K are characteristic elementary abelian subgroups of G. Clearly $H \cap K = 1$. Then $a^{-1}b^{-1}ab \in H \cap K = 1$, implies $ab = ba$. Thus G is abelian, and hence elementary abelian but we already proved that an elementary abelian group has only one non trivial fusion class. Thus G has only one non trivial fusion class. Thus by above lemma, G is elementary abelian. QED

3 Second classification

We will use the following known results in our theorems.

Theorem 2. [4] Suppose that $|G| = mn$ with $(m, n) = 1$, that either $x^m = 1$ or $x^n = 1 \quad \forall x \in G$ and that $\{x \in G | x^n = 1\}$ is normal a subgroup of G. Then G is a Frobenius group with kernel N.

Theorem 3. [4] Suppose G is a Frobenius group with complement H. Let $P \in Syl_p(H)$ then

(1) If $p = 2$, then P is cyclic or generalized quaternion.

(2) If $p \neq 2$, then P is cyclic.

Theorem 4. Let G be a finite group with $K(G) = 3$. Then G is one of the following types:

(1) If $Z(G) \neq 1$, then either G is abelian or G is special p-group of class 2 with $Z(G)$ as an elementary abelian p-group.

(2) If $Z(G) = 1$, then G is a Frobenius group with elementary abelian kernel G' and complement as cyclic group.

Proof. Suppose $G = 1 \cup \overline{cl(a)} \cup \overline{cl(b)}$, $a, b \in G$.

Case 1. Suppose $Z(G) \neq 1$. Let p be a prime divisor of $|Z(G)|$. Suppose $x \in Z(G)$ is an element of order p. Suppose $x \in \overline{cl(a)}$. Therfore $a \in Z(G)$ as $Z(G)$ is characteristic in G and $|a| = p$. If $|b|$ is prime other than p , say q, then there exist an element ab of G having pq, a contradiction, as each non-trivial element of G has order either p or q. Thus $|b|$ is p-power. Suppose G is non abelian. Then G has only two proper characterstic subgroups, namely 1 and $Z(G)$ implying $Z(G) = G' = \Phi(G)$. Suppose $Z(G) = 1 \cup \overline{cl(a)}$ and $H = 1 \cup \overline{cl(b)}$ are two characterstic subgroups of G. Then $G = Z(G) \times H$, and this implies G is abelian, a contradiction.

Case 2. Now Suppose $Z(G) = 1$. Therefore G can not be p- group. Therefore $|G|$ has two prime divisors, p and q. Suppose $|G| = p^m q^n$, where m, n are natural numbers. Suppose $|a| = p$, and $|b| = q$. Thus G is solvable. Thus $G' \neq G$. Suppose $G' = 1 \cup \overline{cl(a)}$. Then G' is an elementary abelian p-group. Therefore every element of order p in G satisfies $x^{p^m} = 1$, every element of order q in G satisfies $x^{q^n} = 1$ and all elements of order p lie in G'. Thus $G' = \{x | x^{p^m} = 1\}$ is a normal subgroup of G. Therefore G is a Frobenius group with kernel G'. Thus $G/G' = 1 \cup \overline{cl(aG')} \cong K$ is an elementary abelian group having order q^n and also sylow q- subgroup of G. Hence $n = 1$. Therefore G is a Frobenius group of order $p^n q$. QED

Lemma 3. Let G be a finite group in which union of any two non-trivial fusion classes with 1 is a subgroup. Then G has exactly two non-trivial fusion classes.

Proof. Suppose $\overline{cl(x)}, \overline{cl(y)}$ and $\overline{cl(z)}$ are any three distinct fusion classes. Consider the characteristics subgroups of G as under

$$H = 1 \cup \overline{cl(x)} \cup \overline{cl(y)}$$
$$K = 1 \cup \overline{cl(y)} \cup \overline{cl(z)}$$

Suppose $|x| = p$ and $|y| = q$ Let $H_1 = H \cap K = 1 \cup \overline{cl(y)}$. Then H_1 is an elementary abelian q-group and characteristic in G. Thus we have, the union of any one non trivial fusion class with 1 become a subgroup in G. Therefore G must be elementary abelian p-group, this implies G has only one non trivial fusion class. A contradiction. QED

Lemma 4. Let G be a finite group such that the union of any three non-trivial fusion classes with 1 becomes a subgroup. Then G has exactly three non-trivial fusion class classes.

Proof. Consider the following characteristic subgroups of G:

$$H = 1 \cup \overline{cl(a)} \cup \overline{cl(b)} \cup \overline{cl(c)}$$
$$K = 1 \cup \overline{cl(a)} \cup \overline{cl(b)} \cup \overline{cl(d)}$$

Let $H_1 = H \cap K = 1 \cup \overline{cl(a)} \cup \overline{cl(b)}$. Thus we have, the union of any two non-trivial fusion classes with 1 become a subgroup in G. By above lemma G has exactly two non trivial fusion classes. A contradiction. QED

Theorem 5. Let G be a group such that the union of any n non trivial fusion classes with 1 become a subgroup. Then G has exactly n non trivial fusion classes.

Proof. We shall prove the result by induction on n. If $n = 1$ then by theoram 1, we have G as an elementary abelian group and so there is just one non trivial fusion class.
We can prove the resullt by an easy induction as in Lemma 3 and 4.

$\boxed{QED}$

Lemma 5. Let G be a finite group in which union of any n non-trivial fusion classes with 1 is a subgroup, then $|G|$ has at the most n distinct primes divisors.

Proof. Follows from above theorem. $\boxed{QED}$

Theorem 6. Let G be a finite group in which union of any two non trivial fusion classes with 1 is a subgroup. Let $|G|$ be divisible by two distinct prime. Then G is a Frobenius group with elementary abelian kernel.

Proof. Result follows from theorem 4 case(2) and lemma 5.

$\boxed{QED}$

Theorem 7. Let G be a finite p- group in which union of any two non trivial fusion classes with 1 is a subgroup, then G is metabelian.

Proof. Proof follows from Lemma 3 an Theorem 4. $\boxed{QED}$

Proposition 3. Let G be a finite p- group with $K(G) = 3$. Then $|L(G)| \leq 2$, and for $|L(G)| = 2$ either $G \cong C_4$ or $G \cong Q_8$.

Proof. Suppose $G = 1 \cup \overline{cl(a)} \cup \overline{cl(b)}$. We claim that G has elements of order either 1, p or p^2, that is $exp(G) \leq p^2$. It is clear that G has elements of order 1 or $|a|$ or $|b|$. Now since $p/|G|$, there exists an element say, $x \in G$ of order p. Suppose $x \in \overline{cl(a)}$. Therefore $|a|$=p. Suppose $|b| = p^m$, where $m > 2$. Then $|b^p| = p^{m-1}$. Therefore $b^p \in \overline{cl(b)}$. Therefore b^p and b have same order. A contradiction. Therefore $m \leq 2$.Clearly $|L(G)| \leq 3$, otherwise G has more than two non trivial fusion classes. Now if $|L(G)| = 3$, then G is a 3-group, Suppose a and b are two non trivial elements of $L(G)$, then $G = 1 \cup \{a\} \cup \{b\}$, thus $G = L(G)$ implying $G \cong C_2$, we get a contradiction. If $|L(G)| = 2$, then G is a 2-group and $G = 1 \cup \{a\} \cup \overline{cl(b)}$, where $1 \neq a \in L(G), 1 \neq b \in G$, $|b| \leq 2^2 = 4$. If $|b| = 4$, then $b^2 \notin \overline{cl(b)}$, hence $b^2 = a$. Thus $G = 1 \cup \{b^2\} \cup \overline{cl(b)}$. In this case either $G \cong C_4$ or Q_8. Hence the result.

$\boxed{QED}$

4 Third Classification

Lemma 6. Let G be a group such that the union of any three non-trivial fusion classes with 1 becomes a subgroup and $|G|$ has three distinct prime divisors. Then every element of G is of prime order.

Proof. Proof is straight forward as G has exactly three non-trivial fusion classes. QED

Proposition 4. If G is a finite p-group, p is a prime, with $K(G) = 4$, then $exp(G) \leq p^3$.

Proof. Suppose $exp(G) \leq p^m$, where $m > 3$. Therefore G has elements of order p, p^2 and p^3. Choose a, b and c elements of G of order p, p^2 and p^3, respectively. Then $\overline{cl(a)}, \overline{cl(b)}$ and $\overline{cl(c)}$ are three distinct non-trivial fusion classes of G. Therefore $G = 1 \cup \overline{cl(a)} \cup \overline{cl(b)} \cup \overline{cl(c)}$
. This implies G has elements of order p, p^2 or p^3 only. A contradiction. QED

Remark 1. The above result can also be generalized.

Proposition 5. Let G be a finite group with $K(G) = 4$. Then G has at most four characteristic subgroups.

Proof. If G is simple or elementary abelian p-group, then G has only two charactristic subgroups 1 and G. Therefore suppose G is not simple and elementary abelian. It is clear that G can not have two distinct characteristic subgroups which are union of two non- trivial fusion classes with 1, as there union is again becomes a subgroup of G which is G itself. Also G can not have two characteristic subgroups in which one is union of two non-trivial fusion classes with 1 and another is union of one non-trivial fusion class with 1 not contained in first one, as their union is again G. Let H_1 be a minimal characteristic subgroup of G. If H_1 is union of two non-trivial fusion classes with 1, then H_1 is the only proper characteristic subgroup of G. Suppose H_1 is union of one non-trivial fusion class with 1 and also suppose there exist another characteristic subgroup H_2 of G not containing H_1. Then H_2 is also union of one non-trivial fusion class with 1. Then $H = H_1 \times H_2$ having non-trivial characteristic subgroups H_1 and H_2 only. QED

In [8], the authors proved the result,

Theorem 8. The characteristic property of A_5 is:

(1) the order of the group contains at least three different prime factors,

(2) the order of every non identity element in the group is prime.

Corollary 1. If G is a non abelian finite simple group and order of every non identity element of G is prime, then G is isomorphic to A_5.

Theorem 9. Let G be a finite abelian group with $K(G) = 4$, then either G is a $p-$group or a direct product of homocyclic groups of order pq with elementary abelian p-group (or q-group).

Proof. Suppose G is not a p-group. Suppose $G = 1 \cup \overline{cl(a)} \cup \overline{cl(b)} \cup \overline{cl(c)}$, a, b and c are elements of G. Suppose $|a|$ =p, and $|b| = q$. Then we must have $|c| = pq$, and $\overline{cl(c)} = \overline{cl(ab)}$. Thus $|G| = p^m q^n$, for some natural numbers m and n. Since G is a finite abelian group, there must exist elements of orders dividing the $|G|$. But G has elements of order either p, q or pq. Thus G is the direct product of homocyclic groups of order pq with elementary abelian p-group (or q-group). $\boxed{QED}$

Example 1. C_8 is a 2-group witk $K(G) = 4$ and C_6 is cyclic group with $K(G) = 4$.

Theorem 10. Let G be a non-abelian finite group with $K(G) = 4$. If $|Z(G)| \neq 1$, then G is a special p-group of class 2.

Proof. Suppose $G = 1 \cup \overline{cl(a)} \cup \overline{cl(b)} \cup \overline{cl(c)}$, where a, b and c are elements of G. Without loss of generality, suppose $a \in Z(G)$ and $|a| = p$. Suppose q is the prime different from p dividing $|G|$. Suppose $|b| = q$. Thus there exists an element $ab \in G$ of order pq. Thus we can suppose $|c| = pq$, and $\overline{cl(c)} = \overline{cl(ab)}$. Thus $|G| = p^m q^n$, for some natural numbers m and n. We claim that $Z(G)$ is the only non-trivial proper characteristic subgroup of G. First it is clear that every characteristic subgroup of G is union of at most two fusion classes. Suppose $Z(G) = 1 \cup \overline{cl(a)}$. Let $H = 1 \cup \overline{cl(b)}$ be another characteristic subgroup of G, but then $G = Z(G) \times H$, implies G is abelian, a contradiction. Thus $1 \neq G' = Z(G) = \Phi(G)$ are elementary abelian p-groups. Now $[x, b] \in G' = Z(G)$ for all $1 \neq x \in G$, implies $b^p \in Z(G)$. We get $b^{p^2} = 1$. A contradiction. Therefore G is a p-group. $\boxed{QED}$

Theorem 11. Let G be a finite group with $K(G) = 4$. If $|Z(G)| = 1$, then G is one of the following types:

(1) G is Frobenius group with kernel G'.

(2) G is Frobenius group with kernel as sylow p-subgroup and which is also elementary abelian p-group.

(3) G is isomorphic to A_5.

Proof. Since $|Z(G)| = 1$, G can not be a p-group. Therefore $|G|$ has at least two distinct prime divisors. We will give the proof in two parts, the first part in which $|G|$ has two distinct prime divisors and second part in which $|G|$ has three distinct prime divisors. Suppose $G = 1 \cup \overline{cl(a)} \cup \overline{cl(b)} \cup \overline{cl(c)}$, where a, b and c are elements of G.

Case 1. Suppose $|G| = p^m q^n$. Suppose $|a| = p$ and $|b| = q$. Suppose G' is the union of two fusion classes. Let $G' = 1 \cup \overline{cl(a)}$. Then G' is elementary abelian p-group. Then $G/G' = 1 \cup \overline{cl(bG')} \cup \overline{cl(cG')}$ and $|bG'| = q$. Now if $|G/G'|$ is divisible by p and q both , then $|cG'| = p$, but then G/G' is an abelian group having each element order as prime number, therefore has element $bG'cG'$ of order pq. A contradiction. Thus $|G/G'|$ is divisible by q only as G is not a p-group. Thus $|G'| = p^m$ is Sylow p-subgroup of G and $G' = \{x | x^{p^m} = 1\}$. Hence G is Frobenius group with kernel G'.
Now suppose G' is the union of three fusion classes. We have one of the following types of G' :

(i) $G' = 1 \cup \overline{cl(a)} \cup \overline{cl(b)}$.
(ii) $G' = 1 \cup \overline{cl(b)} \cup \overline{cl(c)}$.
(iii) $G' = 1 \cup \overline{cl(a)} \cup \overline{cl(c)}$.
Now $|c| = p$ or p^2 or q or q^2 or pq. For (i) Suppose $|c| = pq$. Let N be a minimal characteristic subgroup of G . Then $N = 1 \cup \overline{cl(a)}$(or $N = 1 \cup \overline{cl(b)}$) since if $N = G'$, then G' is abelian and must contain an element of order pq. Without loss of generality, suppose $N = 1 \cup \overline{cl(a)}$. Then $G/N = 1 \cup \overline{cl(bN)} \cup \overline{cl(cN)}$. Therefore G/N is q-group and N is an elementary abelian p-group which is also Sylow p-subgroup. And $\{x | x^{p^m} = 1\} = N$. Thus G is aFrobenius group with sylow p-subgroup as its kernel which is also elementary abelian p-group.

Suppose $|c| = p$ or q , then G has no element of composite order.Thus $C_G(G') \leq G'$. Hence G is Frobenius group with kernel G'. Now suppose $|c| = p^2$(or q^2). Hence G has no element having order divisible by two distinct primes. We get $C_G(G') \leq G'$. Hence G is Frobenius with kernel G'.

For (ii) the possibility of $|c| = p$ or q or q^2. If $|c| = p$, then G has no element of composite order. Thus $C_G(G') \leq G'$. If $|c| = q$ or q^2, then G' is a $q-$ group and G/G' is elementary abelian p-group, and G' is Sylow q-subgroup of G and all elements of G with q-power order belongs to G'. Thus $G' = \{x | x^{q^n} = 1\}$. Hence G is Frobenius with kernel G'.
For (iii), proceed on the similar argument as in the above paragraph.

Case 2. Suppose $|G|$ has three prime divisors p, q and r. Suppose $|a| = p, |b| = q$ and $|c| = r$. Let $|G| = p^l q^m r^n$. Thus each element of G has prime order. We claim that G is a simple group. Firstly, suppose G is not a simple group. First suppose that G is solvable. As we know any mimimal characteristic subgroup of a solvable is elementary abelian. Thus any minimal characteristic subgroup of G is the union of two fusion classes, as G has three distinct non-trivial classes for three different primes. Let $H = 1 \cup \overline{cl(a)}$ be a minimal characteristic subgroup of G of order p^l. Then G/H is the union of three fusion classes. Then $G/H = 1 \cup \overline{cl(bH)} \cup \overline{cl(cH)}$ is Frobenius with kernel $(G/H)' = K/H = 1 \cup \overline{cl(bH)}$ having order $|K/H| = q^m$, and $|G/H| = q^m r$. Now $C_{G/H}(xH) \leq K/H$, for all $H \neq xH \in K/H$. It is easy to check that $C_G(x) \leq K$, for all $1 \neq x \in K$. Hence G is Frobenius group with kernel K. Since kernel of Frobenius group is nilpotent, so K is nilpotent. Also $K = 1 \cup \overline{cl(a)} \cup \overline{cl(b)}$ is also Frobenius implies not nilpotent. A contradiction. Thus G is not solvable. We will prove that G is characteristically simple. Let H be a minimal characteristic subgroup of G. Then H and G/H both are union of atmost three fusion classes, therefore have order divisible by atmost two primes, hence solvable implies G is solvable. A contradiction. Thus G is characteristically simple. Let $G \cong X_1 \times X_2 \times \cdots \times X_t$, where all X_i are simple and pairwise isomorphic. Choose $x \in X_1$, $y \in X_2$ of order p and q, respectively. Then $|xy| = pq$, but every element of G has prime order. Thus $t = 1$, implies G is simple. Now the result follows from Corollary 1. $\boxed{QED}$

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