A STRONG BARRELLEDNESS PROPERTY FOR SPACES $C(X,E)$

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Abstract. *A locally convex space (lcs) E is called s-barrelled [DiK] if every sequentially closed linear map, i.e. with sequentially closed graph, of E into a Fréchet space, i.e. a metrizable and complete lcs, is continuous. Let E be a lcs, X a locally compact topological space,* βX *its Stone-Čech compactification. If* $C(\beta X,E)$ *is s-barrelled, then* $C(X,E)$ *is s-barrelled iff X is realcompact, where all spaces of continuous functions are provided with the compact-open topology. Some remarks and corollaries are also included.*

1. INTRODUCTION

Let E be a lcs, X a completely regular Hausdorff space, βX the Stone-Čech compactification and νX the realcompactification of X. By $C(X,E)$ and $C(\beta X,E)$ we understand the space of all continuous E-valued functions on X and βX, respectively, endowed with the compact-open topology; $C(X)$ means the space $C(X,R)$. In [Me] Mendoza proved that whenever X is locally compact and $C(X)$ and $C(\beta X,E)$ are bornological (ultrabornological), then $C(X,E)$ is bornological (ultrabornological), cf. also [Sch] for all background.

In this note we extend this result to the class of s-barrelled spaces. Clearly within metrizable spaces the notions of barrelledness and s-barrelledness coincide. Every ultrabornological space is s-barrelled and every s-barrelled space is barrelled, the converse fails, [DiK], Example 2.6. There exist however s-barrelled spaces which are not bornological, [DiK], Example 2.7. We know also that there exist Baire bornological spaces which are not s-barrelled, [DiK], Example 2.4. In [DiK] we collected also a few properties of s-barrelled spaces which essentially distinguish this class from the barrelled one: the product of s-barrelled spaces $(E_\gamma)_{\gamma\in\Gamma}$ is s-barrelled iff Γ is non-measurable; the completion of a s-barrelled space need not be s-barrelled; the s-barrelledness is not a three space property.

We shall say the a lcs E is s*-*barrelled* if in the definition of s-barrelledness "Fréchet space" is replaced by "F-space", i.e. a metrizable and complete topological vector space (tvs). Clearly every s*-barrelled space is s-barrelled; the converse fails: Since every s*-barrelled space is ultrabarrelled (in the sense of Iyahen, cf. [Iy]), then any uncountable dimensional vector space E endowed with the finest locally convex topology (under which E is not ultrabarrelled, cf. [AeK], p. 8) provides such an example.

Using S. Dierolf example, cf. [Sch], IV.2.3, one deduces that even for a compact space X and a compactly regular inductive limit E of Banach spaces the space $C(X,E)$ need not be s-barrelled. Neverthless one gets the following

Theorem A. *If X is locally compact and* $C(\beta X,E)$ *is s-barrelled, then* $C(X,E)$ *is s-barrelled iff X is realcompact, i.e.* $X=\nu X$.

Theorem A combined with Mendoza result, cf. [Sch], IV.7.9, yelds

Corollary. *If X is locally compact and E is metrizable and barrelled, then* $C(X,E)$ *is s-barrelled iff X is realcompact.*

It is well-known, cf. [CaB], 10.1.26, that whenever X is locally compact and paracompact, then $C(X)$ is Baire. We already know that Baire property does not imply s-barrelledness and s-barrelledness does not imply s*-barrelledness. We have however

Theorem B. *Let X be locally compact and paracompact and E a Fréchet space. Then $C(X,E)$ is Baire and $C(X,E)$ is s-barrelled iff $C(X,E)$ is s*-barrelled iff X is realcompact.*

We discuss also s-barrelledness of the space $c_0(E)$ of all null-sequences in E endowed with the uniform topology.

A bounded absolutely convex subset of a tvs $E = (E,\tau)$ is called a *Banach disc* if the linear span E_B of B endowed with the topology τ_B defined by the Minkowski functional is a Banach space. By $\mathcal{F}(\mathcal{E})$ we denote the filter of all neighbourhoods of zero in E. Recall, that a lcs E is *ultrabornological* if every absolutely convex subset of E which absorbs the compact absolutely convex sets of E is a neighbourhood of zero.

2. PROOF OF THEOREMS

Proof of Theorem A depends heavily on the following characterization of s-barrelledness (Lemma 1; for its proof see [DiK] or [KSW]) and an extension of Banach-Mackey theorem for CS-closed sequences (Lemma 2).

A sequence (U_j) of non-empty subsets of a tvs E such that $U_{j+1}+U_{j+1} \subset U_j, j \in N$, will be called *CS-closed*, if whenever $k \in N$ and $x_j \in U_j, j > k$, and $x = \sum_{j=k+1}^{\infty} x_j$ exists in E, then $x \in U_k$. If U is a convex open or sequentially closed subset of E, then $(2^{-n}U)$ is *CS*-closed.

Lemma 1, [DiK]. *A lcs E is s-barrelled iff every CS-closed sequence (U_j) of absolutely convex and absorbing subsets of E is topological, i.e. $U_j \in \mathcal{F}(\mathcal{E})$ for all $j \in N$.*

Lemma 2. *Let (U_j) be a CS-closed sequence of balanced and absorbing sets in a tvs E. Then every U_j absorbs all Banach discs of E.*

Proof. Let B be a Banach disc in E. Fix $k \in N$. Then $E_B = \bigcup_{n=1}^{\infty} nU_{k+j+1} \cap E_B$. Since $T_j = \overline{U_{k+j} \cap E_B} \in \mathcal{F}(E_B)$ (the closure in E_B) for every $i \in N$, it is enough to show that $T_1 \subset U_k \cap E_B$. Let U be the closed unit ball in E_B and $y \in T_1$. There exists a sequence $x_j \in E_B \cap U_{k+j+1}$ such that $y - \sum_{j=1}^{n} x_j \in U_{k+n+1} \cap 2^{-n}U$. Then $\sum_{j=1}^{n} x_j \in U_k$ and $y - \sum_{j=1}^{n} x_j \to 0$ in E_B. Since $(U_{k+j} \cap E_B)_{j\in N}$ is *CS*-closed in E_B one gets $y \in U_k \cap E_B$.

Proof of THEOREM A. If $C(X,E)$ is s-barrelled, then $C(X)$ is s-barrelled; hence every sequentially continuous linear functional over $C(X)$ is continuous. This implies however that X is realcompact, cf. [CaB], 10.1.15 and 10.1.12. For the converse, let (U_j) be a *CS*-closed sequence in $C(X,E)$ of absolutely convex and absorbing sets. Fix $k \in N$. Since $(U_{k+j} \cap C(\beta X,E))_{j\in N}$ is *CS*-closed in $C(\beta X,E)$, there exists a continuous seminorm p on E such that $\{f \in C(\beta X,E) : p_X(f) \leq 1\} \subset U_k \cap C(\beta X,E)$. Since (U_j) absorbs Banach discs in $C(X,E)$ (Lemma 2) and X is realcompact, the support $s(U_k)$ of U_k is contained in X, [Sch], II.2.5. Now it is enough to proceed as in the proof of Mendoza theorem, cf. [Sch], IV.4.3, to get $\{f \in C(X,E) : p_{s(U_k)}(f) < 1\} \subset U_k$. This proves that (U_j) is topological in $C(X,E)$.

Proof of THEOREM B. By assumption $X = \bigoplus_{\gamma\in\Gamma} X_\gamma$, where X_γ are locally compact and σ-compact. Similarly as in the real case, cf. [CaB], 10.1.26, one gets that $C(X,E)$ and the product $Y = \prod_{\gamma\in\Gamma} C(X_\gamma,E)$ are isomorphic. Then $C(X,E)$ is Baire as the product of Fréchet spaces $C(X_\gamma, E)$. On account of Corollary it is enough to show that s-barrelledness of $C(X,E)$ implies $s*$-barrelledness. If $C(X,E)$ is s-barrelled, then by our Proposition 3.2 of [DiK] the set Γ is non-measurable. Let τ be the product topology of Y. Since Γ is non-measurable and $C(X_\gamma, E)$ are Fréchet spaces, Robertson theorem of [Ro] applies to show that every bounded linear map of (Y,τ) into any tvs is continuous. Let $\mathcal{B}$ be the set of all Banach discs of (Y,τ) and ξ the $*$-inductive limit topology (in the sens of Iyahen [Iy]) of Banach spaces $(Y_B, \tau_B)_{B\in\mathcal{B}}$. Then $\tau \le \xi$ and τ, ξ have the same bounded sets (since τ is complete). Hence $\tau = \xi$, so (Y,τ) is $s*$-barrelled.

By [MaS] and [Me$_2$] the space $c_0(E)$ is (quasi)barrelled iff E is (quasi)barrelled and E'_b has property (B) of Pietsch; for information on (B) see [Me$_1$], [Pi]. Our Lemmas 1 and 2 can also be applied to obtain the following

Proposition 1. *Let E be the inductive limit space of metrizable and barrelled spaces $(E_\gamma)_{\gamma\in\Gamma}$ such that $c_0(E) = \bigcup_{\gamma\in\Gamma} c_0(E_\gamma)$. Then $c_0(E)$ is s-barrelled iff E'_b has property (B). Moreover, if every E_γ is a Banach space, then $c_0(E)$ is s-barrelled iff $c_0(E)$ is ultrabornological iff E'_b has property (B).*

Note that E'_b satisfies (B) if Γ is countable, cf. [Sch], IV.6.10.

Proof. If $c_0(E)$ is s-barrelled, then it is barrelled and E'_b satisfies (B). Now assume that E'_b satisfies (B). Since $c_0(E) = \bigcup_{\gamma\in\Gamma} c_0(E_\gamma)$ and $c_0(E_\gamma)$ is metrizable (hence bornological) for every $\gamma \in \Gamma$, we can apply Theorem 13 of [DeG] to show that $c_0(E)$ is bornological. Let (U_j) be a CS-closed sequence of absolutely convex and absorbing sets in $c_0(E)$ and B a bounded subset of E. Since (U_j) absorbs Banach discs (Lemma 2), the argument used by Defant and Govaerts, cf. also [CaB], 4.8.9, applies also to show that for every $j \in N$ there exists $p_j \in N$ such that $\{(x_n) \in B^N \cap c_0(E) : x_k = 0, 1 \le k \le p_j - 1\}$ is absorbed by U_j. Then U_j absorbs $B^N \cap c_0(E)$. Since $c_0(E)$ is bornological one gets that $U_j \in \mathcal{F}(c_0(E))$. Hence (U_j) is topological. If every E_γ is Banach we proceed similarly replacing "a CS-closed sequence" by "an absolutely convex sets absorbing every absolutely convex and compact subset".

We conclude with the following

Proposition 2. *Every s-barrelled space E is isomorphic to a closed subspace of the ultrabornological space $C(X)$, where $X = (E', \sigma(E', E))$.*

Proof. First observe that X is realcompact: Let Y be a completely regular space and $f : Y \to X$ a continuous map. Define continuous linear maps $j : E \to C_s(Y), j(x)(t) = f(t)(x), x \in E$ and $j^\nu : E \to C_s(\nu Y), j^\nu(x) = (j(x))^\nu$, where $C_s(Y)$ denotes the space $C(Y)$ endowed with the simple topology. Then j^ν is sequentially continuous: If $x_n \in E$ and $x_n \to 0$, then for $x \in \nu Y$ there exists $t \in Y$ such that $(j(x_n))^\nu(x) = j(x_n)(t), n \in N$. Then $j(x_n)(t) = f(t)(x_n) \to 0$. Since E is s-barrelled, j^ν is continuous. Now define $f^\nu : \nu Y \to X$ by $f^\nu(z)(x) = j^\nu(x)(z), z \in \nu Y$. Hence Y is realcompact. Since X is quasicomplete and E is Mackey, E is isomorphic to the topological dual X' endowed with the topology of $C(X)$. Clearly X' is closed in $C(X)$. Finally, Schmets theorem, cf. [CaB], 10.1.12, implies that $C(X)$ is ultrabornological.

Questions. (1) Let X be a compact topological space and E an s-barrelled space such that E'_b satisfies property (B). Is then $C(X,E)$ an s-barrelled space?

(2) By [MaS] and [Me$_2$] the space $c_0(E)$ is barrelled whenever E is an (LB)-space. Is also $c_0(E)$ s-barrelled in this case?

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