GEOODULAR AXIOMATICS OF AFFINE SPACES

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Abstract. *Any flat geoodular space can be treated as an affine space and vice versa. A purely algebraic proof of the fact is presented here. It gives us a new axiomatics of affine spaces. Moreover, such an approach permits us to consider affine spaces over arbitrary rings and to regard an affine space as a universal algebra.*

Any algebraic system $M = <M, L, (\omega_t)_{t\in R}>$ equipped with a ternary operation $L(x, a, y) = L^a_\omega y = x \cdot y$ and a collection of binary operations $\omega_t(a, b) = t_a b$ is called a geoodular space if:

1. M is a left loop [1] with respect to the operation $x, y \in M \to x \cdot y \in M$ and a is its right neutral element [1].
2. $t_a x \cdot u_a x = (t + u)_a x, (x \in M), (t, u \in \mathbb{R})$,
3. $t_a(u_a x) = (tu)_a x, (X \in M), (t, u \in \mathbb{R})$,
4. $1_a x = x, (x \in M)$,
5. $L^{t_a b}_{u_a b} \circ L^a_{t^b_a} = L^a_{u_a b}, (a, b \in M), (t, u \in \mathbb{R})$ (the first geoodular identity),
6. $L^a_b \circ t_a = t_b \circ L^a_b, (a, b \in M), (t \in \mathbb{R})$ (the second geoodular identity).

Remark. The properties 1-4 mean that $\mathcal{M}^a = <M, \underset{a}{\cdot}, a, (t_a)_{t\in\mathbb{R}}>$ is a left $\mathbb{R}$-odule.

A geoodular space $\mathcal{M} = <M, L, (\omega_t)_{t\in\mathbb{R}}>$ is said to be of trivial curvature (or of zero curvature) if

$$L^b_c \circ L^a_b = L^a_c \quad (a, b, c \in M). \tag{1}$$

This condition is stronger than the first geoodular identity.

1. Definition. A geoodular space $\mathcal{M} = <M, L, (\omega_t)_{t\in\mathbb{R}}>$ of trivial curvature is said to be flat, if for any $a \in M, \mathcal{M}^a = <M, \cdot_a, a, (t_a)_{t\in\mathbb{R}}>$ is a vector space over $\mathbb{R}$ (with a zero element a).

Remark. In the flat case it is more suitable to use the notation $\underset{a}{+}$ instead of $\underset{a}{\cdot}$. Henceforth we follow this convention. Due to our conditions we have evidently

$$L^a_p \circ L^a_q = L^a_q \circ L^a_p = L^a_{p \underset{a}{+} q} = L^a_{q \underset{a}{+} p}. \tag{2}$$

From now and onward we consider flat geoodular spaces only.

1. Proposition.

$$L^a_b = L^c_d \Longleftrightarrow d = L^a_b c. \tag{3}$$

Proof.

$$L^a_b = L^c_d \Longleftrightarrow L^a_b \circ L^a_c = L^c_d \circ L^a_c \Longleftrightarrow$$

$$L^a_{b \underset{a}{+} c} = L^a_d \Longleftrightarrow b \underset{a}{+} c = d \Longleftrightarrow L^a_b c = d.$$

[1]Proofs not corrected by the author.

2. Proposition.

$$L_q^p = L_b^a \circ L_q^p \circ (L_b^a)^{-1} = L_{L_b^a q}^{L_b^a p}. \tag{4}$$

Proof. We shall show that the first part of the equality (4) follows from Proposition 1. Indeed, Proposition 1 shows $L_q^p = L_{L_q^p a}^a$, and in virtue of (2) we obtain

$$L_b^a \circ L_q^p \circ (L_b^a)^{-1} = L_b^a \circ L_{L_q^p a}^a \circ (L_b^a)^{-1} = L_{L_q^p a}^a \circ L_b^a \circ (L_b^a)^{-1} = L_{L_q^p a}^a = L_q^p.$$

As to the second part of the equality (4) we can use (2) again. Then

$$L_b^a \circ L_q^p \circ (L_b^a)^{-1} = L_b^a \circ L_q^a \circ (L_p^a)^{-1} \circ (L_b^a)^{-1} = L^a_{b \overset{+}{a} q} \circ (L^a_{b \overset{+}{a} p})^{-1} =$$

$$= L^{b \overset{+}{a} p}_{b \overset{+}{a} q} = L_{L_b^a q}^{L_b^a p}.$$

3. Proposition.

$$L_b^a \circ t_c = t_{L_b^a c} \circ L_b^a. \tag{5}$$

Proof. By means of Proposition 1 we have $L_b^a = L_{L_b^a c}^c$. Consequently, due to the second geoodular identity

$$L_b^a \circ t_c = L_{L_b^a c}^c \circ t_c = t_{L_b^a c} \circ L_b^a.$$

Remark. The properties

$$L_b^a \circ L_q^p \circ (L_b^a)^{-1} = L_{L_b^a q}^{L_b^a p}, \qquad L_b^a \circ t_c \circ (L_b^a)^{-1} = t_{L_b^a c} \tag{6}$$

are called identities of reductivity [1].

Let $V = \{L_b^a\}_{a,b \in M}$. Then we can introduce for any $f, g \in V$ the operation

$$f + g \overset{def}{=} f \circ g \tag{7}$$

It is easily verified that $f \circ g \in V$ again. Indeed, if $f = L_p^a, g = L_q^b$, then due to Proposition 1 g can be represented in the form $g = L_{L_q^b a}^a$. Consequently, $f \circ g = L_p^a \circ L_{L_q^b a}^a = L^a_{p \overset{+}{a} L_q^b} \in V$ and, moreover, $f \circ g = g \circ f$. Thus, the operation + is commutative and evidently associative. We have zero element $O_V = L_q^a$ and for any $f = L_b^a$ there exists an opposite element $(-f) = L_a^b$. Thus, we obtain the proposition:

4. Proposition. *The set $V = \{L_b^a\}_{a,b \in M}$ constitutes a commutative group with respect to the operation $f + g \overset{def}{=} f \circ g (f, g \in V)$ with zero element $O_V = L_q^a (\forall_a)$ and the opposite element $(-L_b^a) \overset{def}{=} L_a^b$.*

Now we introduce the multiplication by scalars

$$tL_b^a \overset{def}{=} L_{t_a b}^a, \quad (a, b \in M), (t \in \mathbb{R}). \tag{8}$$

One should verify that this definition is correct. This means that $L_b^a = L_d^c \Rightarrow L_{t_a b}^a = L_{t_c d}^c$ should be satisfied. Due to Proposition 1 we have $L_{t_a b}^a = L_{t_c d}^c \iff t_c d = L_{t_a b}^a c$. Or $t_c d = t_a b \overset{+}{a} c$

$= c \overset{+}{_a} t_a b = L^a_c t_a b = t_c L^a_c b = t_c(c \overset{+}{_a} b) = t_c(b \overset{+}{_a} c) = t_c L^a_b c$. But in virtue of Proposition 1, $L^a_b = L^d_c \iff d = L^a_b c$. Consequently, we have shown $L^a_b = L^c_d \implies L^a_{t_a b} = L^c_{t_c d}$ and our definition is correct.

It is easily verified that the group $< V, +, -(\ ), O_V >$, equipped with multiplication by scalars generates the vector space $\mathcal{V} = < V, +, O_V, -(\ ), (t)_{t \in \mathbb{R}} >$:

$$(t+u)L^a_b = L^a_{(t+u)_a b} = L^a_{(t_a b) \overset{+}{_a} (u_a b)} = L^a_{t_a b} \circ L^a_{u_a b} =$$

$$(tL^a_b) + (uL^a_b),$$

$$t(L^a_b + L^p_q) = t(L^a_b + L^a_{L^p_q a}) = t(L^a_{b \overset{+}{_a} L^p_q a}) = L^a_{t_a(b \overset{+}{_a} L^p_q a)} =$$

$$= L^a_{(t_a b)} + L^a_{(t_a L^p_q a)} = (tL^a_b) + (tL^a_{L^p_q a}) = tL^a_b + tL^p_q,$$

$$(tu)L^a_b = L^a_{(tu)_a b} = L^a_{t_a(u_a b)} = tL^a_{u_a b} = t(uL^a_b),$$

$$1 \cdot L^a_b = L^a_{(1)_a b} = L^a_b.$$

The vector group $< V, +, -(\ ), O_V >$ acts on M transitively, since, for any $x, y \in M$, $L^x_y x = y$. Let us show, that this action is simply transitive. Suppose that $L^p_q a = b$. Using the proposition 1 we can write $L^p_q = L^a_{L^p_q a}$ and $L^p_q a = L^a_{L^p_q a} a = b$, or $L^p_q a = b$. Finally we obtain $L^p_q = L^a_b$. Last one shows, that there exists one and only one transformation in V, namely $f = L^a_b$, such that $fa = b$. Thus, our action is simply transitive.

Taking into account (4) and (5), we have the following Proposition 5.

5. Proposition. *The vector group $< V, +, -(\ ), O_V >$ acts on M simply transitively and keeps the structure of a flat geoodular space invariant, that is,*

$$f \circ L^p_q \circ f^{-1} = L^{fp}_{fq}, f \circ t_c = t_{fc} \circ f (f \in V).$$

Moreover, any flat geoodular space can be considered as an affine space.

Remark. One can reconstruct the flat geoodular space knowing its vector space

$$\mathcal{V} = < V, +, -(), O_V, (t)_{t \in \mathbb{R}} >$$

Indeed, if $fx = y (f \in V)$, then $L^x_y = f$ and $t_x y = (tL^x_y)x$.

6. Proposition. *Given any simply transitive action of the vector group $< V, +, -(\ ), O_V >$ of some vector space $\mathcal{V} = < V, +, (), O_V, (t)_{t \in \mathbb{R}} >$ on a set M, one can construct in unique manner a flat geoodular space $\mathcal{M} = < M, L, (\omega_t)_{t \in \mathbb{R}} >$ such one that its vector space is the same as originally given.*

Proof. For this purpose we use the construction from the remark above. If $fx = y$ $(f \in V)$, then $f = L^x_y$ and $t_x y = (tL^x_y)x$. In such a way we get the structure

$$\mathcal{M} = < M, L, (\omega_t)_{t \in \mathbb{R}} >, \qquad L(x, a, y) = L^a_x y, \omega_t(a, b) = t_a b.$$

Let us check up that $\mathcal{M}$ is a flat geoodular space. The identity (1) is obvious since $L_c^b \circ L_b^a$ and L_c^a translate a into c, both, and coincide due to the simple transitivity. In the same way $L_p^a \circ L_q^a = L^a_{p \overset{+}{a} q}$ and $L_p^a \circ L_q^a = L_q^a \circ L_p^a$ implies $L^a_{p \overset{+}{a} q} a = L^a_{q \overset{+}{a} p} a$, or commutativity $p \overset{+}{a} q = q \overset{+}{a} p$. Analogously, $L_p^a \circ (L_q^a \circ L_r^a) = (L_p^a \circ L_q^a) \circ L_q^a$ implies $L^a_{p \overset{+}{a} (q \overset{+}{a} r)} a = L^a_{(p \overset{+}{a} q) \overset{+}{a} r} a$, that is, associativity $p \overset{+}{a} (q \overset{+}{a} r) = (p \overset{+}{a} q) \overset{+}{a} r$. Further, $(t+u)_x y = ((t+u)L_y^x)x = (tL_y^x + uL_y^x)x = (tL_y^x)\,[(uL_y^x)x] = (tL_y^x)u_x y - L^x_{t_x y} u_x y = t_x y \overset{+}{x} u_x y$ (we represented $tL_y^x = L_y^x$, then $t_x y = (tL_y^x)x = L_q^x x = q$, thus $tL_y^x = L^x_{t_x y}$). Further, $t_x(y \overset{+}{x} z) = (tL^x_{y \overset{+}{x} z})x = (t[L_y^x + L_z^x])x = (tL_y^x + tL_z^x)x = (L^x_{t_x y} + L^x_{t_x z})x = (L^x_{t_x y \overset{+}{x} t_x z})x = t_x y \overset{+}{x} t_x z$. Further $(tu)_x y = ((tu)L_y^x)x = (t[uL_y^x])x = (tL^x_{u_x y})x = (L^x_{t_x u_z y})x = t_z u_x y$. And finally, $(1)_x y = (1 \cdot L_y^x)x = L_y^x z = y$.

Now we shall show that the second geoodular identity is satisfied. Analogously to the case of the proposition 1 we can prove that $L_b^a = L_d^c \iff d = L_b^a c (\forall a, b, c \in M)$. Further

$$L_b^a = L^c_{L_b^a c} \Longrightarrow tL_b^a = tL^c_{L_b^a c} \Longrightarrow L^a_{t_a b} = L^a_{t_c L_b^a c} \Longrightarrow$$

$$L^a_{t_a b} c = t_c L_b^a c \Longrightarrow L_c^a t_a b = t_c L_c^a b \Longrightarrow L_c^a \circ t_a = t_c \circ L_c^a$$

(that is the second geoodular identity).

Thus any affine space can be considered as flat geoodular space.

Remark. We note that in presentation above given one can take an arbitrary skew field instead of $\mathbb{R}$. All results will be correct in that case.

For the first time the idea to treat affine spaces as universal algebras was announced as hypothesis by Malcev [2]. But at that time the concept of a geoodular space did not exist.

REFERENCES

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