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Maximal visibility and unions of orthogonally starshaped sets

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Abstract. Let S be an orthogonal polygon in the plane. For each point x in S, let V_x denote the set of points which x sees via staircase paths, and let $M_x = \{y : V_y = V_x\}$. For S simply connected, S is starshaped via staircase paths (i.e., orthogonally starshaped) if and only if S contains exactly one such closed set M_x, and when this occurs M_x is the staircase kernel of S. In general, if S contains exactly k such distinct closed set $M_{x_1}, \ldots, M_{x_k}$, then S is a union of k (or possibly fewer) orthogonally starshaped sets chosen from $V_{x_1}, \ldots, V_{x_k}$.

Keywords: orthogonal polygons, starshaped via staircase paths.

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1 Introduction

We begin with some definitions from [1]. Let S be a nonempty set in the plane. Set S is called an *orthogonal polygon* (*rectilinear polygon*) if and only if S is a connected union of finitely many convex polygons (possibly degenerate) whose edges are parallel to the coordinate axes. Set S is said to be *horizontally convex* if and only if for each x, y in S with $[x, y]$ horizontal, it follows that $[x, y] \subseteq S$. *Vertically convex* is defined analogously. Set S is *orthogonally convex* if and only if S is an orthogonal polygon which is both horizontally and vertically convex.

Let λ be a simple polygonal path in the plane whose edges $[w_{i-1}, w_i]$, $1 \leq i \leq n$, are parallel to the coordinate axes. Path λ is called a *staircase path* if and only if the associated vectors alternate in direction. That is, for an appropriate labeling, for i odd the vectors $\overrightarrow{w_{i-1}w_i}$ have the same horizontal direction and for i even the vectors $\overrightarrow{w_{i-1}w_i}$ have the same vertical direction. Edge $[w_{i-1}, w_i]$ will be called north, south, east, or west according to the direction of vector $\overrightarrow{w_{i-1}w_i}$. Similarly, we use the terms north, south, east, west, northeast, northwest, southeast, southwest to describe the relative position of points.

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For points x and y in set S, we say *x sees y via staircase paths* (x is *visible* from y via staircase paths) if and only if there is a staircase path in S which contains both x and y. For each point x in S, we define its *visibility set* in S by $V_x - \{ y : x \text{ sees } y \text{ via staircase paths}\}$. By [8, Lemma 1], orthogonal polygon S is orthogonally convex if and only if every two of its points see each other via staircase paths. Similarly, set S is *starshaped via staircase paths* (*orthogonally starshaped*) if and only if for some point p in S, p sees each point of S via starshaped paths, and the set of all such points p is the *staircase kernel* of S, denoted $\operatorname{Ker} S$.

Many results in convexity that involve the usual notion of visibility via straight line segments have interesting analogues that employ the idea of visibility via staircase paths. (See [1] for a list of related references.) Results in [9], [4], and [2] use points of locally maximal visibility to describe certain starshaped sets and their unions in a linear topological space, and here we seek an analogous result for an orthogonal polygon S. For set S the local property above is not very useful, however, since every point x of S has locally maximal visibility in S. That is, for each x in S, points near x see no more than x sees (via staircase paths) and may well see less. Instead, we examine those points x whose visibility sets are *maximal* in S. That is, those points x for which V_x is not a proper subset of V_y for any y in S. It turns out that, for such an x, the corresponding set $M_x = \{y : V_y = V_x\}$ is closed and (for S simply connected) orthogonally convex. Moreover, these M_x sets function as kernels for appropriate subsets of S, yielding a decomposition of S into starshaped sets.

Throughout the paper we will use the following notation: $\operatorname{int} S$, $\operatorname{cl} S$, and $\operatorname{conv} S$ will denote the interior, closure, and convex hull, respectively, of set S. If λ is a polygonal path containing points s and t, $\lambda(s,t)$ will represent the subpath of λ from s to t. As discussed above, for point x in set S, V_x will be its visibility set in S, with $M_x = \{y : V_y = V_x\}$.

The reader may refer to Valentine [10], to Lay [7], to Danzer, Grünbaum, Klee [5], and to Eckhoff [6] for discussions concerning visibility via segments and starshaped sets.

2 The Results.

We begin with some preliminary lemmas.

1 Lemma. *Let S be an orthogonal polygon in the plane. There are finitely many distinct visibility sets V_x, x in S, and finitely many associated sets $M_x = \{y : V_y = V_x\}$.*

Proof. As in [3], let $\mathcal{L}$ be the family of lines determined by edges of S. Then

$\mathcal{L}$ gives rise to a collection $\mathcal{T}$ of non-degenerate closed rectangular regions such that each member T of $\mathcal{T}$ is minimal and $\cup\{T : T \text{in} \mathcal{T}\} = \text{cl}(\text{int}\, S)$. Let $\mathcal{B}$ be the finite family $\{\text{int}\, T : T \text{in} \mathcal{T}\} \cup \{(s,t) : [s,t] \text{ an edge of } T, T \text{in} \mathcal{T}\} \cup \{(s,t) : [s,t]$ an edge of S and $(s,t) \cap \text{cl}(\text{int}\, S) - \phi\}$. Certainly for any B in $\mathcal{B}$, all points of B have the same visibility set. Moreover, only finitely many points of S fail to belong to any B set. Thus there are finitely many distinct visibility sets V_x, x in S, and finitely many associated sets M_x as well. QED

2 Lemma. *Let S be a simply connected orthogonal polygon in the plane. For each x in S, the associated set $M_x = \{y : V_y = V_x\}$ is orthogonally convex.*

Proof. Let y, z belong to M_x to show that M_x contains a staircase $y - z$ path. In fact, we will show that M_x contains every staircase $y - z$ path in S. Since $V_y = V_z, y$ sees z via staircase paths in S, and we let λ denote such a path. For $w \epsilon \lambda$, we will show that $w \epsilon M_x$. That is, $V_w = V_x$: Certainly $V_x = V_y = V_z \subseteq V_w$, for if y and z both see some point s (via staircase paths), then by [3, Lemma 2], all points of λ see s (via staircase paths) as well. To show that $V_w \subseteq V_x$, assume that w sees some point t of S. Without loss of generality, assume that z is northeast of y. If t is northeast of w, then y sees t (via staircase paths), and $t \epsilon V_y = V_x$, the desired result. Similarly, if t is southwest of w, then z sees t (via staircase paths), again the desired result. Hence without loss of generality assume that t is northwest of w. Let μ be a staircase $w - t$ path, and let w' be the last point of $\mu(w,t)$ seen by y and z. (See Figure 1.) Observe that μ is west of the vertical line at z and north of the horizontal line at y. If $w' \neq t$ and $\mu(w',t)$ begins with a north segment, then y sees this segment, contradicting our choice of w'. Likewise, if $w' \neq t$ and $\mu(w',t)$ begins with a west segment, then z sees this segment, again impossible. Thus $w' = t, t \epsilon V_y = V_z = V_x$ and $V_w \subseteq V_x$. We conclude that $V_w = V_x$. Hence $\mu \subseteq M_x$ and M_x is orthogonally convex. QED

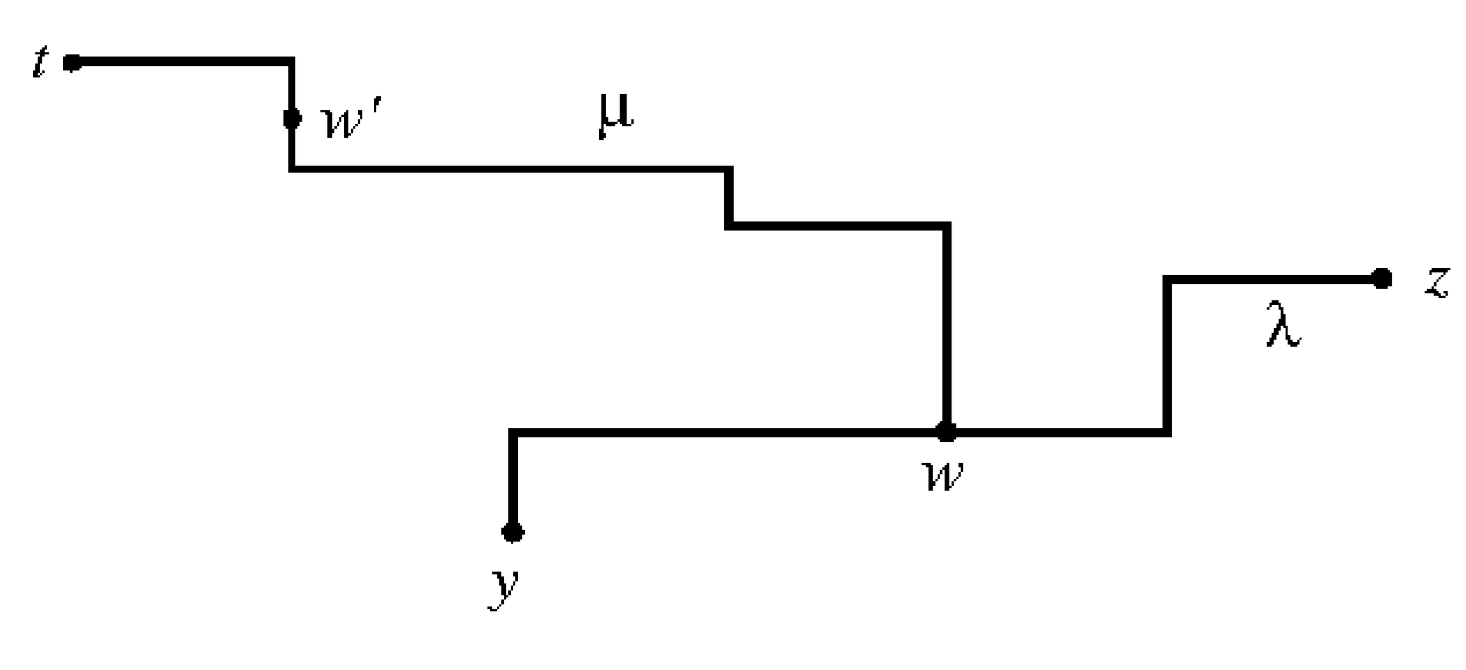

Figure 1.

It is easy to see that Lemma 2 fails when we delete the simple connectedness requirement. Consider the following example.

1 Example. Set S be the boundary of a rectangle having vertices $x_i, 1 \leq i \leq 4$. Certainly $M_{x_1} = \{x_i : 1 \leq i \leq 4\}$ is not orthogonally convex.

3 Lemma. *Let S be a simply connected orthogonal polygon in the plane. For point x in S, visibility set V_x is maximal if and only if the associated set $M_x = \{y : V_y = V_x\}$ is closed.*

Proof. The necessity is easy and does not require simple connectedness: If V_x is maximal, choose any y in $\operatorname{cl} M_x$ to show that $y \epsilon M_x$. Clearly $V_x \subseteq V_y$ (since visibility sets are closed), and since V_x is maximal, V_x cannot be a proper subset of V_y. Hence $V_x = V_y, y \epsilon M_x$, and M_x is closed.

For the sufficiency, we use a contrapositive argument. Assume that for some x set V_x is not maximal to show that M_x is not closed. If V_x is not maximal, then for some y in S, V_x is a proper subset of V_y. Since $V_x \subseteq V_y, y$ sees x via staircase paths in S, and we let $\lambda(x, y)$ be a staircase $x - y$ path in S. For future reference, observe that for every t in V_x, both x and y see t via staircase paths, and by [3, Lemma 2], each point s of $\lambda(x, y)$ sees t via staircase paths as well. Thus $V_x \subseteq V_s$ for every $s \epsilon \lambda(x, y)$.

Let A be the component of $M_x \cap \lambda(x, y)$ at x. Then A is a subpath of $\lambda(x, y)$ (possibly degenerate) with endpoints x and p for some p in $\lambda(x, y)$. We will show that $p \notin A$ and hence A is not closed: If $p = y$, then since $V_x \neq V_y, p = y \notin M_x$ and M_x is not closed, the desired result. Hence we assume that $p \neq y$. Then since A is a component of $M_x \cap \lambda(x, y)$, every $\frac{1}{n}$ -neighborhood of p must contain some point p_n in $\lambda(p, y) \backslash M_x$. Moveover, since there are only finitely many visibility sets V_{p_n} in S, we may choose the sequence $\{p_n\}$ so that $V_{p_1} = V_{p_n}$ for all $n \geq 1$. Since $V_{p_n} \neq V_x$, one of these two visibility sets is not a subset of the other. By an observation above, $V_x \subseteq V_{p_1}$, so we must have $V_{p_1} \not\subseteq V_x$. Thus for some $w \notin V_x, p_1$ (and in fact each p_n) sees w via staircase paths. Since $\{p_n\}$ converges to p, p sees w as well. Hence $V_p \neq V_x, p \notin M_x$, and M_x is not closed. This establishes the sufficiency and finishes the proof of Lemma 3. QED

The lemmas yield the following results for orthogonally starshaped sets and their unions, with the M_x sets functioning as staircase kernels.

1 Theorem. Let S be a simply connected orthogonal polygon in the plane, and for each x in S let $M_x = \{y : V_y = V_x\}$. Set S is orthogonally starshaped if and only if S contains exactly one such set M_x which is closed. When this occurs, $M_x = \operatorname{Ker} S$.

Proof. When S is orthogonally starshaped, then visibility set V_x is maximal for x in S if and only if $x \epsilon \operatorname{Ker} S$. By Lemma 3 it follows that M_x is closed for x in

S if and only if $x \epsilon \operatorname{Ker} S$. For such an x, set $M_x = \{y : V_y = V_x = S\}$ is unique and is, of course, $\operatorname{Ker} S$.

The converse does not require the simple connectedness condition. We assume that S contains exactly one set M_x which is closed, to show that S is orthogonally starshaped. Certainly S is a finite union of orthogonally starshaped sets, say $V_{x_1}, \dots, V_{x_k}$. Without loss of generality, we assume that each of these visibility sets is maximal and hence (by the first part of Lemma 3) the corresponding sets M_{x_i} are closed, $1 \leq i \leq k$. However, this means that all the sets M_{x_i} are the same and hence $V_{x_i} = V_{x_1}$ for $1 \leq i \leq k$. Thus $S = V_{x_1}, S$ is starshaped, and $M_{x_1} = \operatorname{Ker} S$, finishing the proof. QED

The importance of the simple connectedness condition, both for Lemma 3 and for Theorem 1, will be addressed in Example 3. Without simple connectedness, Lemma 3 and Theorem 1 yield the following corollary.

4 Corollary. *Let S be an orthogonal polygon in the plane. If for some x in S the corresponding visibility set V_x is maximal, then the associated $M_x = \{y : V_y = V_x\}$ is closed. If S contains exactly one such set M_x which is closed, then S is orthogonally starshaped with $M_x = \operatorname{Ker} S$.*

Theorem 2 provides a similar result for unions of orthogonal polygons.

2 Theorem. Let S be an orthogonal polygon in the plane, and for each x in S let $M_x = \{y : V_y = V_x\}$. If S contains exactly k distinct closed M_x sets $M_{x_1}, \dots, M_{x_k}$ for some $k \geq 1$, then S is a union of k or fewer starshaped sets chosen from $V_{x_1}, \dots, V_{x_k}$.

Proof. Certainly set S is a finite union of distinct orthogonally starshaped sets, say $V_{y_1}, \dots V_{y_n}$, where each set V_{y_i} is maximal, $1 \leq i \leq n$. Hence by the first part of Lemma 3, the associated sets $M_{y_1}, \dots M_{y_n}$ are closed, $1 \leq i \leq n$, and so each M_{y_i} is one of the k sets $M_{x_1}, \dots M_{x_k}$. Since the visibility sets V_y are distinct, so are the associated sets M_y, and thus each M_y is a different M_x. Therefore $n \leq k$, and we may relabel the M_x sets if necessary so that $M_{y_i} = M_{x_i}$ for $1 \leq i \leq n$. Clearly $V_{y_i} = V_{x_i}, 1 \leq i \leq n$, and $S = V_{x_1} \cup \dots \cup V_{x_n}$, finishing the argument. QED

In the proof of Theorem 2, certainly $n \leq k$. In fact, n may be strictly less than k, as the following example illustrates.

2 Example. Let S be the polygonal path in Figure 2. Using the terminology in Theorem 2, set S contains exactly three distinct closed M_x sets: $M_{x_i} = [x_i, y_i], 1 \leq i \leq 3$. However, S is a union of two (and no fewer) orthogonally starshaped sets V_{x_1} and V_{x_2}.

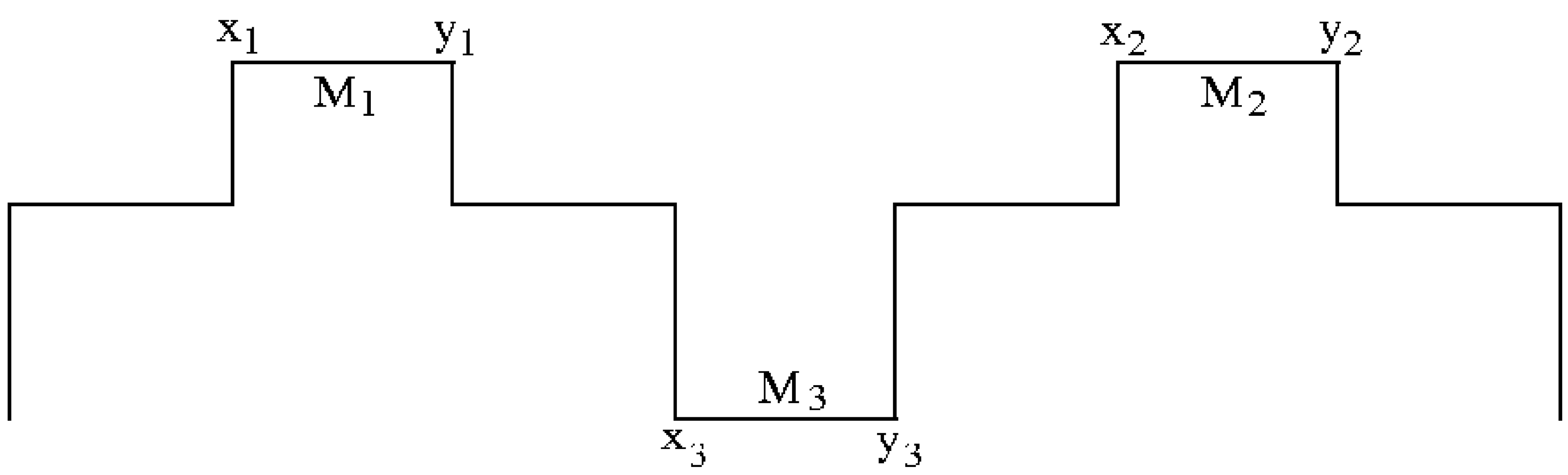

Figure 2.

We conclude with some other examples. First, we observe that without the simple connectedness condition for set S, portions of Lemma 3 and Theorem 1 fail. Consider the following example.

3 Example. Let S be the union of the polygonal paths in Figure 3. Using our earlier notation, set $M_x = \{x\}$ is closed. However, $\operatorname{Ker} S = \{y\}$, so V_x is a proper subset of V_y, and V_x is not maximal. Thus the sufficiency in Lemma 3 fails. Similarly, set S is orthogonally starshaped although S contains distinct closed sets $M_x = \{x\}$ and $M_y = \{y\}$, violating the necessity in Theorem 1.

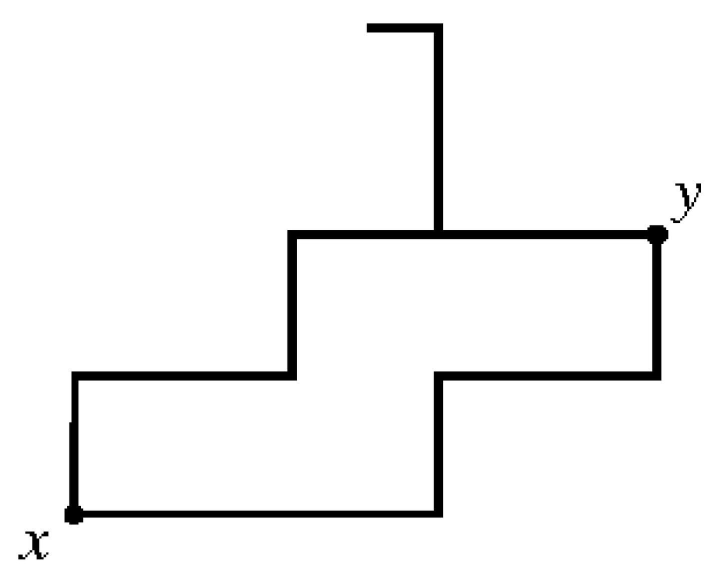

Figure 3.

Further, analogous results fail for visibility via segments, even for a set which is closed and simply connected in the plane. Of course, using our previous notation, when visibility (via segments) set V_x is maximal, then the associated set M_x is closed by an argument like the one in Lemma 3. However, set M_x may be closed although V_x is not maximal. For example, if S is the familiar five-pointed star, then $M_x = \{x\}$ for every x in S outside the kernel. Thus M_x is closed for every x in S, while V_x will be maximal only for x in the kernel. A more interesting example (below) reveals a similar situation for nontrivial sets M_x.

4 Example. In Figure 4, segments $[a,b],[c,d]$ are tangent to circle C at x,y, respectively. Let S be the closed, simply connected set whose boundary consists of the minor arc from x to y in C together with polygonal path $[y,c] \cup [c,c'] \cup [c',a'] \cup [a',a] \cup [a,x]$. Then both sets $M_x = [a,x]$ and $M_y = [c,y]$ are nontrivial and closed. However, both $V_x = \text{conv}\{a,a',b\}$ and $V_y = \text{conv}\{c,c',d\}$ are proper subsets of V_s for any s in the kernel of $S, \text{conv}\{z,b,d\}$. Hence neither V_x nor V_y is maximal. Moreover, set S is starshaped although for every x in S, the corresponding M_x is closed..

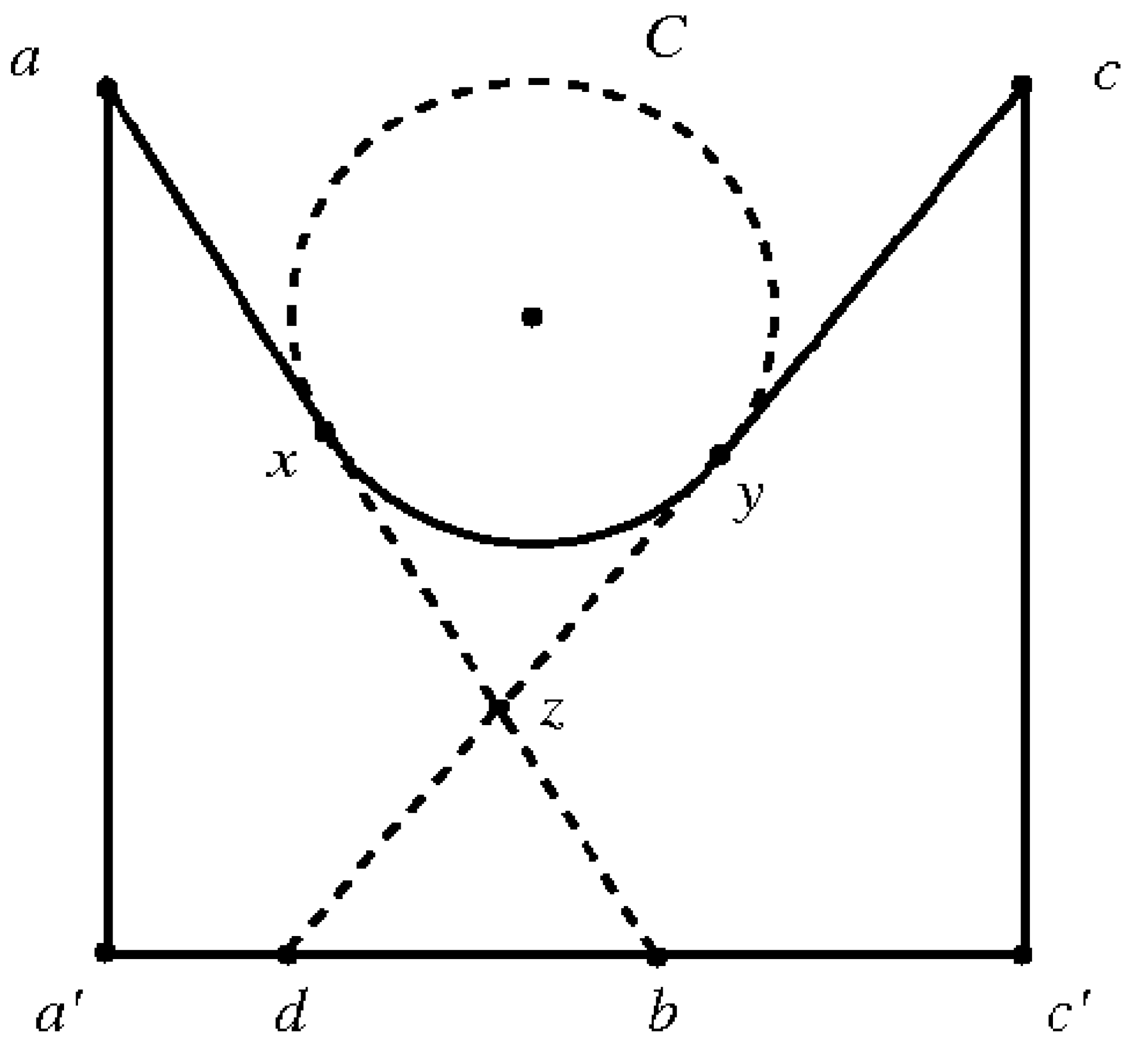

Figure 4.

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