ON SOME H-STRUCTURE MANIFOLDS WITH CONSTANT HOLOMORPHIC SECTIONAL CURVATURE

CR. CHRISTOPHORIDOU and PH.J. XENOS

1. INTRODUCTION

Structures on differentiable manifolds by introducing vector-valued linear functions satisfying some algebraic equations have been studied by a number of mathematicians. K.L. Duggal in [1] defined on a differentiable manifold the GF-structure which is more general than almost complex, almost product and almost tangent structures.

Let M be a n-dimensional differentiable manifold of class C^∞. A vector-valued linear function F of class C^∞ is defined on M such that

$$F^2(X) = \alpha^2 X \tag{1.1}$$

where X is an arbitrary vector field and α is any real or purely imaginary number. Then, F is said to give a differentiable structure called GF-structure on M defined by (1.1). If $\alpha \neq 0$ we have the known π-structure [3], if $\alpha = 0$ we have an almost tangent structure. For $\alpha = \pm 1$ or $\alpha = \pm\sqrt{-1}$ we obtain an almost product structure or an almost complex structure respectively.

Suppose further that M admits a Hermitian metric g satisfying

$$g(\bar{X}, \bar{Y}) + \alpha^2 g(X, Y) = 0 \tag{1.2}$$

where $\bar{X} = FX$ and X, Y are vector fields on M. Then, we say that (g, F) gives to M an H-structure and M is called H-structure manifold.

If the structure tensor F is parallel (i.e. $\nabla_X F)Y = 0$ where ∇ is the Riemannian connection), then M is called K-manifold.

An H-structure manifold M will be called nearly K-manifold (briefly NK-manifold) if the structure tensor F satisfies the condition $(\nabla_X F)X = 0$, for arbitrary vector field X on M.

In the present article we deal with some $2m$-dimensional H-structure manifolds. In the second paragraph we shall study an H-structure manifold admiting pointwise constant holomorphic sectional curvature. In the third paragraph we obtain the main result of the present paper on NK-manifolds.

2. ON H-STRUCTURE MANIFOLDS

On a $2m$-dimensional H-structure manifold M we consider a $(0,2)$ tensor such that:

$$\Phi(X, Y) = g(\bar{X}, Y) = -g(X, \bar{Y}) \tag{2.1}$$

It is easy to prove the following results:

$$\Phi(X, Y) + \Phi(Y, X) = 0 \tag{2.2}$$

$$\Phi(\bar{X},\bar{Y})+\alpha^2\Phi(X,Y)=0 \tag{2.3}$$

$$(\nabla_X\Phi)(Y,Z)+(\nabla_X\Phi)(Z,Y)=0 \tag{2.4}$$

$$(\nabla_X\Phi)(\bar{Y},\bar{Z})=\alpha^2(\nabla_X\Phi)(Y,Z) \tag{2.5}$$

We denoty by $(W,X,Y,Z)=g((\nabla_W F)X,(\nabla_Y F)Z)$ and because of (2.2), (2.3) we obtain:

$$(W,X,Y,Z)=(Y,Z,W,X),(W,\bar{X},Y,\bar{Z})=-\alpha^2(W,X,Y,Z),(W,\bar{X},Y,Z)=-(W,X,Y,\bar{Z}). \tag{2.6}$$

We assume that the curvature tensor R is defined by

$$R(X,Y)Z=\nabla_{[X,Y]}Z-[\nabla_X,\nabla_Y]Z,$$

and

$$R(W,X,Y,Z)=g(R(W,X)Y,Z)$$

for arbitrary vector fields W,X,Y and Z on M.

The holomorphic sectionarl curvature $H(x)$ is defined by

$$H(x)=R(x,\bar{x},x,\bar{x})\,/\,g(x,x)g(\bar{x},\bar{x}) \tag{2.7}$$

for $x\epsilon T_p(M),(p\epsilon M)$ where $T_p(M)$ is the tangent space of M, at p.

Theorem 2.1. *Let M be an H-structure manifold of pointwise constant holomorphic sectional curvature $c(p)$. Then*

$$\begin{aligned}&4\alpha^2c(p)[2\Phi(x,y)\Phi(z,w)-\Phi(x,w)\Phi(y,z)+\Phi(x,z)\Phi(y,w)+\\&+\alpha^2g(x,w)g(y,z)-\alpha^2g(x,z)g(y,w)]=\\&=-3\alpha^4R(w,x,y,z)-3R(\bar{w},\bar{x},\bar{y},\bar{z})+\alpha^2R(\bar{w},\bar{x},y,z)+\alpha^2R(w,x,\bar{y},\bar{z})-\\&-\alpha^2R(\bar{w},x,\bar{y},z)+3\alpha^2R(\bar{w},x,y,\bar{z})+3\alpha^2R(w,\bar{x},\bar{y},z)-\alpha^2R(w,\bar{x},y,\bar{z})+\\&3\alpha^4R(w,y,x,z)+3R(\bar{w},\bar{y},\bar{x},\bar{z})-\alpha^2R((\bar{w},\bar{y},x,z)-\alpha^2R(w,y,\bar{x},\bar{z})+\\&+\alpha^2R(\bar{w},y,\bar{x},z)-3\alpha^2R(\bar{w},y,x,\bar{z})-3\alpha^2R(w,\bar{y},\bar{x},z)+\alpha^2R(w,\bar{y},x,\bar{z}).\end{aligned} \tag{2.8}$$

Proof. Since $H(x)=c(p)$, (2.7) takes the form

$$R(x,\bar{x},x,\bar{x})=c(p)g(x,x)g(\bar{x},\bar{x}). \tag{2.9}$$

By linearizing (2.9) and using Bianchi identity we get

$$\begin{aligned}&4\alpha^2c[g(x,y)g(z,w)+g(x,z)g(y,w)+g(x,w)g(y,z)]=\\&=R(\bar{w},\bar{x},y,z)-2R(\bar{w},x,\bar{y},z)+R(\bar{w},x,y,\bar{z})+R(w,\bar{x},\bar{y},z)-\\&-2R(w,\bar{x},y,\bar{z})+R(w,x,\bar{y},\bar{z})+R(\bar{w},\bar{y},x,z)-2R(\bar{w},y,\bar{x},z)+\end{aligned}$$

$$+R(\bar{w},y,x,\bar{z})+R(w,\bar{y},\bar{x},z)-2R(w,\bar{y},x,\bar{z})+R(w,y,\bar{x},\bar{z}). \qquad (2.10)$$

In (2.10) we replace Y and W by $\bar{Y}$ and $\bar{W}$ and in the resulting equation we replace X and Y by Y and X respectively. Adding the last two equations we obtain (2.8).

We can choose an orthonormal frame field $\{E_1,\dots,E_m,E_{m+1},\dots,E_{2m}$ such that $E_{m+i}=\sqrt{-1}\bar{E}_i\,/\,\alpha, i=1,\dots,m\}$.

We denote by r and $r*$ the Ricci tensor and the Ricci *tensor of M, respectively. The Ricci *tensor r* is defined by

$$r*(x,y)=traceof(z\to R(\bar{z},x)\bar{y}),$$

for $x,y,z\in T_p(M)$.

Lemma 2.2. *If M is an H-structure manifold and $\{E_i\}$ is an orthonormal frame field, for arbitrary vector fields X,Y on M we have:*

$$\sum_{i=1}^{2m}R(X,\bar{E}_i,Y,\bar{E}_i)=-\alpha^2\sum_{i=1}^{2m}R(X,E_i,Y,E_i),$$

$$\sum_{i=1}^{2m}R(X,E_i,Y,\bar{E}_i)=-\sum_{i=1}^{2m}R(X,\bar{E}_i,Y,E_i).$$

Proof. The proof depends on the above way of the determination of the orthonormal frame field $\{E_i\}$.

We can easily prove the following.

Lemma 2.3. *Let M be an H-structure manifold. Then, for arbitrary vector fields X,Y, on M we have:*

$$r(X,Y)=r(Y,X), r*(\bar{X},\bar{Y})=-\alpha^2r*(Y,X), r*(\bar{X},Y)=-r*(\bar{Y},X).$$

*We denote by s and $s*$ the scalar and the *scalar curvature of M respectively. Then, using the theorem 2.1 and the lemmas 2.2 and 2.3 we obtain.*

Proposition 2.4. *Let M be a 2m-dimensional H-structure manifold of pointwise constant holomorphic sectional curvature $c(p)$. Then, for arbitrary vector fields X,Y on M, we have.*

$$\alpha^2r(X,Y)-r(\bar{X},\bar{Y})-3[r*(X,Y)+r*(Y,X)]=4(m+1)c(p)\alpha^2g(X,Y),$$

$$\alpha^2s-3s*=4m(m+1)\alpha^2c(p).$$

The main results of the second paragraph (thm 2.1 and propos. 2.4) for $\alpha^2=-1$ have been obtained by G.B. Rizza in [4] (fundamental identity (11) and thm 1).

3. ON NEARLY K-MANIFOLDS

We denote by $(W,X,Y,Z)=g((\nabla_WF)X,(\nabla_YF)Z)$. By definition of the NK-manifold and the curvature tensor R we obtain that: $R(W,X,Y,Z)-R(W,X,\bar{Y},\bar{Z})$ depends on the quantities: $(W,X,Y,Z), (W,Y,X,Z), (W,Z,X,Y), (W,X,Y,\bar{Z}), (W,Y,X,\bar{Z})$ and $(W,Z,X,\bar{Y})$. Applying the fundamental properties of $R(W,X,Y,Z)$ we obtain.

Proposition 3.1. *Let M be a NK-manifold. If W, X, Y and Z are arbitrary vector fields on M, then*

$$R(W,X,Y,Z) = \frac{1}{\alpha^2 - 3}[2(W,X,Y,Z) + (W,Y,X,Z) - (W,Z,X,Y)],$$

$$R(W,X,\bar{Y},\bar{Z}) = \frac{1}{\alpha^2 - 3}\alpha^2[2(W,X,Y,Z) - (W,Y,X,Z) + (W,Z,X,Y)].$$

Using the proposition 3.1 and the definitions of the Ricci tensor and Ricci *tensor we get the following.

Lemma 3.2. *For arbitrary vector fields X and Y on a NK-manifold it holds:*

$$r(X,Y) = \frac{1}{\alpha^2 - 3}\sum_{i=1}^{2m}(X,E_i,Y,E_i),$$

$$r(\bar{X},\bar{Y}) = -\alpha^2 r(X,Y), \quad r*(X,Y) = r*(Y,X),$$

$$r*(X,Y) = \frac{1}{\alpha^2 - 3}\alpha^2 r(X,Y).$$

By virtue of the first relation of proposition 2.4, the lemma 3.2 and [2] (p.292) we can obtain the main result:

Theorem 3.3. *If M is a 2m-dimensional connected NK-manifold of pointwise constant holomorphic sectional curvature, then M is an Einstein manifold.*

For NK-manifolds of small dimension we can state the following.

Proposition 3.4. *A NK-manifold M of dimension $n = 2, 4$ is a K-manifold.*

Proof. It is clear that a 2-dimensional NK-manifold is a K-manifold.

If M is a 4-dimensional NK-manifold, we choose an orthonormal frame field on an open subset of M to be of the from

$$\left\{E_1, E_2, \frac{\sqrt{-1}}{\alpha}\bar{E}_1, \frac{\sqrt{-1}}{\alpha}\bar{E}_2\right\}.$$

We can easily prove that $(\nabla_{E_1}F)E_2$ is perpendicular to E_1 and E_2. Because of:

$$(\nabla_X\Phi)(\bar{Y},\bar{Z}) = \alpha^2(\nabla_X\Phi)(Y,Z) = -\alpha^2(\nabla_X\Phi)(Z,Y).$$

it is proved that $(\nabla_{E_i}F)E_2$ is perpendicular to $\frac{\sqrt{-1}}{\alpha}\bar{E}_1$ and $\frac{\sqrt{-1}}{\alpha}\bar{E}_2$.

Hence:

$$(\nabla_{E_i}F)E_j = 0 \quad , \quad (i,j = 1,2).$$

REFERENCES

[1] K.L. DUGGAL, *On differentiable structures defined by algebraic equations*, I. Nijenhuis tensor. Tensor N.S.22 (1971), 238-242.

[2] S. KOBAYASHI and K. NOMIZU, *Foundations of Differential Geometry, I*, J. Wiley, New York, 1969.

[3] G. LEGRAND, *Sur les variétés à structure de presque prouit complexe*, C.R. Acad. Sc.Paris 242 (1956), 335-337.

[4] G.B. RIZZA, *On almost Hermitian manifolds with constant holomorphic curvature at a point*, Tensor N. S. 50 (1991), 79-89.

Received January 18, 1993 and in revised from July 25, 1993
Mathematics Division
School of Technology
Aristotle University of Thessaloniki
Thessaloniki 540 06 - GREECE