# ON A TYPE OF CONTACT MANIFOLD

U.C. DE and J.C. GHOSH
(Dedicated to the memory of Professor K. Yano)

## 1. THE OBJECT OF THE PAPER

Contact Riemannian manifolds satisfying $R(X, \xi).R = 0$ where $\xi$ belongs to the $K$-nullity distribution or a condition similar to it have been studied by various authors ([3], [4], [6]).

In the present paper we consider contact manifolds with characteristic vector field $\xi$ belonging to the $K$-nullity distribution satisfying the condition

$$R(\xi, X).P = 0$$

where $P$ is the Weyl projective curvature tensor and $R(\xi, X)$ is considered as a derivation of the tensor algebra at each point of the tangent space.

It is proved that either the contact manifold $M^{2m+1}$ is locally isometric to the product manifold $E^{m+1}XS^m$ (4) or $M^{2m+1}$ is an Einstein manifold. In the last section of this paper the contact metric manifolds satisfying div $P = 0$ where 'div' denotes divergence are studied.

## 2. PRELIMIANRIES

A contact manifold is a $C^\infty(2m+1)$ manifold $M^{2m+1}$ equipped with a global $l$-form $\eta$ such that $\eta \wedge (d\eta)^m \neq 0$ everywhere on $M^{2m+1}$. $\eta$ induces a unique vector field $\xi$ on $M^{2m+1}$ satisfying $\eta(\xi) = 1$ and $d\eta(\xi, X) = 0$ for every vector field $X$ on $M^{2m+1}$. A Riemannian metric $g$ is said to be associated with a contact manifold if there exists a tensor field $\phi$ of type (1,1) such that $d\eta(X, Y) = g(X, \phi Y), \eta(X) = g(X, \xi)$ and $\phi^2 = -I + \eta \otimes \xi$ and the manifold $M^{2m+1}$ with a contact metric structure $(\phi, \xi, \eta, g)$ is usually called a contact metric manifold [1]. Also a tensor field $h$ is defined by $h = 1/2L_\xi\phi$ where $L$ denotes Lie differentiation and $h$ satisfies $\phi h = -h\phi$. Thus, if $\lambda$ is an eigenvalue of $h$ with eigenvector $X$, $-\lambda$ is also an eigenvalue with eigenvector $\phi X$. Also we have $Trh = Tr\phi h = 0$ and $h\xi = 0$. Moreover if $\nabla$ denotes the Riemannian connection of $g$, the following relations hold

$$\nabla_X \xi = -\phi X - \phi h X \tag{2.1}$$

$$\nabla_\xi \phi = 0 \tag{2.2}$$

$$g(\phi X, \phi Y) = g(X, Y) - \eta(X)\eta(Y). \tag{2.3}$$

The vector field $\xi$ is a killing vector with respect to $g$ if and only if $h = 0$. A contact metric manifold $M^{2m+1}(\phi, \eta, \xi, g)$ for which $\xi$ is a killing vector is said to be a $K$-contact manifold.

If the almost complex structure $J$ on $M^{2m+1}XR$ defined by $J(X, f\frac{d}{dt}) = (\phi X - f\xi, \eta(X)\frac{d}{dt})$, where $f$ is a real-valued function, is integrable, then the structure is said to be normal and $M^{2m+1}(\phi, \eta, \xi, g)$ is said to be Sasakian. If $R$ denotes the curvature tensor of the manifold, a Sasakian manifold may be characterized by $R(X, Y)\xi = \eta(Y)X - \eta(X)Y$.

It is known that a Sasakian manifold is a $K$-contact manifold but the converse is not necessarily true unless dim $M^{2m+1} = 3$.

The $K$-nullity distribution [7] of a Riemannian manifold $(M, g)$ for a real number $K$ is a distribution

$$N(K) : p \to N_p(K) = \{Z : T_pM \,/\, R(X, Y)Z = K[g(Z, Y)X - g(X, Z)Y]\}$$

for any $X, Y \in T_pM$.

Next, suppose that $M^{2m+1}(\phi, \eta, \xi, g)$ is a contact metric manifold with $\xi$ belonging to the $K$-nullity distribution i.e.

$$R(X, Y)\xi = K[\eta(Y)X - \eta(X)Y]. \tag{2.4}$$

From (2.4) we have

$$S(X, \xi) = 2mK\eta(X) \tag{2.5}$$

where

$$S(X, Y) = \sum_{i=1}^{(2m+1)} g(R(e_i, X)Y, e_i), \tag{2.6}$$

is the Ricci tensor and $\{e_i\}$ is an orthonormal basis of the tangent space at each point of the manifold.

Also from (2.4), since

$$g(R(X, Y)\xi, Z) = g(R(\xi, Z)X, Y),$$

we have

$$R(\xi, Z)X = K[g(X, Z)\xi - \eta(X)Z]. \tag{2.7}$$

## 3. CONTACT MANIFOLD SATISFYING $R(\xi, X).P = 0$

The first author and N. Guha in their paper [5] considered a Sasakian manifold $M^{2m+1}$ satisfying $R(X, Y).P = 0$. In this paper the weaker hypothesis $R(\xi, Y).P = 0$ instead of $R(X, Y).P = 0$ is considered.

Let us suppose that

$$R(\xi, X).P = 0 \tag{3.1}$$

where

$$P(X, Y)Z = R(X, Y)Z - \frac{1}{2m}[S(Y, Z)X - S(X, Z)Y]. \tag{3.2}$$

From (3.2) it follows that

$$P(X, Y)Z = -P(Y, X)Z \tag{3.3}$$

$$g(P(X, Y)\xi, \xi) = 0, \text{ by } (2.5) \tag{3.4}$$

$$\sum_i g(P(e_i, V)W, e_i) = 0, \text{ where } \{e_i\} \text{ is defined in } (2.6) \tag{3.5}$$

$$g(P(\xi,Y)Z,\xi)=Kg(Y,Z)-\frac{1}{2m}S(Y,Z),\ \text{by (3.2) and (2.7)}. \tag{3.6}$$

Also, we know that

$$(R(\xi,Y).P)(U,V)W=R(\xi,Y)P(U,V)W-P(R(\xi,Y)U,V)W-$$
$$-P(U,R(\xi,Y)V)W-P(U,V)R(\xi,Y)W. \tag{3.7}$$

In virtue of (3.1) we get from (3.7) that

$$g(R(\xi,Y)P(U,V)W,\xi)-g(P(R(\xi,Y)U,V)W,\xi)-g(P(U,R(\xi,Y)V)W,\xi)-$$
$$-g(P(U,V)R(\xi,Y)W,\xi)=0. \tag{3.8}$$

Now putting $Y=U=e_i$ in (3.8), $\{e_i\}, i=1,2,\ldots,2m+1$ being an orthonormal basis of the tangent space at any point of the manifold, in the relation (3.8) we get

$$\sum_i\{g(R(\xi,e_i)P(e_i,V)W,\xi)-g(P(R(\xi,e_i)e_i,V)W,\xi)$$
$$-g(P(e_i,R(\xi,e_i)V)W,\xi)-g(P(e_i,V)R(\xi,e_i)W,\xi)\}=0 \tag{3.9}$$

But

$$\sum_i g(R(\xi,e_i)P(e_i,V)W,\xi) \tag{3.10}$$
$$=\sum_i g[K\{g(P(e_i,V)W,e_i)\xi-\eta(P(e_i,V)W)e_i\},\xi],\ \text{by (2.7)}$$
$$=\sum_i[Kg(P(e_i,V)W,e_i)g(\xi,\xi)-Kg(P(e_i,V)W,\xi)g(e_i,\xi)]$$
$$=\sum_i -Kg(P(e_i,V)W,\xi)g(e_i,\xi),\ \text{by (3.5)}$$
$$=-Kg(P(\xi,V)W,\xi)$$

$$\sum_i g(P(R(\xi,e_i)e_i,V)W,\xi) \tag{3.11}$$
$$=\sum_i g[P\{K(g(e_i,e_i)\xi-\eta(e_i)e_i),V\}W,\xi],\ \text{by (2.7)}$$
$$=(2m+1)Kg(P(\xi,V)W,\xi)-\sum_i K[g(e_i,\xi)g(P(e_i,V)W,\xi)]$$
$$=(2m+1)Kg(P(\xi,V)W,\xi)-Kg(P(\xi,V)W,\xi)$$
$$=2mKg(P(\xi,V)W,\xi)$$

$$\sum g(P(e_i,R(\xi,e_i)V)W,\xi) \tag{3.12}$$

$$= \sum_i g[P(e_i, K\{g(V,e_i)\xi - \eta(V)e_i\})W, \xi], \text{ by } (2.7)$$

$$= \sum_i [Kg(g(V,e_i)P(e_i,\xi) - Kg(g(V,\xi)P(e_i,e_i)W,\xi)]$$

$$= - Kg(P(\xi, V)W, \xi)]$$

$$\sum_i g(P(e_i, V)R(\xi, e_i)W, \xi) \tag{3.13}$$

$$= \sum_i g[P(e_i, V)\{K(g(W,e_i)\xi - \eta(W)e_i)\}, \xi], \text{ by } (2.7)$$

$$= \sum_i Kg[(P(e_i, V)\xi, \xi)g(W, e_i)] - \sum_i Kg(P(e_i, V)e_i, \xi)\eta(W)$$

$$=(2m+1)K^2\eta(V)\eta(W) - \frac{Kr}{2m}\eta(V)\eta(W), \text{ by } (3.4) \text{ and } (3.2)$$

where $r$ denotes the scalar curvature.

From (3.9), using (3.10), (3.11), (3.12) and (3.13), we get

$$-2mKg(P(\xi, V)W, \xi) - (2m+1)K^2\eta(W)\eta(V) + \frac{rK}{2m}\eta(W)\eta(V) = 0$$

and using (3.6) we have

$$K[\eta(W)\eta(V)\{-(2m+1)K + \frac{r}{2m}\} - 2mKg(V,W) + S(V,W)] = 0.$$

Then either $K = 0$, or

$$S(V,W) = 2mKg(V,W) + \eta(V)\eta(W)[(2m+1)K - \frac{r}{2m}]. \tag{3.14}$$

If $K = 0$, then from (2.4) we get

$$R(X,Y)\xi = 0. \tag{3.15}$$

If $K \neq 0$, putting $V = W = e_i$ in (3.14) we get

$$r = K(2m+1)2m$$

and (3.14) becomes

$$S(V,W) = 2mKg(V,W). \tag{3.16}$$

Now we use the following result due to Blair [2]

**Result 1.** *Let $M^{2m+1}(\phi, \eta, \xi, g)$ be a contact metric manifold with $R(X,Y)\xi = 0$ for all vector fields $X, Y$. Then $M^{2m+1}$ is locally the Riemannian product of a flat $(m+1)$-dimensional manifold and m-dimensional manifold of positive curvature 4.*

Then we get from (3.5) and (3.16) the following theorem:

**Theorem 1.** *Let $M^{2m+1}(\phi, \eta, \xi, g)$ be a contact metric manifold with $\xi$ belonging to the K-nullity distribution satisfying $R(\xi, X).P = 0$. Then either $M^{2m+1}$ is locally the product of a flat $(m+1)$-dimensional Riemannian manifold and an m-dimensional manifold of positive curvature 4 or $M^{2m+1}$ is an Einstein manifold.*

If $K = 1$, then by (3.16), we can state the following Corollary.

**Corollary.** A Sasakian manifold $M^{2m+1}$ satisfying $R(\xi, X).P = 0$ is an Einstein manifold.

## 4. CONTACT METRIC MANIFOLD SATISFYING DIV $P = 0$

From (3.2) we get

$$(divP)(X, Y)Z = \frac{(2m-1)}{2m}[(\nabla_X S)(Y, Z) - (\nabla_Y S)(X, Z)].$$

Then $divP = 0 \Longleftrightarrow (\nabla_X Q)Y = (\nabla_Y Q)X$ where $S(X, Y) = g(QX, Y)$.

Hence using Theorem 3.1 of [3] we can state the following theorem:

**Theorem 2.** *Let $M^{2m+1}$ be a contact metric manifold with $\xi$ belonging to the K-nullity distribution satisfying $divP = 0$. Then either $M^{2m+1}$ is locally the product of a flat $(m+1)$-dimensional Riemannian manifold and an m-dimensional manifold of constant curvature 4 or $M^{2m+1}$ is an Einstein Sasakian manifold.*

Acknowledgement: The authors are grateful to the referee for his valuable suggestions in the improvement of the paper.

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Received January 18, 1995
Department of Mathematics
University of Kalyani
Kalyani - 741 235
West Bengal, India.