# SOME COMMUTATIVITY THEOREMS THROUGH A STREB'S CLASSIFICATION

H.A.S. ABUJABAL, M. ASHRAF

**Abstract.** *In the present paper we investigate commutativity of rings with unity satisfying any one of the properties* $\{1-(x^m y)g(x^m y)\}\,[x^m y - x^r f(x^m y)x^s, x]\,\{1-(x^m y)h(x^m y)\} = 0$, $\{1-(x^m y)g(x^m y)\}\,[yx^m - x^r f(x^m y)\,x^s, x]\,\{1-(x^m y)h(x^m y)\} = 0$, $x^t[x^k, y] = g(y)[x, f(y)]\,h(y)$ *and* $[x^k, y]x^t = g(y)\,[x, f(y)]h(y)$, *for some* $f(X)$ *in* $X^2Z[X]$ *and* $g(X), h(X)$ *in* $Z[X]$, *where* $m \geq 0$, $r \geq 0$, $s \geq 0$, $k>0$, $t>0$ *are non-negative integers. Finally, under different appropriate constraints on commutators, commutativity of R has been established.*

## 1. INTRODUCTION

Throughout the present paper $R$ will represent an associative ring (may be without unity 1), $Z(R)$ the center of $R$, $N(R)$ the set of nilpotent elements of $R$ and $C(R)$ the commutator ideal of $R$. The symbol $[x, y]$ will denote the commutator $xy - yx$. As usual $Z[X]$ is the totality of polynomials in $X$ with coefficients in $Z$, the ring of integers. Consider the following ring properties:

$(P_1)$ For all $x, y$ in $R$ there exist polynomials $f(X)$ in $X^2Z[X]$ and $g(X), h(X)$ in $XZ[X]$ such that $\{1-g(x^m y)\}\,[x^m y - x^r f(x^m y)x^s, x]\,\{1-h(x^m y)\} = 0$, where $m \geq 0$, $r \geq 0$, $s \geq 0$ are fixed integers.

$(P_1)^*$ For all $x, y$ in $R$ there exist integers $m \geq 0$, $r \geq 0$, $s \geq 0$ and polynomials $f(X)$ in $X^2Z[X]$ and $g(X), h(X)$ in $XZ[X]$ such that $\{1-g(x^m y)\}\,[x^m y - x^r f(x^m y)x^s, x]\,\{1-h(x^m y)\} = 0$.

$(P_2)$ For all $x, y$ in $R$ there exist polynomials $f(X)$ in $X^2Z[X]$ and $g(X)$, $h(X)$ in $XZ[X]$ such that $\{1-g(x^m y)\}\,[yx^m - x^r f(x^m y)\,x^s, x]\,\{1-h(x^m y)\} = 0$, where $m \geq 0$, $r \geq 0$, $s \geq 0$ are fixed integers.

$(P_2)^*$ For all $x, y$ in $R$ there exist integers $m \geq 0$, $r \geq 0$, $s \geq 0$ and polynomials $f(X)$ in $X^2Z[X]$ and $g(X), h(X)$ in $XZ[X]$ such that $\{1-g(x^m y)\}\,[yx^m - x^r f(x^m y)\,x^s, x]\,\{1-h(x^m y)\} = 0$.

$(P_3)$ For all $y$ in $R$ there exist polynomials $f(X)$ in $X^2Z[X]$ and $g(X)$, $h(X)$ in $Z[X]$ such that $x^t[x^m, y] = g(y)\,[x, f(y)]\,h(y)$ and $x^t[x^n, y] = g(y)\,[x, f(y)]\,h(y)$, for all $x$ in $R$, where $t \geq 1$, $m \geq 1$, $n \geq 1$ are fixed integers with $(m, n) = 1$.

$(P_3)^*$ For all $x, y$ in $R$ there exist integers $t \geq 1$, $m \geq 1$, $n \geq 1$ with $(m, n) = 1$ and polynomials $f(X)$ in $X^2Z[X]$, $g(X)$, $h(X)$ in $Z(X)$ such that $x^t[x^m, y] = g(y)\,[x, f(y)]\,h(y)$ and $x^t[x^n, y] = g(y)\,[x, f(y)]h(y)$.

$(P_4)$ For all $y$ in $R$ there exist polynomials $f(X)$ in $X^2Z[X]$, $g(X), h(X)$ in $Z[X]$ such that $[x^m, y]x^t = g(y)\,[x, f(y)]\,h(y)$ and $[x^n, y]x^t = g(y)[x, f(y)]\,h(y)$, for all $x$ in $R$, where $t \geq 1$, $m \geq 1$, $n \geq 1$ are fixed integers with $(m, n) = 1$.

$(P_4)^*$ For all $x, y$ in $R$ there exist integers $t \geq 1$, $m \geq 1$, $n \geq 1$ with $(m, n) = 1$ and polynomials $f(X)$ in $X^2Z[X]$, $g(X)$, $h(X)$ in $Z[X]$ such that $[x^m, y]x^t = g(y)[x, f(y)]h(y)$ and $[x^n, y]x^t = g(y)[x, f(y)]h(y)$.

$(CH)$ For each $x, y$ in $R$ there exist $f(X), g(X)$ in $X^2Z[X]$ such that $[x - f(x), y - g(y)] = 0$.

Recently in an attempt to generalize famous Jacobson's "$x^{n(x)} = x$ theorem" Searcóid and MacHale [16] established commutativity of ring satisfying the condition $(xy)^{n(x,y)} = xy$ with

$n(x, y) > 1$. It is natural to consider the related ring properties like $xy = p(xy)$ or $xy = p(yx)$, where $p(X) \in X^2Z[X]$. Tominaga and Yaqub [18, Theorem 2] obtained commutativity results for such rings. Further, Bell et al. (cf. [4, Theorem 2], [5, Theorem 1]) studied the commutativity of the rings with unity 1 satisfying polynomial identities of the form $[xy - p(xy), x] = 0$ and $[xy - q(xy), x] = 0$, where $p(X), q(X) \in X^2Z[X]$. Our first purpose in the present paper is to establish commutativity of rings with unity 1 satisfying either of the properties $(p_1)$ or $(P_2)$. Next we shall consider the properties $(P_1)^*$ and $(P_2)^*$, where integral exponents are allowed to vary with the pair of ring's elements $x, y$ and ring also satisfies the Chacron's condition $(CH)$. In fact our theorems generalize many known results to mention a few [4, Theorem 2], [5, Theorem 1], [13 Theorem 1 $(i)$], [16, Theorem] and [18, Theorem] etc. Our second aim is to investigate commutativity of rings with unity 1 satisfying any one of the conditions $(P_3)$, $(P_4)$, $(P_3)^*$ and $(P_4)^*$. There are numerous results in the existing literature concerning commutativity of rings with unity 1 satisfying certain special cases of these conditions (cf. [1, Theorem], [3, Theorems 5 & 6], [7, Theorem B], [10, Theorem 1], [14, Theorem] and [15, Theorems 1&2]). In the present paper we shall confine mainly our attention to the case when polynomials in the underlying conditions are varying with the pair of ring's elements $x, y$, which offer simultaneous extensions of these results for rings with unity 1. Finally, some related cases of conditions $(P_3)$ and $(P_4)$ have been considered and commutativity of rings has been investigated under appropriate torsion restrictions on commutators. The method of the proofs presented in the last section is based on some iteration techniques developed by Tong [19].

## 2. COMMUTATIVITY OF RINGS WITH UNITY

**Theorem 2.1.** *Let $R$ be a ring with unity 1 satisfying any one of the properties $(P_1)$ and $(P_2)$. Then $R$ is commutative (and conversely).*

**Theorem 2.2.** *Let $R$ be a ring with unity 1 satisfying any one of the properties $(P_3)$ and $(P_4)$. Then $R$ is commutative (and conversely).*

In order to develop the proofs of the above theorem, we consider the following types of rings.

$(a)_l$ $\begin{pmatrix} GF(p) & GF(p) \\ 0 & 0 \end{pmatrix}$, $p$ a prime.

$(a)_r$ $\begin{pmatrix} 0 & GF(p) \\ 0 & GF(p) \end{pmatrix}$, $p$ a prime.

$(a)$ $\begin{pmatrix} GF(p) & GF(p) \\ 0 & GF(p) \end{pmatrix}$, $p$ a prime.

$(b)$ $M_\sigma(K) = \left\{ \begin{pmatrix} a & \beta \\ 0 & \sigma(a) \end{pmatrix} / a, \beta \in K \right\}$, where $K$ is a finite field with a non-trivial automorphism $\sigma$.

(c) A non-commutative division ring.

(d) $S = <1> + T$, $T$ is a non-commutative radical subring of $S$.

(e) $S = <1> + T$, $T$ is a non-commutative subring of $S$ such that $T[T, T] = [T, T]T = 0$.

Recently, Streb [17] classified non-commutative rings, which has been used effectively as a tool by several authors to prove a number of commutativity theorems (cf.[2], [11], [12] & [13]). It follows easily from the proof of [17, Corollary 1] that if $R$ is a non-commutative ring with unity 1, then there exists a factorsubring of $R$ which is of type $(a)$, $(b)$, $(c)$, $(d)$ or $(e)$.

This observation yields the following result which plays the key role in our subsequent study (cf. [13, Lemma 1]).

**Lemma 2.1.** *Let P be a ring property which is inherited by factor subrings. If no rings of type (a), (b), (c), (d) or (e) satisfy P, then every ring with unity 1 and satisfying P is commutative.*

For easy reference we present the following lemmas, which are essentially proved in [8], [9] and [11, Corollary 1].

**Lemma 2.2.** *Let R be a ring in which for every $x, y$ in R, there exists polynomial $f(X)$ in $X^2Z[X]$ such that $[x - f(x), y] = 0$, then R is commutative.*

**Lemma 2.3.** *Let f be a polynomial in non-commuting indeterminates $x_1, x_2, \dots, x_n$ with relatively prime integral coefficients. Then the following are equivalent.*

(*i*) For any ring $R$ satisfying the polynomial identity $f = 0$, $C(R)$ is a nil ideal.

(*ii*) For every prime $p$, $(GF(p))_2$ fails to satisfy $f = 0$.

(*iii*) Every semi prime ring satisfying $f = 0$ is commutative.

**Lemma 2.4.** *Suppose that a ring R with unity 1 satisfies (CH). If R is non-commutative, then there exists a factorsubring of R which is of type (a) or (b).*

Now we prove the following:

**Lemma 2.5.** *Let R be a division ring satisfying any one of the properties $(P_1)$ and $(P_2)$. Then R is commutative.*

**Proof.** Suppose that $R$ satisfies $(P_1)$. If $u$ is a unit in $R$, then for every $y$ in $R$ choose polynomial $f(X)$ in $X^2Z[X]g(X), h(X)$ in $XZ[X]$ such that $\{1 - g(u^m u^{-m} y)\}\ [u^m u^{-m} y - u^r f\ (u^m u y^{-m} y)\ u^s, u]$ $\{1 - h(u^m u^{-m} y)\} = 0 -$ *i.e.* $\{1 - g(y)\}\ [y - u^r f\ (y) u^s, u]\ \{1 - h(y)\} = 0$. This implies that either $1 - g(y) = 0$, $1 - h(y) = 0$ or $[y - u^r f(y) u^s, u] = 0$. In the first two cases we find that $y - yg(y) = 0$, $y - yh(y) = 0$, and $R$ is commutative by Lemma 2.2. Hence, we assume that for unit $u$ and arbitrary $y$,

$$[y - u^r f(y) u^s, u] = 0, \text{ where } f(X) \in X^2Z[X]. \tag{2.1}$$

Now choose polynominal $f(X)$ in $X^2Z[X]$ such that $[y - u^{-r} f(y) u^{-s}, u^{-1}] = 0$. This implies that $[y - u^{-r} f(y) u^{-s}, u] = 0 -$ *i. e.*

$$u^r [u, y] u^s = [u, f(y)]. \tag{2.2}$$

Again in view of (2.1), we can choose the polynomial $p(X)$ in $X^2Z[X]$ such that $[f(y) - u^r p(f(y)) u^s, u] = 0$, Thus for $q(X) = p(f(X)) \in X^2Z[X]$, we find that

$$[u, f(y)] = u^r [u, q(y)] u^s. \tag{2.3}$$

Compare (2.2) and (2.3), to get $u^r [u, y] u^s = u^r\ [u, q(y)] u^s$. But since $u$ is a unit in $R$, hence $[u, y - q(y)] = 0$ for $q(X) \in X^2Z[X]$ Hence, again by Lemma 2.2, $R$ is commutative.

Further, if $R$ satisfies $(P_2)$, then let $u$ be a unit in $R$ and for arbitrary element $y$ in $R$, we find polynomials $f(X)$ in $X^2Z[X]$, $g(X), h(X)$ in $XZ[X]$ such that $\{1 - g(u^m u^{-m} y)\}\ [u^{-m} y u^m - u^r f\ (u^m u^{-m} y)\ u^s, u]\ \{1 - h(u^m u^{-m} y)\} = 0 -$ *i. e.* $\{1 - g(y)\}\ [u^{-m} y u^m - u^r f(y)\ u^s, u]\ \{1 - h(y)\} = 0$.

This implies that either $y - yg(y) = 0$, $y - yh(y) = 0$ or $[u^{-m}yu^m - u^r f(y)u^s, u] = 0$. In the first two cases $R$ is commutative by Herstein's theorem. Henceforth, we shall assume the remaining possibility that $[u^{-m}yu^m - u^r f(y)u^s, u] = 0$ – *i.e.*

$$[u, y]u^m = u^{m+r}[u, f(y)]u^s, \text{ where } f(X) \in X^2Z[X]. \tag{2.4}$$

Now, we choose polynomial $f(X)$ in $X^2Z[X]$ such that $[u^{-1}, y]u^{-m} = \bar{u}^{(m+r)}[u^{-1}, f(y)]\, u^{-s}$, which yields that

$$u^{m+r}[u, y]u^s = [u, f(y)]u^m. \tag{2.5}$$

In view of (2.4), we can find a polynomial $p(X)$ in $X^2Z[X]$ such that $[u, f(y)]u^m = u^{m+r}[u, p(f(y))]\, u^s$. Thus, for $q(X) = p(f(X))$ in $x^2Z[X]$, (2.5) yields that $u^{m+r}\, [u, y]u^s = u^{m+r}[u, q(y)]u^s$. But since $u$ is a unit, hence we find that $[u, y - q(y)] = 0$. Again using Lemma 2.2, we get the required result.

**Lemma 2.6.** *Let $t \geq 1, k \geq 1$ be fixed integers and $R$ a ring with unity 1 in which for every $y$ in $R$ there exist polynomials $f(X)$ in $X^2Z[X]$ and $g(X), h(X)$ in $Z[X]$ such that either $x^t[x^k, y] = g(y)\, [x, f(y)]\, h(y)$ or $[x^k, y]\, x^t = g(y)\, [x, f(y)$, for all $x$ in $R$. Then $C(R) \subseteq N(R)$.*

**Proof.** Suppose that $R$ satisfies $x^t[x^k, y] = g(y)\, [x, f(y)]\, h(y)$. Replace $x$ by $1 + x$ in the given property, to get $(1+x)^t\, [(1+x)^k, y] = x^t[x^k, y]$. This is a polynomial identity and $x = e_{12} - e_{22}$, $y = e_{12}$ fail to satisfy this equality in $(GF(p))_2, p$ a prime. Hence by Lemma 2.3, $R$ has nil commutator ideal.

A similar arguments can be used to obtain the result if $R$ satisfies the property $[x^k, y]x^t = g(y)\, [x, f(y)]h(y)$.

**Proof of Theorem 2.1.** Let $R$ be a ring of the type $(a)$. If $R$ satisfies $(P_1)$, then in $(GF(p))_2, p$ a prime, we see that $\{1 - g(e_{11}e_{12})\}\, [e_{11}e_{12} - e_{11}f\,(e_{11}e_{12})\, e_{11}, e_{11}]\, \{1 - h(e_{11}e_{12})\} = -e_{12} \neq 0$ for every $f(X) \in X^2Z[X]$, $g(X), h(X) \in XZ[X]$.

If $R$ satisfies $(P_2)$, then $\{1 - g(e_{22}e_{12})\}\, [e_{12}e_{22} - e_{22}f(e_{22}e_{12})\, e_{22}, e_{22}]\, \{1 - h(e_{22}e_{12})\} = e_{12} \neq 0$ for every $f(X)$ in $X^2Z[X]$ and $g(X), h(X)$ in $XZ[X]$. Thus, in both the cases we find a contradiction and hence, no rings of type $(a)$ satisfy $(P_1)$ and $(P_2)$.

Next, consider the ring $M_\sigma(K)$, a ring of type $(b)$. If $R$ satisfies $(P_1)$, then choose $x = \begin{pmatrix} a & 0 \\ 0 & \sigma(a) \end{pmatrix}$ $(a \neq \sigma(a)), y = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ such that $\{1 - g(x^m y)\}\, [x^m y - x^r f(x^m y)\, x^s, x]\, \{1 - h\,(x^m y)\} = a^m(\sigma(a) - a)\, e_{12} \neq 0$ for all $f(X)$ in $X^2Z[X]$ and $g(X)$, $h(X)$ in $XZ[X]$. Also, if $R$ satisfies $(P_2)$, then with the same choice of $x$ and $y$, we find that $\{1 - g(x^m y)\, [yx^m - x^r f\,(x^m y)\, x^s, x]\, \{1 - h\,(x^m y)\} = (\sigma(a) - a)\, (\sigma(a))^m\, e_{12} \neq 0$. Hence, in both the cases, $R$ can not be of type $(b)$. Further, if $R$ is of type $(c)$, then in view of Lemma 2.5, we find a contradiction.

Now suppose that $R$ is of type $(d)$. If $R$ satisfies either of the properties $(P_1)$ or $(P_2)$, then a careful scrutiny of the proof of Lemma 2.5 yields that there exist unit $u$ and arbitrary $y$ in $R$ such that either $y - yg(y) = 0, y - yh(y) = 0$ or $[u, y - q(y)] = 0$ for all $q(X)$ in $X^2Z[X]$ and $g(X)$, $h(X)$ in $XZ[X]$. Now in the present case if $t_1, t_2 \in T$, then $u = 1 + t_1$ is a unit and there exist $q(X) \in X^2Z[X]$ and $g(X)$, $h(X)$ in $XZ[X]$ such that either $t_2 - t_2g(t_2) = 0, t_2 - t_2h(t_2) = 0$ or $[t_2 - q(t_2),\, 1 + t_1] = 0$. Hence, in every case $T$ is commutative by Lemma 2.2, a contradiction.

Finally, assume that $R$ is of type $(e)$. Let $t_1, t_2 \in T$ such that $[t_1, t_2] \neq 0$. If $R$ satisfies $(P_1)$, then there exist polynomials $f(X)$ in $X^2Z[X]$ and $g(X)$, $h(X)$ in $XZ[X]$ such that

$$\begin{aligned} 0 &= \{1 - g((1+t_1)^m t_2)\}[(1+t_1)^m t_2 - \\ &\quad (1+t_1)^r f((1+t_1)^m t_2)(1+t_1)^s, (1+t_1)]\{1 - h((1+t_1)^m t_2)\} \\ &= \{1 - g((1+t_1)^m t_2)\}[(1+t_1)^m t_2, 1 + t_1]\{1 - h((1+t_1)^m t_2)\} \\ &= \{1 - g((1+t_1)^m t_2)\}(1+t_1)^m [t_2, t_1]\{1 - h((1+t_1)^m t_2)\} \\ &= [t_2, t_1], \end{aligned}$$

a contradiction. Similarly, we get a contradiction if $R$ satisfies $(P_2)$.

Thus we have seen that no rings of type $(a), (b), (c), (d)$ or $(e)$ satisfy $(P_1)$ and $(P_2)$ and by Lemma 2.1, $R$ is commutative.

**Proof of Theorem 2.2.** In view of Lemma 2.3 and 2.6, $R$ can not be of type $(c)$ or $(d)$. Now, if $R$ is assumed to be of type $(a)$, then in $(GF(p))_2$, $p$ a prime, we find that $e_{11}[e_{11}, e_{12}]$ $-g(e_{12})[e_{11}, f(e_{12})]\, h(e_{12}) = e_{12} \neq 0$, for all $f(X)$ in $X^2Z[X]$ and $g(X)$, $h(X)$ in $Z[X]$. Thus in both the cases we find a contradiction.

Next, consider the ring $M_\sigma(k)$, a ring of type $(b)$. Let $R$ satisfy $(P_3)$. Then note that $N(M_\sigma(K)) = Ke_{12}$. Thus for any $a$ in $N(M_\sigma(K))$ and arbitrary unit $u$ there exist polynomials $f(X)$ in $X^2Z[X]$ and $g(X)$, $h(X)$ in $Z[X]$ such that $u^t[u^m, a] - g(a)\,[u, f(a)]h(a) = 0$. But $a^2 = 0$ and $u$ be a unit, hence $[u^m, a] = 0$. Similary, it can also be shown that $[u^n, a] = 0$. But, since $(m, n) = 1$, we find that $[u, a] = 0$. Hence for non-central element $a = e_{12}$ and arbitrary unit $u$ we get $[u, e_{12}] = 0$, which forces a contradiction that $e_{12}$ is central. Using a similar arguments we get a contradiction, if $R$ satisfies $(P_4)$.

Finally suppose that $R$ is a ring of type $(e)$. Let $R$ satisfy $(P_3)$. Assume that $t_1, t_2 \in T$ such that $[t_1, t_2] \neq 0$. Then there exist polynomials $f(X)$ in $X^2Z[X]$ and $g(X)$, $h(X)$ in $Z[X]$ such that $m[t_1, t_2] = (1+t_1)^t\, [(1+t_1)^m, t_2] = g(t_2)\, [1 + t_1, f(t_2)]\, h(t_2) = 0$. Similarly, it can be shown that $n[t_1, t_2] = 0$. This implies that $[t_1, t_2] = 0$, a contradiction. Similarly, we can find a contradiction if $R$ satisfies $(P_4)$.

Thus no rings of type $(a), (b), (c), (d)$ or $(e)$ satisfy $(P_3)$ and $(P_4)$ and hence by Lemma 2.1, $R$ is a commutative.

**Remark 2.1.** Let $R = \left\{ \begin{pmatrix} a & b & c \\ 0 & a & d \\ 0 & 0 & a \end{pmatrix} / a, b, c, d \in GF(2) \right\}$. Then $R$ is a non-commutative ring with unity, and it can be easily verified that for $n = 4$, $R$ satisfies the conditions $x^t[x^n, y]$ $= y^r[x, y^n]y^s$ and $[x^n, y]x^t = y^r[x, y^n]y^s$ for any integers $t \geq 0, r \geq 0, s \geq 0$. This shows that in the hypotheses of Theorem 2.2, the existence of both the conditions in the properties $(P_3)$ and $(p_4)$ is not superfluous.

A careful scrutiny of the proofs of Theorems 2.1 and 2.2 reveals that if $R$ satisfies any one of the properties $(P_1)^*$, $(P_2)^*$, $(P_3)^*$, and $(P_4)^*$ then in every case $R$ has no factorsubrings of type $(a)$ or $(b)$. Hence, combining this fact with the Lemma 2.4, we have the following:

**Theorem 2.3.** *Let $R$ be a ring with unity 1 satisfying (CH). Suppose, further that $R$ satisfies any one of the properties $(P_1)^*$ and $(P_2)^*$. Then $R$ is commutative (and conversely).*

**Theorem 2.4.** *Let R be a ring with unity 1 satisfying (CH). Suppose further that R satisfies any one of the properties $(P_3)^*$ and $(P_4)^*$. Then R is commutative (and conversely).*

## 3. COMMUTATIVITY OF TORSION-FREE RINGS

In view of Remark 2.1, it is natural to ask that what additional hypothesis is required to prove the commutativity of ring $R$, if we merely assume $x^t[x^m, y] = g(y)\,[x,f(y)]\,[x,f(y)]\,h(y)$ and $[x^m, y]x^t = g(y)[x,f(y)]h(y)$ in the properties $(P_3)$ and $(P_4)$ respectively. To this end, it is tempting to conjecture that an $m$-torsion free ring with unity 1 satisfying any one of the above properties must be commutative. However, under certain appropriate constraints on the commutators involved in the underlying conditions, we can establish some interesting cases of the conjecture. In fact, we shall consider the following related ring properties:

$(P_5)$ For all $y$ in $R$ there exist polynomials $f(X)$, $g(X)$, $h(X)$ in $Z[X]$ such that $[x^m, y^n] = g(y)[x,f(y)]h(y)$ for all $x$ in $R$, where $m > 1$, $n \geq 1$ are fixed integers.

$(P_6)$ For all $y$ in $R$ there exist polynomials $f(X)$, $g(X)$, $h(X)$ in $Z[X]$ such that either $x^m[x, y^n] = g(y)\,[x,f(y)]\,h(y)$ or $[x, y^n]\,x^m = g(y)\,[x,f(y)]\,h(y)$ for all $x$ in $R$, where $m \geq 1$, $n \geq 1$ are fixed integers.

$(P_7)$ For all $y$ in $R$ there exist polynomials $f(X)$, $g(X)$, $h(X)$ in $Z[X]$ such that either $y^n[x^m, y] = g(y)\,[x,f(y)]\,h(y)$ or $[x^m, y]\,y^n = g(y)\,[x,f(y)]\,h(y)$ for all $x$ in $R$, where $m > 1, n \geq 0$ are fixed integers.

To establish the commutativity of ring $R$ with the above properties we shall assume some extra conditions on commutators in $R$, like the following property.

$Q(d)$ For all $x, y$ in $R$, $d[x, y] = 0$ implies $[x, y] = 0$, where $d$ is a positive integer.

Our method of the proof uses some iteration techniques, which is based on the following lemma due to Tong [19, Lemma 1].

**Lemma 3.1.** *Let R be a ring with unity 1, and $I_0^r(x) = x^r$. if $k \geq 1$, let $I_k^r(x) = I_{k-1}^r(x+1) - I_{k-1}^r(x)$ for all x in R. Then $I_{r-1}^r(x) = 1/2(r-1)\,r! + r!x$; $I_r^r(x) = r!$ and $I_j^r(x) = 0$ for $j > r$.*

**Theorem 3.1.** *Let R be a ring with unity 1 satisfying any one of the properties $(P_5)$ — — — — $(P_7)$. Further, if R satisfies $Q((max\{m, n\})!)$, then R is commutative.*

**Proof.** Suppose that $R$ satisfies $(P_5)$. First we shall apply iteration to $x^m$. Following notations of Lemma 3.1, we set $I_p(x) = I_p^m(x)$, for $p = 0, 1, 2, 3, \ldots\ldots\ldots$ Then property $(P_5)$ can be written as

$$[I_0(x), y^n] = g(y)[x,f(y)]h(y). \tag{3. 1}$$

Replace $x$ by $x + 1$ in the above expression and use Lemma 3.1, to get $[I_0(x) + I_1(x), y^n] = g(y)[x,f(y)]\,h(y)$, and hence in view of (3.1), whe have

$$[I_1(x), y^n] = 0 \text{ for all } x, y \text{in } R. \tag{3. 2}$$

Again replacing $x$ by $x+1$ and using Lemma 3.1, we obtain $[I_1(x+1), y^n] = [I_1(x)+I_2(x), y^n] = 0$. This in view of (3.2) implies that $[I_2(x), y^n] = 0$. Thus it is now clear that replacing $x$ by $x + 1$ and iterating $(m - 1)$-times, we have $[I_{m-1}(x), y^n] = 0-$ *i. e.* $m![x, y^n] = 0$. Finally, replacing $y$ by $y + 1$ and using the same techniques as above, we get $m!n!\,[x, y] = 0$, and thus by the property $Q((max\{m, n\})!)$, we establish the commutativity of $R$.

If $R$ satisfies the property $(P_6)$, then using the above notations we find that either $I_0(x)[x, y^n] = g(y)\,[x, f(y)]\,h(y)$ or $[x, y^n]\,I_0(x) = g(x)\,[x, f(y)]\,h(y)$. Replacing $x$ by $x + 1$ and using Lemma 3.1 we find that either $I_1(x)\,[x, y^n] = 0$ or $[x, y^n]\,I_1(x) = 0$, proceeding in the same way, we finally get either $I_m(x)\,[x, y^n] = 0$ or $[x, y^n]\,I_m(x) = 0$. Hence, in both the cases we have $m![x, y^n] = 0$. Now, using a similar technique of replacing $y$ by $y + 1$ and iterating $(n-1)$ -times we find that $m!n![x, y] = 0$, and the property $Q((max\{m, n\})!)$ yields the commutativity of R.

Similarly in the remaining case of $(P_7)$, we can easily get the required result.

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H.A.S. Abujabal
Department of Mathematics
Faculty of Science,
King Abdul Aziz University
P.O. Box. 31464
Jeddah 21497, Saudi Arabia.
M. Ashraf
Department of Mathematics
Aligarh muslim University
Aligarh 202002 (India)