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Characterizations by normal coordinates of special points and conics of a triangle

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Abstract. In [6], we associated with a given triangle $A_1A_2A_3$ and with each point P of the Euclidean plane a pencil of homothethic ellipses or hyperbolas with center P, which are determined by the loci of the points of the plane for which the distances d_1, d_2, d_3 to the sides of the triangle $A_1A_2A_3$ are related by $\frac{l_1}{d_1^0}d_1^2 + \frac{l_2}{d_2^0}d_2^2 + \frac{l_3}{d_3^0}d_3^2 = s$ (s variable in $\mathbb{R}$), where l_1, l_2, l_3 are the lengths of the sides of the triangle and where (d_1^0, d_2^0, d_3^0) are normal coordinates of P relative to the triangle $A_1A_2A_3$ (see section 1). In particular, a construction for the axes of these conics is given. Several special cases are treated, where P is the orthocenter H, the Lemoine point K, the incenter I, the centroid Z, and the circumcenter O of $A_1A_2A_3$ (for a summary of these results, see section 2).

In the present paper, we construct another pencil of conics with center P, using again normal coordinates relative to $A_1A_2A_3$ and look again for the axes, especially in the cases where $P = H, K, I, Z$, or O.

Keywords: Euclidean plane, triangle center, trilinear coordinates

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1 Normal coordinates

We work in the extended complexified Euclidean plane; extended means that we have a line at infinity with equation $z = 0$ in homogeneous rectangular coordinates (x, y, z). Assume that $A_1A_2A_3$ is a fixed general triangle (it is not rectangular and its sides have distinct lengths $l_1 = A_2A_3, l_2 = A_3A_1$, and $l_3 = A_1A_2$) and suppose that (a_i^1, a_i^2) are the non-homogeneous rectangular coordinates of A_i, $i = 1, 2, 3$. Normal or trilinear coordinates relative to the triangle $A_1A_2A_3$ are homogeneous projective coordinates where the vertices A_i are the base points and the incenter I of the triangle is the unit point.

Since the position vector $\vec{r}$ of the incenter I is given by ($\vec{r}_i$ is the position vector of $A_i, i = 1, 2, 3$) : $\vec{r} = \frac{l_1\vec{r}_1 + l_2\vec{r}_2 + l_3\vec{r}_3}{l_1+l_2+l_3}$, the relation between normal coordinates (x_1, x_2, x_3) and homogeneous rectangular coordinates (x, y, z) is given by:

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} l_1 a_1^1 & l_2 a_2^1 & l_3 a_3^1 \\ l_1 a_1^2 & l_2 a_2^2 & l_3 a_3^2 \\ l_1 & l_2 & l_3 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}.$$

The Lemoine point K (for constructions and properties of this point see [1], [4]), the centroid Z, the circumcenter O, the orthocenter H, and the incenter I of the triangle $A_1A_2A_3$ have normal coordinates (l_1, l_2, l_3), $\left(\frac{1}{l_1}, \frac{1}{l_2}, \frac{1}{l_3}\right)$, $(\cos \hat{A}_1, \cos \hat{A}_2, \cos \hat{A}_3)$, $\left(\frac{1}{\cos \hat{A}_1}, \frac{1}{\cos \hat{A}_2}, \frac{1}{\cos \hat{A}_3}\right)$, and $(1, 1, 1)$, respectively. In normal coordinates, the equation of the line at infinity, of the circumcircle and of the Steiner ellipse (through A_1, A_2, A_3 and with center Z) of $A_1A_2A_3$ are: $l_1x_1+l_2x_2+l_3x_3 = 0$, $l_1x_2x_3 + l_2x_3x_1 + l_3x_1x_2 = 0$, and $l_1l_2x_1x_2 + l_2l_3x_2x_3 + l_3l_1x_3x_1 = 0$, respectively.

If (x_1, x_2, x_3) are normal coordinates of a point P relative to $A_1A_2A_3$ and F is the area of this triangle, then $d_i = (2Fx_i)/(l_1x_1 + l_2x_2 + l_3x_3), i = 1, 2, 3,$ are the absolute normal coordinates of P relative to $A_1A_2A_3$. It is well-known that d_i is the distance from P to the side a_i of the triangle with positive or negative sign, according that P lies at the same side or opposite side as A_i, relative to a_i, $i = 1, 2, 3$. It is clear that absolute normal coordinates (d_1, d_2, d_3) are related by $l_1d_1 + l_2d_2 + l_3d_3 = 2F$.

The equation of the locus of the points of the plane for which the distances d_1, d_2, d_3 to the sides a_1, a_2, a_3 of the triangle $A_1A_2A_3$ are connected by

$$\frac{l_1}{d_1^0}d_1^2 + \frac{l_2}{d_2^0}d_2^2 + \frac{l_3}{d_3^0}d_3^2 = s, \ d_1^0, d_2^0, d_3^0, s \in \mathbb{R} \tag{1}$$

is clearly given by:

$$4F^2 \left(\frac{l_1}{d_1^0}x_1^2 + \frac{l_2}{d_2^0}x_2^2 + \frac{l_3}{d_3^0}x_3^2\right) - s\,(l_1x_1 + l_2x_2 + l_3x_3)^2 = 0 \tag{2}$$

and for variable s, we find a pencil of homothethic ellipses or hyperbolas which have all the same center P with normal coordinates (d_1^0, d_2^0, d_3^0), the same asymptotes and the same axes. We denote this bundle by $\mathcal{K}(P)$; it consists of ellipses if the product of the absolute normal coordinates of P is positive and it contains two sets of homothethic hyperbolas if this product is negative (see [6]).

2 The axes of the conics of the pencil $\mathcal{K}(P)$

Actually, this investigation started when O. Bottema studied the homothethic ellipses determined by

$$d_1^2 + d_2^2 + d_3^2 = s \tag{3}$$

with center the Lemoine point $K(l_1, l_2, l_3)$ of the triangle $A_1A_2A_3$, and asked for a geometric construction of the axes of these ellipses ([3]). Solutions were given by G.R. Veldkamp and J. Bilo ([7], [2]); this first construction was rather complicated, but Bilo gave a nice solution: the axes are the perpendicular lines through K on the Simson lines of the common points of the Euler line and the circumscribed circle of $A_1A_2A_3$.

In [6], we considered the generalized form (1) of (3) and we proved that, for each point P in the plane, the axes of the conics of $\mathcal{K}(P)$ are the orthogonal lines through P which cut the sides of the triangle $A_1A_2A_3$ in pairs of points whose midpoints are collinear. We called these lines the DNF-lines through P (relative to the triangle $A_1A_2A_3$). The existence of these orthogonal lines through P follows from the theorem of Desargues-Sturm: the two tangent lines through P at each of the parabolas that are tangent at a_1, a_2, a_3, correspond in an involution in the pencil of lines through P and it is not difficult to see that each pair of conjugate lines of this involution cuts the sides of $A_1A_2A_3$ in pairs of points whose midpoints are collinear. For a general point P, this involution contains just one orthogonal pair; but there is one exception: if $P = H$, the orthocenter of $A_1A_2A_3$, then each pair of orthogonal lines through H has this property (the orthocenter of any triangle circumscribed at a parabola lies on the directrix of the parabola and tangent lines from any point of the directrix at the parabola are orthogonal). This last result about H is known in the literature as the theorem of Droz-Noyer-Farny.

In [6] we considered the following special cases for the bundle $\mathcal{K}(P)$. The axes are always the DNF-lines relative to $A_1A_2A_3$ through the center.

I. $d_1^2 + d_2^2 + d_3^2 = s \Rightarrow$ homothethic ellipses with center the Lemoine point $K(l_1, l_2, l_3)$.

II. $l_1d_1^2 + l_2d_2^2 + l_3d_3^2 = s \Rightarrow$ homothethic ellipses with center the incenter $I(1, 1, 1)$.

III. $\frac{l_1}{\cos\hat{A}_1}d_1^2 + \frac{l_2}{\cos\hat{A}_2}d_2^2 + \frac{l_3}{\cos\hat{A}_3}d_3^2 = s$
(or, equivalently, $(\mathrm{tg}\hat{A}_1)d_1^2 + (\mathrm{tg}\hat{A}_2)d_2^2 + (\mathrm{tg}\hat{A}_3)d_3^2 = s'$)
$\Rightarrow$ pencil of ellipses or hyperbolas with center O ($\cos\hat{A}_1$, $\cos\hat{A}_2$, $\cos\hat{A}_3$).

IV. $(l_1 \cos\hat{A}_1)d_1^2 + (l_2 \cos\hat{A}_2)d_2^2 + (l_3 \cos\hat{A}_3)d_3^2 = s$
(or $(\sin 2\hat{A}_1)d_1^2 + (\sin 2\hat{A}_2)d_2^2 + (\sin 2\hat{A}_3)d_3^2 = s'$) $\Rightarrow$ concentric circles with center $H\left(\frac{1}{\cos\hat{A}_1}, \frac{1}{\cos\hat{A}_2}, \frac{1}{\cos\hat{A}_3}\right)$.

V. $l_1^2d_1^2 + l_2^2d_2^2 + l_3^2d_3^2 = s \Rightarrow$ homothethic ellipses with center $Z\left(\frac{1}{l_1}, \frac{1}{l_2}, \frac{1}{l_3}\right)$; $s = 4F^2$ gives the Steiner ellipse.

VI. $l_1^2(l_2^2 - l_3^2)d_1^2 + l_2^2(l_3^2 - l_1^2)d_2^2 + l_3^2(l_1^2 - l_2^2)d_3^2 = s \Rightarrow$ orthogonal hyperbolas with center $S\left(\frac{l_2 l_3}{l_2^2 - l_3^2}, \frac{l_3 l_1}{l_3^2 - l_1^2}, \frac{l_1 l_2}{l_1^2 - l_2^2}\right)$, which is the fourth common point of the circumcircle and the Steiner ellipse; it is the Steiner point.

VII. $\left(l_2^2 - l_3^2\right) d_1^2 + \left(l_3^2 - l_1^2\right) d_2^2 + \left(l_1^2 - l_2^2\right) d_3^2 = s \Rightarrow$ orthogonal hyperbolas with center $T\left(l_1/\left(l_2^2 - l_3^2\right), l_2/\left(l_3^2 - l_1^2\right), l_3/\left(l_1^2 - l_2^2\right)\right)$, which is the fourth common point of the circumcenter and the conic through A_1, A_2, A_3 with center K (the equation of this last conic is $l_3\left(l_1^2 + l_2^2 - l_3^2\right) x_1 x_2 + l_1(l_2^2 + l_3^2 - l_1^2)x_2 x_3 + l_2(l_3^2 + l_1^2 - l_2^2)x_3 x_1 = 0$ or, equivalently, $x_1 x_2 \cos \hat{A}_3 + x_2 x_3 \cos \hat{A}_1 + x_3 x_1 \cos \hat{A}_2 = 0$).

We conclude this section with two remarks:

1. The conics of $\mathcal{K}(P)$ with equations (2) are degenerate for $s = 2F$, if (d_1^0, d_2^0, d_3^0) are the absolute normal coordinates of P.

2. The locus of the points P where the conics of $\mathcal{K}(P)$ are orthogonal hyperbolas, is the circumcircle of $A_1 A_2 A_3$.

3 New pencils of homothethic conics

Let us now consider the locus of the points of the plane for which the absolute normal coordinates (d_1, d_2, d_3) relative to the triangle $A_1 A_2 A_3$, are related by

$$a d_2 d_3 + b d_3 d_1 + c d_1 d_2 = s, s \in \mathbb{R}, \tag{4}$$

with $a, b, c \in \mathbb{R}\backslash\{0\}$.
Of course, this gives a pencil of conics. The first question is: for what values of a, b, c these conics will have center P with normal coordinates (d_1^0, d_2^0, d_3^0)?

1 Theorem. *The points of the plane which have absolute normal coordinates (d_1, d_2, d_3) relative to $A_1 A_2 A_3$ related by*

$$d_1^0(-l_1 d_1^0 + l_2 d_2^0 + l_3 d_3^0) d_2 d_3 + d_2^0(l_1 d_1^0 - l_2 d_2^0 + l_3 d_3^0) d_3 d_1 + d_3^0(l_1 d_1^0 + l_2 d_2^0 - l_3 d_3^0) d_1 d_2 = s, \tag{5}$$

describe, for variable $s \in \mathbb{R}$, homothethic conics of a pencil and all these conics have the same center $P(d_1^0, d_2^0, d_3^0)$, the same asymptotes and the same axes. Let us denote this pencil by $\mathcal{L}(P)$.

PROOF. The equation in normal coordinates, corresponding with (5), is:

$$\mathcal{F}(x_1, x_2, x_3, s) = 4F^2(d_1^0(-l_1 d_1^0 + l_2 d_2^0 + l_3 d_3^0)x_2 x_3 + d_2^0(l_1 d_1^0 - l_2 d_2^0 + l_3 d_3^0) x_3 x_1 + d_3^0(l_1 d_1^0 + l_2 d_2^0 - l_3 d_3^0)x_1 x_2) - s(l_1 x_1 + l_2 x_2 + l_3 x_3)^2 = 0. \tag{6}$$

Since

$$\left(\frac{\partial \mathcal{F}}{\partial x_i}\right)_{\begin{cases} x_1=d_1^0 \\ x_2=d_2^0 \\ x_3=d_3^0 \end{cases}} = 2l_i(4F^2 d_1^0 d_2^0 d_3^0 - s(l_1 d_1^0 + l_2 d_2^0 + l_3 d_3^0)),\ i = 1,2,3,$$

the polar line of $P(d_1^0, d_2^0, d_3^0)$ relative to (6) becomes $l_1x_1 + l_2x_2 + l_3x_3 = 0$ or is indeterminate (if (d_1^0, d_2^0, d_3^0) are absolute normal coordinates and if $s = 2Fd_1^0d_2^0d_3^0$: in this case the conic (6) is degenerate). This completes the proof. QED

2 Theorem. *There exists a value s_0 of s such that the equation of the conic of $\mathcal{L}(P)$ corresponding with s_0 becomes:*

$$\begin{aligned} \mathcal{F}(x_1,x_2,x_3,s_0) &= k_1(-l_1x_1 + l_2x_2 + l_3x_3)^2 + k_2(l_1x_1 - l_2x_2 + l_3x_3)^2 \\ &+ k_3(l_1x_1 + l_2x_2 - l_3x_3)^2 = 0 \end{aligned} \tag{7}$$

with $k_1 = \frac{-F^2}{2l_1l_2l_3}(l_1d_1^0 - l_2d_2^0 + l_3d_3^0)(l_1d_1^0 + l_2d_2^0 - l_3d_3^0)$, and k_2, k_3 the cyclic permutations of k_1.

PROOF. From (6) and (7) we find the conditions:

$$\begin{cases} k_1 + k_2 + k_3 &= -s \\ l_1l_2(k_1+k_2) &= -F^2d_3^0(l_1d_1^0 + l_2d_2^0 - l_3d_3^0) \\ l_2l_3(k_2+k_3) &= -F^2d_1^0(-l_1d_1^0 + l_2d_2^0 + l_3d_3^0) \\ l_3l_1(k_3+k_1) &= -F^2d_2^0(l_1d_1^0 - l_2d_2^0 + l_3d_3^0). \end{cases}$$

And after a straightforward calculation, we obtain the given values of k_1, k_2, and k_3. Remark that k_1, k_2, k_3 are homogeneous and that we may omit the common factor $-F^2/2l_1l_2l_3$. This completes the proof. QED

Next, we see from (7) that the triangle $A_1'A_2'A_3'$ with sides $a_1' = A_2'A_3', a_2' = A_3'A_1'$, and $a_3' = A_1'A_2'$ given by the equations $-l_1x_1 + l_2x_2 + l_3x_3 = 0, l_1x_1 - l_2x_2 + l_3x_3 = 0$, and $l_1x_1 + l_2x_2 - l_3x_3 = 0$, respectively, is a polar triangle of the conic (7) of $\mathcal{L}(P)$. Analogously, the triangle $A_1A_2A_3$ is a polar triangle for the conic of $\mathcal{K}(P)$ which corresponds with $s = 0$ (see (2)) and it was this property that led us in [6] to the conclusion that the axes of the conics of $\mathcal{K}(P)$ are the DNF-lines through P, relative to $A_1A_2A_3$.

Let us now further look at the polar triangle $A_1'A_2'A_3'$ of the conic 7 of $\mathcal{L}(P)$: the line a_1' clearly contains the points $(l_2, l_1, 0)$ and $(l_3, 0, l_1)$, which are the midpoints of A_1A_2 and A_1A_3, and analogously for the sides a_2' and a_3'. Thus $A_1'(0, l_3, l_2)$, $A_2'(l_3, 0, l_1)$, and $A_3'(l_2, l_1, 0)$ are the midpoints of the sides of the triangle $A_1A_2A_3$. Remark that $A_i'A_j'$ is parallel with $A_iA_j, i,j = 1,2,3, \quad i \neq j$ and that the length of the side a_i' is $l_i' = \frac{l_i}{2}, i = 1,2,3$.

3 Theorem. *The formulas for the coordinate transformations between normal coordinates* (x_1, x_2, x_3) *relative to* $A_1A_2A_3$ *and normal coordinates* (x'_1, x'_2, x'_3) *relative to* $A'_1A'_2A'_3$ *are given by*

$$\begin{pmatrix} l_1x'_1 \\ l_2x'_2 \\ l_3x'_3 \end{pmatrix} = \begin{pmatrix} -l_1 & l_2 & l_3 \\ l_1 & -l_2 & l_3 \\ l_1 & l_2 & -l_3 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \tag{8}$$

and

$$\begin{pmatrix} l_1x_1 \\ l_2x_2 \\ l_3x_3 \end{pmatrix} = \begin{pmatrix} 0 & l_2 & l_3 \\ l_1 & 0 & l_3 \\ l_1 & l_2 & 0 \end{pmatrix} \begin{pmatrix} x'_1 \\ x'_2 \\ x'_3 \end{pmatrix}.$$

PROOF. It follows from (8) that $x'_1 = 0$ ($x'_2 = 0, x'_3 = 0$, respectively) corresponds with the equation of a'_1 (a'_2, a'_3, respectively). Moreover, the triangles $A_1A_2A_3$ and $A'_1A'_2A'_3$ have the same centroid $Z = Z'$. Relative to $A_1A_2A_3$, Z has normal coordinates $\left(\frac{1}{l_1}, \frac{1}{l_2}, \frac{1}{l_3}\right)$ and since $l'_i = \frac{l_i}{2}, i = 1, 2, 3$, Z' has the same normal coordinates relative to $A'_1A'_2A'_3$. From (8), we get indeed that $(x_1, x_2, x_3) = \left(\frac{1}{l_1}, \frac{1}{l_2}, \frac{1}{l_3}\right)$ gives $(x'_1, x'_2, x'_3) = \left(\frac{1}{l_1}, \frac{1}{l_2}, \frac{1}{l_3}\right)$ and this completes the proof. QED

4 Theorem. *The axes of the conics of the bundle* $\mathcal{L}(P)$ *are the DNF-lines through* P *relative to the triangle* $A'_1A'_2A'_3$. *Moreover, the conics of* $\mathcal{L}(P)$ *are ellipses if the product of the absolute normal coordinates of* P *relative to* $A'_1A'_2A'_3$ *is positive and hyperbolas otherwise.*

PROOF. In normal coordinates (x'_1, x'_2, x'_3) relative to $A'_1A'_2A'_3$, the equation (7) of the conic of $\mathcal{L}(P)$ becomes (with (d'^0_1, d'^0_2, d'^0_3) the normal coordinates of P relative to $A'_1A'_2A'_3$):

$$l_2d'^0_2l_3d'^0_3(l_1x'_1)^2 + l_3d'^0_3l_1d'^0_1(l_2x'_2)^2 + l_1d'^0_1l_2d'^0_2(l_3x'_3)^2 = 0,$$

or, equivalently:

$$\frac{l_1}{d'^0_1}x'^2_1 + \frac{l_2}{d'^0_2}x'^2_2 + \frac{l_3}{d'^0_3}x'^2_3 - 0,$$

which is the equation of the conic determined by:

$$\frac{l_1}{d'^0_1}d'^2_1 + \frac{l_2}{d'^0_2}d'^2_2 + \frac{l_3}{d'^0_3}d'^2_3 = 0,$$

with d'_1,d'_2,d'_3 the distances to the sides of the triangle $A'_1A'_2A'_2$. This completes the proof. QED

Let us denote (with the notations of the last proof) the pencil determined by $\frac{l_1}{d'^0_1}d'^2_1 + \frac{l_2}{d'^0_2}d'^2_2 + \frac{l_3}{d'^0_3}d'^2_3 = s', s' \in \mathbb{R}$, by $\mathcal{K}'(P)$ and, as before, the bundle determined by $d^0_1(-l_1d^0_1 + l_2d^0_2 + l_3d^0_3)d_2d_3 + d^0_2(l_1d^0_1 - l_2d^0_2 + l_3d^0_3)d_3d_1 + d^0_3(l_1d^0_1 + l_2d^0_2 - l_3d^0_3)d_1d_2 = s, s \in \mathbb{R}$, by $\mathcal{L}(P)$. Then $\mathcal{K}'(P) = \mathcal{L}(P)$.

4 Special cases for the pencil $\mathcal{L}(P) = \mathcal{K}'(P)$

In this section (d_1, d_2, d_3) are absolute normal coordinates of a variable point of the plane and (d_1^0, d_2^0, d_3^0) are normal coordinates of P relative to the triangle $A_1A_2A_3$; (d'_1, d'_2, d'_3) are normal coordinates of a variable point of the plane, and (d'^0_1, d'^0_2, d'^0_3) are normal coordinates of P relative to $A'_1A'_2A'_3$.

Moreover, l_1, l_2, l_3 are the lengths of the sides of $A_1A_2A_3$, and twice these lengths for $A'_1A'_2A'_3$. The following results are straightforward corollaries of the foregoing theorems and of the results of [6]. The considered point P is always the center and the DNF-lines relative to $A'_1A'_2A'_3$ through P are always the axes of the conics of the pencil $\mathcal{L}(P) = \mathcal{K}'(P)$.

Suppose that s,s',s_1,s'_1, etc... are real numbers. We look for the locus of the points for which the absolute normal coordinates (d_1, d_2, d_3) relative to $A_1A_2A_3$ (or the distances d'_1, d'_2, d'_3 to the sides of $A'_1A'_2A'_3$) are connected by the following equations:

4.1 $P = I$, the incenter of $A_1A_2A_3$

4.1.1 In normal coordinates relative to $A_1A_2A_3$

Normal coordinates of I are $(d_1^0, d_2^0, d_3^0) = (1, 1, 1)$ and the following relation between absolute normal coordinates (d_1, d_2, d_3) gives, for variable s, a pencil $\mathcal{L}(I)$ of homothethic ellipses with center I and axes the DNF-lines through I relative to $A'_1A'_2A'_3$:

$$(-l_1 + l_2 + l_3)d_2d_3 + (l_1 - l_2 + l_3)d_3d_1 + (l_1 + l_2 - l_3)d_1d_2 = s.$$

Remark that $s = 0$ gives us the equation of the ellipse through A_1, A_2, A_3, and with center I:

$$(-l_1 + l_2 + l_3)x_2x_3 + (l_1 - l_2 + l_3)x_3x_1 + (l_1 + l_2 - l_3)x_1x_2 = 0.$$

4.1.2 In normal coordinates relative to $A'_1A'_2A'_3$

Normal coordinates of I are

$$(d'^0_1, d'^0_2, d'^0_3) = \left(\frac{-l_1 + l_2 + l_3}{l_1}, \frac{l_1 - l_2 + l_3}{l_2}, \frac{l_1 + l_2 - l_3}{l_3}\right)$$

and the same pencil $\mathcal{K}'(I) = \mathcal{L}(I)$ is determined by

$$\frac{l_1^2}{(-l_1 + l_2 + l_3)}d'^2_1 + \frac{l_2^2}{(l_1 - l_2 + l_3)}d'^2_2 + \frac{l_3^2}{(l_1 + l_2 - l_3)}d'^2_3 = s', \text{ with } d'_1, d'_2, d'_3$$

the distances to the sides of $A'_1A'_2A'_3$.

4.2 $P = K$, the Lemoine point of $A_1A_2A_3$

4.2.1 Relative to $A_1A_2A_3$

Normal coordinates of K are $(d_1^0, d_2^0, d_3^0) = (l_1, l_2, l_3)$. The following relation between absolute normal coordinates (d_1, d_2, d_3):

$$l_1(-l_1^2 + l_2^2 + l_3^2)d_2d_3 + l_2(l_1^2 - l_2^2 + l_3^2)d_3d_1 + l_3(l_1^2 + l_2^2 - l_3^2)d_1d_2 = s$$

or, equivalently

$$(\cos \hat{A}_1)d_2d_3 + (\cos \hat{A}_2)d_3d_1 + (\cos \hat{A}_3)d_1d_2 - s_1 \left(- \frac{s}{2l_1l_2l_3}\right).$$

gives a pencil $\mathcal{L}(K)$ of ellipses or hyperbolas (depending on the location of K) with center K. Remark that $s = 0$ (or $s_1 = 0$) gives us the conic through A_1, A_2, A_3 with center K.

4.2.2 Relative to $A'_1A'_2A'_3$

Normal coordinates of K are: $\left({d'}_1^0, {d'}_2^0, {d'}_3^0\right) = \left(\frac{-l_1^2+l_2^2+l_3^2}{l_1} = \frac{2l_2l_3\cos\hat{A}_1}{l_1}, \frac{l_1^2-l_2^2+l_3^2}{l_2} = \frac{2l_3l_1\cos\hat{A}_2}{l_2}, \frac{l_1^2+l_2^2-l_3^2}{l_3} = \frac{2l_1l_2\cos\hat{A}_3}{l_3}\right) \sim \left(\frac{\cos\hat{A}_1}{l_1^2}, \frac{\cos\hat{A}_2}{l_2^2}, \frac{\cos\hat{A}_3}{l_3^2}\right)$. And the pencil $\mathcal{K}'(K) = \mathcal{L}(K)$ is determined by the following relations between the distances d'_1, d'_2, d'_3 to the sides of $A'_1A'_2A'_3$:

$$\frac{l_1^2}{-l_1^2 + l_2^2 + l_3^2}{d'}_1^2 + \frac{l_2^2}{l_1^2 - l_2^2 + l_3^2}{d'}_2^2 + \frac{l_3^2}{l_1^2 + l_2^2 - l_3^2}{d'}_3^2 = s'$$

or, equivalently

$$\frac{l_1^3}{\cos\hat{A}_1}{d'}_1^2 + \frac{l_2^3}{\cos\hat{A}_2}{d'}_2^2 + \frac{l_3^3}{\cos\hat{A}_3}{d'}_3^2 = s'_1(= 2l_1l_2l_3s')$$

or

$$l_1^2(\text{tg } \hat{A}_1){d'}_1^2 + l_2^2(\text{tg } \hat{A}_2){d'}_2^2 + l_3^2(\text{tg } \hat{A}_3){d'}_3^2 = {s''}_1.$$

4.3 $P = O$, the circumcenter of $A_1A_2A_3$, which is also the orthocenter H' of $A'_1A'_2A'_3$

4.3.1 Relative to $A_1A_2A_3$

Normal coordinates of O are $(d_1^0, d_2^0, d_3^0) = (\cos\hat{A}_1, \cos\hat{A}_2, \cos\hat{A}_3)$ and the bundle $\mathcal{L}(O)$ is determined by:

$$\begin{aligned}\cos\hat{A}_1(-l_1\cos\hat{A}_1 + l_2\cos\hat{A}_2 + l_3\cos\hat{A}_3)d_2d_3 \\ + \cos\hat{A}_2(l_1\cos\hat{A}_1 - l_2\cos\hat{A}_2 + l_3\cos\hat{A}_3)d_3d_1 \\ + \cos\hat{A}_3(l_1\cos\hat{A}_1 + l_2\cos\hat{A}_2 - l_3\cos\hat{A}_3)d_1d_2 = s\end{aligned}$$

or, equivalently

$$l_1d_2d_3 + l_2d_3d_1 + l_3d_1d_2 = s_1\left(= \frac{s}{2\cos\hat{A}_1\cos\hat{A}_2\cos\hat{A}_3}\right)$$

or

$$(\sin\hat{A}_1)d_2d_3 + (\sin\hat{A}_2)d_3d_1 + (\sin\hat{A}_3)d_1d_2 = s'_1.$$

Since $O = H'$, any two orthogonal lines through O are conjugate diameters of the conics of the pencil $\mathcal{L}(O)$, which means that these conics are concentric circles with center $O = H'$.

Remark that $s = 0$ (or $s_1 = 0, s'_1 = 0$) gives the circumscribed circle of $A_1A_2A_3$ with equation $l_1x_2x_3 + l_2x_3x_1 + l_3x_1x_2 = 0$.

4.3.2 Relative to $A'_1A'_2A'_3$

Normal coordinates of $O = H'$ are:

$$(d'^0_1, d'^0_2, d'^0_3) = (1/\cos\hat{A}_1, 1/\cos\hat{A}_2, 1/\cos\hat{A}_3).$$

And these concentric circles of $\mathcal{L}(O)$ are also determined by:

$$(l_1\cos\hat{A}_1)d'^2_1 + (l_2\cos\hat{A}_2)d'^2_2 + (l_3\cos\hat{A}_3)d'^2_3 = s'$$

or, equivalently

$$(\sin 2\hat{A}_1)d'^2_1 + (\sin 2\hat{A}_2)d'^2_2 + (\sin 2\hat{A}_3)d'^2_3 = s''.$$

4.4 $P = H$, the orthocenter of $A_1A_2A_3$

4.4.1 Relative to $A_1A_2A_3$

Normal coordinates of H are $(d_1^0, d_2^0, d_3^0) = (1/\cos\hat{A}_1, 1/\cos\hat{A}_2, 1/\cos\hat{A}_3)$ and $\mathcal{L}(H)$ is given by:

$$\frac{1}{\cos\hat{A}_1}\left(-\frac{l_1}{\cos\hat{A}_1}+\frac{l_2}{\cos\hat{A}_2}+\frac{l_3}{\cos\hat{A}_3}\right)d_2d_3+\frac{1}{\cos\hat{A}_2}\left(\frac{l_1}{\cos\hat{A}_1}-\frac{l_2}{\cos\hat{A}_2}+\frac{l_3}{\cos\hat{A}_3}\right)d_3d_1+\frac{1}{\cos\hat{A}_3}\left(\frac{l_1}{\cos\hat{A}_1}+\frac{l_2}{\cos\hat{A}_2}-\frac{l_3}{\cos\hat{A}_3}\right)d_1d_2=s.$$

This determines a pencil of ellipses or hyperbolas (depending on the location of H) with center H. With $s = 0$ corresponds the conic through A_1, A_2, A_3, and with center H.

4.4.2 Relative to $A'_1A'_2A'_3$

Normal coordinates of H are

$$\left(d'^0_1, d'^0_2, d'^0_3\right)=\left(\frac{1}{l_1}\left(-\frac{l_1}{\cos\hat{A}_1}+\frac{l_2}{\cos\hat{A}_2}+\frac{l_3}{\cos\hat{A}_3}\right),\frac{1}{l_2}\left(\frac{l_1}{\cos\hat{A}_1}-\frac{l_2}{\cos\hat{A}_2}+\frac{l_3}{\cos\hat{A}_3}\right),\frac{1}{l_3}\left(\frac{l_1}{\cos\hat{A}_1}+\frac{l_2}{\cos\hat{A}_2}-\frac{l_3}{\cos\hat{A}_3}\right)\right)$$

and $\mathcal{K}'(H)$ is determined by:

$$\frac{l_1^2}{\left(-\frac{l_1}{\cos\hat{A}_1}+\frac{l_2}{\cos\hat{A}_2}+\frac{l_3}{\cos\hat{A}_3}\right)}d_1'^2+\frac{l_2^2}{\left(\frac{l_1}{\cos\hat{A}_1}-\frac{l_2}{\cos\hat{A}_2}+\frac{l_3}{\cos\hat{A}_3}\right)}d_2'^2+\frac{l_3^2}{\left(\frac{l_1}{\cos\hat{A}_1}+\frac{l_2}{\cos\hat{A}_2}-\frac{l_3}{\cos\hat{A}_3}\right)}d_3'^2=s'.$$

4.5 $P = Z$ (the centroid of $A_1A_2A_3$) $= Z'$ (the centroid of $A'_1A'_2A'_3$)

4.5.1 Relative to $A_1A_2A_3$

$$(d_1^0, d_2^0, d_3^0)=\left(\frac{1}{l_1},\frac{1}{l_2},\frac{1}{l_3}\right)$$

The conics of the pencil $\mathcal{L}(Z)$ are the homothethic ellipses with center $Z = Z'$ determined by

$$l_2l_3d_2d_3 + l_3l_1d_3d_1 + l_1l_2d_1d_2 = s$$

and also by

$$l_1^2d_1^2 + l_2^2d_2^2 + l_3^2d_3^2 = s_1.$$

Remark that $s = 0$ gives the Steiner ellipse of $A_1A_2A_3$ with equation $l_2l_3x_2x_3 + l_3l_1x_3x_1 + l_1l_2x_1x_2 = 0$.

4.5.2 Relative to $A'_1A'_2A'_3$

$$\left(d'^0_1, d'^0_2, d'^0_3\right) = \left(\frac{1}{l_1}, \frac{1}{l_2}, \frac{1}{l_3}\right)$$

.

And $\mathcal{K}'(Z = Z')$ is determined by: $l_1^2d'^2_1 + l_2^2d'^2_2 + l_3^2d'^2_3 = s'$. The axes of the ellipses of $\mathcal{L}(Z) = \mathcal{K}'(Z')$ are the DNF-lines through $Z = Z'$ relative to $A_1A_2A_3$ and to $A'_1A'_2A'_3$.

4.6 $P = O'$, the circumcenter of $A'_1A'_2A'_3$, which is also the center of the nine-point circle of $A_1A_2A_3$

4.6.1 Relative to $A'_1A'_2A'_3$

$$\left(d'^0_1, d'^0_2, d'^0_3\right) = \left(\cos\hat{A}_1, \cos\hat{A}_2, \cos\hat{A}_3\right)$$

.

The bundle $\mathcal{K}'(O')$ is determined by:

$$\frac{l_1d'^2_1}{\cos\hat{A}_1} + \frac{l_2d'^2_2}{\cos\hat{A}_2} + \frac{l_3d'^2_3}{\cos\hat{A}_3} = s' \text{ or } (tg\hat{A}_1)d'^2_1 + (tg\hat{A}_2)d'^2_2 + (tg\hat{A}_3)d'^2_3 = s''$$

and it consists of ellipses or hyperbolas (depending on the location of O') with center O'.

4.6.2 Relative to $A_1A_2A_3$

Normal coordinates of the center of the nine-point circle of $A_1A_2A_3$ are

$$(d_1^0, d_2^0, d_3^0) = \left(\frac{l_2\cos\hat{A}_2 + l_3\cos\hat{A}_3}{l_1}, \frac{l_3\cos\hat{A}_3 + l_1\cos\hat{A}_1}{l_2}, \frac{l_1\cos\hat{A}_1 + l_2\cos\hat{A}_2}{l_3}\right).$$

The bundle $\mathcal{L}(O')$ is given by:

$$((l_2 \cos \hat{A}_2 + l_3 \cos \hat{A}_3) \cos \hat{A}_1)d_2d_3 + ((l_3 \cos \hat{A}_3 + l_1 \cos \hat{A}_1) \cos \hat{A}_2)d_3d_1 + ((l_1 \cos \hat{A}_1 + l_2 \cos \hat{A}_2) \cos \hat{A}_3)d_1d_2 = s$$

or, equivalently, by:

$$((\sin 2\hat{A}_2 + \sin 2\hat{A}_3) \cos \hat{A}_1)d_2d_3 + ((\sin 2\hat{A}_3 + \sin 2\hat{A}_1) \cos \hat{A}_2)d_3d_1 + ((\sin 2\hat{A}_1 + \sin 2\hat{A}_2) \cos \hat{A}_3)d_1d_2 = s_1.$$

And, as always, $s = 0$ gives the conic of $\mathcal{L}(P)$ through A_1, A_2, A_3, and with center P, thus now with center the center of the nine-point circle of $A_1A_2A_3$.

4.7 $P = K'$, the Lemoine point of $A'_1A'_2A'_3$, (X(83) in [5])

4.7.1 Relative to $A'_1A'_2A'_3$

K' has normal coordinates $\left(d'^0_1, d'^0_2, d'^0_3\right) = (l_1, l_2, l_3)$ and the bundle $\mathcal{K}'(K')$ of homothethic ellipses with center K' is determined by:

$$d'^2_1 + d'^2_2 + d'^2_3 = s'.$$

4.7.2 Relative to $A_1A_2A_3$

K' has normal coordinates $d^0_1, d^0_2, d^0_3 = (l_2l_3(l_2^2+l_3^2), l_3l_1(l_3^2+l_1^2), l_1l_2(l_1^2+l_2^2))$ and the bundle $\mathcal{L}(K')$ is given by:

$$l_1(l_2^2 + l_3^2)d_2d_3 + l_2(l_3^2 + l_1^2)d_3d_1 + l_3(l_1^2 + l_2^2)d_1d_2 = s.$$

The ellipse through A_1, A_2, A_3, and with center K' corresponds with $s = 0$.

4.8 $P = I'$, the incenter of $A'_1A'_2A'_3$ (Spieker center of $A_1A_2A_3$)

4.8.1 Relative to $A'_1A'_2A'_3$

I' has normal coordinates $(d'^0_1, d'^0_2, d'^0_3) = (1, 1, 1)$ and $\mathcal{K}'(I')$ is the bundle consisting of homothethic ellipses with center I' given by:

$$l_1d'^2_1 + l_2d'^2_2 + l_3d'^2_3 = s' \text{ or also } (\sin \hat{A}_1)d'^2_1 + (\sin \hat{A}_2)d'^2_2 + (\sin \hat{A}_3)d'^2_3 = s''.$$

4.8.2 Relative to $A_1A_2A_3$

I' has normal coordinates $(d^0_1, d^0_2, d^0_3) = (l_2l_3(l_2+l_3), l_3l_1(l_3+l_1), l_1l_2(l_1+l_2))$ and the bundle $\mathcal{L}(I')$ is determined by:

$$(l_2 + l_3)d_2d_3 + (l_3 + l_1)d_3d_1 + (l_1 + l_2)d_1d_2 = s.$$

The ellipse through A_1, A_2, A_3, and with center I' corresponds with $s = 0$.

4.9

Finally, let us consider the bundle determined by (relative to $A_1A_2A_3$):

$$d_2d_3 + d_3d_1 + d_1d_2 = s.$$

It is not difficult to see that all the conics of this bundle have the center $(d_1^0, d_2^0, d_3^0) = (-l_1 + l_2 + l_3, l_1 - l_2 + l_3, l_1 + l_2 - l_3)$ (a straightforward calculation shows that the polar line of this point relative to the conic $x_2x_3 + x_3x_1 + x_1x_2 = 0$, which belongs to the bundle, is indeed the line at infinity).

The normal coordinates of this center relative to $A_1'A_2'A_3'$ are:

$$\left(d'^0_1, d'^0_2, d'^0_3\right) = \left(\frac{1}{l_1}\left(-l_1 + l_2 + l_3\right), \frac{1}{l_2}\left(l_1 - l_2 + l_3\right), \frac{1}{l_3}\left(l_1 + l_2 - l_3\right)\right),$$

and the bundle is given by:

$$l_1^2(-l_1 + l_2 + l_3)d'^2_1 + l_2^2(l_1 - l_2 + l_3)d'^2_2 + l_3^2(l_1 + l_2 - l_3)d'^2_3 = s'.$$

The axes of the conics of the bundle are, as always, the DNF-lines through the center relative to $A_1'A_2'A_3'$; but where lies the center of these conics?

5 Lemma. *The center of the conics of the pencil determined by $d_2d_3 + d_3d_1 + d_1d_2 - s$ is the Lemoine point K'' of the triangle $I_1I_2I_3$, where $I_1(-1, 1, 1)$, $I_2(1, -1, 1)$ and $I_3(1, 1, -1)$ are the excenters of the triangle $A_1A_2A_3$ (the centers of the escribed circles of the triangle; the coordinates are normal coordinates relative to $A_1A_2A_3$).*

PROOF. a. The coordinate transformations between normal coordinates (x_1, x_2, x_3) relative to $A_1A_2A_3$ and (x_1'', x_2'', x_3'') relative to $I_1I_2I_3$ are given by:

$$\begin{pmatrix} x_1'' \\ x_2'' \\ x_3'' \end{pmatrix} = \begin{pmatrix} 0 & \frac{1}{\sin\frac{\hat{A}_1}{2}} & \frac{1}{\sin\frac{\hat{A}_1}{2}} \\ \frac{1}{\sin\frac{\hat{A}_2}{2}} & 0 & \frac{1}{\sin\frac{\hat{A}_2}{2}} \\ \frac{1}{\sin\frac{\hat{A}_3}{2}} & \frac{1}{\sin\frac{\hat{A}_3}{2}} & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \tag{9}$$

and

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} -\sin\frac{\hat{A}_1}{2} & \sin\frac{\hat{A}_2}{2} & \sin\frac{\hat{A}_3}{2} \\ \sin\frac{\hat{A}_1}{2} & -\sin\frac{\hat{A}_2}{2} & \sin\frac{\hat{A}_3}{2} \\ \sin\frac{\hat{A}_1}{2} & \sin\frac{\hat{A}_2}{2} & -\sin\frac{\hat{A}_3}{2} \end{pmatrix} \begin{pmatrix} x_1'' \\ x_2'' \\ x_3'' \end{pmatrix}. \tag{10}$$

First, it is obvious from (9) and (10) that the coordinates $(x_1, x_2, x_3) = (-1, 1, 1)$, ($(1, -1, 1)$, and $(1, 1, -1)$, respectively) corresponds with $(x_1^{''}, x_2^{''}, x_3^{''}) = (1, 0, 0)$, ($(0, 1, 0)$, and $(0, 0, 1)$, respectively). Second, it is not difficult to see that the angles of the triangle $I_1I_2I_3$ are given by $\hat{I}_i = 90° - \frac{\hat{A}_i}{2}, i = 1, 2, 3$. Moreover, the incenter I of $A_1A_2A_3$ is clearly the orthocenter $H^{''}$ of $I_1I_2I_3$, and thus, has normal coordinates

$$(x_1^{''}, x_2^{''}, x_3^{''}) = \left(\frac{1}{\cos \hat{I}_1}, \frac{1}{\cos \hat{I}_2}, \frac{1}{\cos \hat{I}_3}\right) = \left(\frac{1}{\sin \frac{\hat{A}_1}{2}}, \frac{1}{\sin \frac{\hat{A}_2}{2}}, \frac{1}{\sin \frac{\hat{A}_3}{2}}\right)$$

relative to $I_1I_2I_3$.

From (9) and (10) it follows that these coordinates correspond with:

$$(x_1, x_2, x_3) = (1, 1, 1),$$

and this completes the proof of a.

b. Next, it is an easy exercise to prove that the lengths of the sides of the triangle $I_1I_2I_3$ are: $I_1I_2 = \frac{l_3}{\sin \frac{\hat{A}_3}{2}}$, $I_2I_3 = \frac{l_1}{\sin \frac{\hat{A}_1}{2}}$, and $I_3I_1 = \frac{l_2}{\sin \frac{\hat{A}_2}{2}}$. From this, it follows that the Lemoine point $K^{''}$ of $I_1I_2I_3$ has normal coordinates relative to $I_1I_2I_3$:

$$(x_1^{''}, x_2^{''}, x_3^{''}) = \left(\frac{l_1}{\sin \frac{\hat{A}_1}{2}}, \frac{l_2}{\sin \frac{\hat{A}_2}{2}}, \frac{l_3}{\sin \frac{\hat{A}_3}{2}}\right)$$

and substituting these coordinates in (10) gives us $(x_1, x_2, x_3) = (-l_1 + l_2 + l_3, l_1 - l_2 + l_3, l_1 + l_2 - l_3)$. This completes the proof of the lemma. QED

The coordinates of $K^{''}$ can also be found in [4], page 90 and in Kimberling's list as $X(9)$, the mittenpunkt ([5]).

5 Other properties

1. The incenter I, the Lemoine point K of $A_1A_2A_3$, and the Lemoine point $K^{''}$ of $I_1I_2I_3$ are collinear.

   Proof. The coordinates of I, K, and $K^{''}$, relative to $A_1A_2A_3$, are $(1, 1, 1)$, (l_1, l_2, l_3) and $(-l_1 + l_2 + l_3, l_1 - l_2 + l_3, l_1 + l_2 - l_3)$, and

$$\begin{vmatrix} 1 & 1 & 1 \\ l_1 & l_2 & l_3 \\ -l_1 + l_2 + l_3 & l_1 - l_2 + l_3 & l_1 + l_2 - l_3 \end{vmatrix} = 0.$$

QED

2. The equation of the nine-point circle of $A_1A_2A_3$ in normal coordinates relative to $A_1A_2A_3$ is (see also [4], page 10):

$$(l_1 \cos \hat{A}_1)x_1^2 + (l_2 \cos \hat{A}_2)x_2^2 + (l_3 \cos \hat{A}_3)x_3^2 - l_3x_1x_2 - l_1x_2x_3 - l_2x_3x_1 = 0.$$

PROOF. The nine-point circle of $A_1A_2A_3$ is the circumscribed circle of $A'_1A'_2A'_3$ and has, in normal coordinates relative to $A'_1A'_2A'_3$, the equation $l_1x'_2x'_3 + l_2x'_3x'_1 + l_3x'_1x'_2 = 0$. Transforming this equation by (8), we find $l_1^2(l_1^2-l_2^2-l_3^2)x_1^2+l_2^2(l_2^2-l_3^2-l_1^2)x_2^2+l_3^2(l_3^2-l_1^2-l_2^2)x_3^2+2l_1l_2l_3^2x_1x_2+2l_2l_3l_1^2x_2x_3+$ $+2l_3l_1l_2^2x_3x_1 = 0$, which is proportional to the given equation. QED

3. The nine-point circle of $A_1A_2A_3$ is the locus of the points P such that the conics of the pencil $\mathcal{L}(P)$ are orthogonal hyperbolas. This follows from the fact that the nine-point circle is the circumscribed circle of the triangle $A'_1A'_2A'_3$ (see remark 2, at the end of section 2).

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