RICCI CURVATURES OF LEFT INVARIANT METRICS ON LIE GROUPS WITH ONE-DIMENSIONAL COMMUTATOR-GROUPS

VALERIUS KAISER

Abstract. *Any possible signatures of the quadratic Ricci form of arbitrary left invariant Riemannian metrics on connected Lie groups with one-dimensional commutator-groups are investigated.*

1. FORMULATIONS OF THE RESULTS

J. Milnor [1] has proved, that all possible 3-dimensional Lie groups do not admit a left invariant Riemannian metrics with the following signatures of the quadratic Ricci form : $(+,+,0),(+,-,0)$ and $(+,+,-)$. He has also raised the question of the existence of any similar restrictions for arbitrary higher dimensions. This question was investigated for two-step solvable unimodular [1] Lie groups in [2]. Let G be some such group and let (n,p,z) be the signature of the Ricci form of certain left invariant metric g on the group G, so that n,p and z are the numbers of negative, positive and zero values of principal Ricci curvatures of the metric g. It was proved in [2], that it holds either $p>0$, $n>1$ or $p=0, z>1$. Any these signatures are accessible, that is there is a two-step solvable unimodular Lie group, which admits a left invariant metric with such prescribed signature of the Ricci form.

In this note all possible signatures of the quadratic Ricci form of left invariant Riemannian metrics on connected Lie groups with one-dimensional commutator-group are investigated. Any such groups are isomorphic to the products $G=H\times K$ of Lie groups H and K to within a local isomorphism, where the factor K is commutative. This note therefore prolongs the investigations of the article [3].

There are exactly $\left[\frac{N+1}{2}\right]$ different N-dimensional Lie groups with 1-dimensional commutator-group, which are not locally isomorphic. These are the groups $G=H\times K$, where the group H is one from the Heisenberg groups $H_m, m=1,2,\dots,\left[\frac{N-1}{2}\right]$ or the affine group $A(1)$, and K is a commutative Lie group of the corresponding dimension, where H_m is the nilpotent $(2m+1)$-dimensional Heisenberg group (see [4]) of all $(m+2)\times(m+2)$-matrices with the ones on the principal diagonal, whose another possibly not vanishing elements are in the first row and in the last column only, and $A(1)$ is the solvable 2-dimensional group of all affine transformations of the straight line R. The Lie algebra of the Heisenberg group H_m admits a basis $\{u_1,u_2,\dots,u_m,v_1,v_2,\dots,v_m,e\}$ with the properties $[u_i,v_i]=e,(i=1,2,\dots,m)$.

For the two-step nilpotent groups $G=H_m\times K$ the problem of the description of Ricci curvatures of left invariant metrics is solved in [4]. It follows from the results of the article [4], that every left invariant metric on the N-dimensional group $G=H_m\times K$ possesses only the signature (n,p,z) of the Ricci-Form with $p=1, n=2m, z=N-2m-1$. An another proof of this result is given here.

For the group $G=A(1)\times K$ we prove the following statement.

Proposition 1. *Let $A(1)$ be the 2-dimensional non-commutative Lie group of all affine transformations of the straight line R. The group $G=A(1)\times R^{N-2}$ admits the left invariant* *metrics with the signatures* $(-,-,+,0,0,\dots,0)$ *and* $(-,-,0,0,\dots,0)$ *of the Ricci form only.*

This Proposition 1 is a generalisation of statement 2 in [3].

Since the groups $H_m \times K$ are unimodular [1] and the group $A(1) \times K$ is non-unimodular [1], the Theorem 2 stated below follows from the Proposition 1 and from the result of the work [4] mentioned above.

Theorem 2. *Let* n, p, z *are the numbers of the negative, positive and zero values of a signature* (n, p, z). *Each left invariant Riemannian metric on a N-dimensional Lie group* G $(N > 2)$ *with a one-dimensional commutator-group possesses only the signatures* (n, p, z) *of Ricci form with* $p = 1$, $n = 2m$ $(m = 1, 2, \dots, \left[\frac{N-1}{2}\right])$, *if the group G is unimodular, and with* $p = 1, n = 2$ *or with* $p = 0, n = 2$, *if G is non-unimodular. All these signatures are accessible, that is there exists one of the above mentioned groups, which admits a left invariant metric with such prescribed signature of the Ricci form.*

This Theorem supplements the result from [2] mentioned at the beginning of this note.

It follows from the Theorem 2, that the group $A(1) \times R^{N-2}$, which is the unique to within of local isomorphism non-unimodular group in the class of all N-dimensional Lie groups with 1-dimensional commutator-group offers more possibilities for the selection of the signature of the Ricci form of left invariant metrics (in comparison with any unimodular group).

The comparison with the result of [2] mentioned above show, that the N-dimensional groups with 1-dimensional commutator-group for $N > 3$ do not admit new signatures of the Ricci form of left invariant metrics.

2. LIE GROUPS WITH ONE-DIMENSIONAL COMMUTATOR-GROUPS AND RICCI CURVATURES OF LEFT INVARIANT METRICS

The Proposition 1 and the Theorem 2 are proved here. This gives a new proof of the result of [4] mentioned above.

Let G' be the one-dimensional commutant of a N-dimensional Lie group G and let $\mathcal{G}'$ be the corresponding one-dimensional commutator-algebra of the Lie algebra $\mathcal{G}$ of the group G.

We choose a unit vector e from the ideal $\mathcal{G}'$ of Lie algebra $\mathcal{G}$. Let Γ be the hyperplane, which is orthogonal to the vector e with respect to a given left invariant metric g on the Lie Group G. Then it holds $[\mathbf{u}, \mathbf{e}] = \varphi(\mathbf{u})\mathbf{e}$ for some linear funktion $\varphi : \Gamma \to R$. Let $\mathbf{a} \in \Gamma$ be the unique element of Γ, such that it holds $\varphi(\mathbf{u}) = <\mathbf{a}, \mathbf{u}>$ for each $\mathbf{u} \in \Gamma$. Then it holds

$$[\mathbf{u}, \mathbf{e}] = <\mathbf{a}, \mathbf{u}> \mathbf{e}, \mathbf{u} \in \Gamma \tag{1}$$

Let be $[\mathbf{u}, \mathbf{v}] = B(\mathbf{u}, \mathbf{v})e$, for $u, v \in \Gamma$, where $B(\mathbf{u}, \mathbf{v})$ is a skew-symmetric bilinear form on Γ. There is an unique skew-adjoint linear transformation $f : \Gamma \to \Gamma$, such that it holds $B(\mathbf{u}, \mathbf{v}) = <f(\mathbf{u}), \mathbf{v})$ for $\mathbf{u}, \mathbf{v} \in \Gamma$. Then it holds

$$[\mathbf{u}, \mathbf{v}] = <f(\mathbf{u}), \mathbf{v}> \mathbf{e}, \mathbf{u}, \mathbf{v} \in \Gamma \tag{2}$$

Lemma 1. *With these notations, the covariant derivative operators* $\nabla_{\mathbf{e}}$ *and* $\nabla_{\mathbf{u}}$ *of the Levi-*

Civita connection of the left invariant metric g satisfy:

$$
\begin{aligned}
\nabla_{\mathbf{e}}\mathbf{e} &= a, \nabla_{\mathbf{e}}\mathbf{u} = -\frac{1}{2}f(\mathbf{u}) - <\mathbf{u},\mathbf{a}>\mathbf{e}, \\
\nabla_{\mathbf{u}}\mathbf{e} &= -\frac{1}{2}f(\mathbf{u}), \nabla_{u}v = \frac{1}{2}<f(\mathbf{u}),\mathbf{v}>\mathbf{e},
\end{aligned} \tag{3}
$$

for each $u, v \in \Gamma$.

Proof. For the computation operators ∇ we apply the formula $<\nabla_X Y, Z> = \frac{1}{2}\{<[X,Y],Z>$ - $<[Y,Z],X>+<[Z,X],Y>\}$ (see [1]), where $X,Y,Z \in \mathcal{G}$. Thus we have for example, $<\nabla_{\mathbf{e}}\mathbf{u},\mathbf{e}> = <[\mathbf{e},\mathbf{u}],\mathbf{e}> = -<\mathbf{u},\mathbf{a}>$, $<\nabla_{\mathbf{e}}\mathbf{u},\mathbf{v}> = -\frac{1}{2}<f(\mathbf{u}),\mathbf{v}>$ for each $\mathbf{u},\mathbf{v} \in \Gamma$; the second equation in (3) follows from here. All other relations in (3) will be analogously proved. □

Lemma 2. *With the notations of the lemma 1, the curvature tensor $R_{XY}Z$ of the Levi-Civita connection ∇ of the left invariant metric g satisfy:*

$$
\begin{aligned}
R_{\mathbf{eu}}u &= -<\mathbf{u},\mathbf{a}>f(\mathbf{u}) - \left(\frac{1}{4}<\mathbf{u},f^2(\mathbf{u}) + <\mathbf{u},\mathbf{a}\overset{2}{>}\right)\mathbf{e}, \\
R_{\mathbf{ue}}\mathbf{e} &= -\frac{1}{4}f^2(\mathbf{u}) - <\mathbf{u},\mathbf{a}>\mathbf{a}, \\
R_{\mathbf{uv}}\mathbf{v} &= <f(\mathbf{u}),\mathbf{v}>\left(\frac{3}{4}f(\mathbf{v}) + <\mathbf{v},\mathbf{a}>\mathbf{e}\right).
\end{aligned} \tag{4}
$$

Proof. It follows easily from the calculating formula $R_{XY}Z = \nabla_X\nabla_Y Z - \nabla_Y\nabla_X - \nabla_{[XY]}Z$ of the curvature tensor $R_{XY}Z$ of the connection ∇. □

Lemma 3. *With the notations of lemmas 1 and 2, the linear Ricci transformation $\hat{r}$ of the connection ∇ satisfy:*

$$
\begin{aligned}
\hat{r}(\mathbf{e}) &= -f(\mathbf{a}) - \left(\frac{1}{4}\,\text{trace}\,(f^2) + |\mathbf{a}|^2\right)\mathbf{e}, \\
\hat{r}(\mathbf{u}) &= \frac{1}{2}f^2(\mathbf{u}) - <\mathbf{u},\mathbf{a}>\mathbf{a} + <f(\mathbf{u}),\mathbf{a}>\mathbf{e}.
\end{aligned} \tag{5}
$$

Proof. We choose a orthonormal basis $u_1, u_2, \dots, u_{N-1}$ of the hyperplane Γ. Then we have

$$
\begin{aligned}
\hat{r}(\mathbf{e}) &= \sum_{i=1}^{N-1} R_{\mathbf{eu}_i}\mathbf{u}_i \\
&= -\sum_{i=1}^{N-1} <\mathbf{u}_i,\mathbf{a}> f(\mathbf{u}_i) - \frac{1}{4}\sum_{i=1}^{N-1} <\mathbf{u}_i, f^2(\mathbf{u}_i)>\mathbf{e} - \sum_{i=1}^{N-1} <\mathbf{u}_i,\mathbf{a}\overset{2}{>}\mathbf{e} \\
&= -f\left(\sum_{i=1}^{N-1} <\mathbf{a},\mathbf{u}_i>\mathbf{u}_i\right) - \frac{1}{4}\text{ trace }(f^2)\mathbf{e} - |\mathbf{a}|^2\mathbf{e} \\
&= -f(\mathbf{a}) - \left(\frac{1}{4}\text{ trace }(f^2) + |\mathbf{a}|^2\right)\mathbf{e}, \\
\hat{r}(\mathbf{u}) &= R_{\mathbf{ue}}\mathbf{e} + \sum_{i=1}^{n-1} r_{\mathbf{uu}_i}\mathbf{u}_i \\
&= -\frac{1}{4}f^2(\mathbf{u}) - <\mathbf{u},\mathbf{a}>\mathbf{a} \\
&+ \frac{3}{4}f\left(\sum_{i=1}^{N-1} <f(\mathbf{u}),\mathbf{u}_i>\mathbf{u}_i\right) + \sum_{i=1}^{N-1} <f(\mathbf{u}),\mathbf{u}_i><\mathbf{u}_i,\mathbf{a}>\mathbf{e} \\
&= \frac{1}{2}f^2(\mathbf{u}) - <\mathbf{u},\mathbf{a}>\mathbf{a} + <f(\mathbf{u}),\mathbf{a}>\mathbf{e}. \ \square
\end{aligned}
$$

It follows from (1) and (2), that *trace* $ad_{\mathbf{e}} = 0$, *trace* $ad_{\mathbf{u}} = <\mathbf{u},\mathbf{a}>$ holds for each $\mathbf{u} \in \Gamma$. Therefore the group G is unimodular [1], if and only if $\mathbf{a} = 0$. We are starting at first from this case. Then it follows from the lemma 3, that

$$\hat{r}(\mathbf{e}) = -\frac{1}{4}\text{ trace }(f^2)\mathbf{e}, \hat{r}(\mathbf{u}) = -\frac{1}{2}f^2(\mathbf{u}). \tag{6}$$

Let be *rank* $(f) = 2m$, so that $1 \le k \le \left[\frac{N-1}{2}\right]$. It is well known [5], that *rank* $(f^2) =$ *rank* (f) and that there is a orthonormal basis $\mathbf{u}_1, \mathbf{u}_2, \dots, \mathbf{u}_{N-1}$ of the huperplane Γ, so that the skew-symmetric bilinear form $B(\mathbf{u},\mathbf{v}) = <f(\mathbf{u}),\mathbf{v}>$ has a canonical form, so that it hold: $B(\mathbf{u},\mathbf{v}) = \sum_{i=0}^{m}\sigma_i(u_{2i-1}v_{2i} - u_{2i}v_{2i-1})$. It holds $f^2 = diag\,(-\sigma_1^2, -\sigma_1^2, -\sigma_2^2, -\sigma_2^2, \dots, -\sigma_m^2, -\sigma_m^2, 0, 0, \dots, 0)$ in this basis. It follows from (6), that the quadratic Ricci form has the diagonal form in that basis and has the signature (n,p,z) with $p = 1, n = 2m, z = N - 2m - 1$, where $1 \le m \le \left[\frac{N-1}{2}\right]$. We denote $\mathbf{a}_i = \mathbf{u}_{2i-1} / \sigma_i, \mathbf{b}_i = \mathbf{u}_{2i}$, $i = 1, 2, \dots, m, \mathcal{H} = \mathcal{S}\sqrt{\dashv}\backslash\backslash\{\mathbf{a}_1, \mathbf{a}_2, \dots, \mathbf{a}_{\Updownarrow}, \mathbf{b}_1, \mathbf{b}_2, \dots \mathbf{b}_{\Updownarrow}, \mathbf{e}\}, \mathcal{K} = Spann\ \{\mathbf{u}_{2m+1}, \dots, \mathbf{u}_{N-1}\}$. It follows from (1) and (2), that the relations $[\mathbf{a}_i, \mathbf{b}_i] = \mathbf{e}, [\mathcal{H},\mathcal{K}] = 0, [\mathcal{K},\mathcal{K}] = 0$ hold. This means, that the Lie algebra $\mathcal{H}$ is isomorphic to the Lie algebra of $(2m+1)$-dimensional Heisenberg group and that the ideal $\mathcal{K}$ is commutative. By this the first part of the Theorem 2 is proved.

In order to prove the second part of this theorem we assume, that the group G is non-unimodular. Then it holds : $\mathbf{a} \neq 0$. Then for each vectors $\mathbf{u},\mathbf{v},\mathbf{w} \in \Gamma$ the Jacobi-identity $[\mathbf{u},[\mathbf{v},\mathbf{w}]] + [\mathbf{v},[\mathbf{w},\mathbf{u}]] + [\mathbf{w},[\mathbf{u},\mathbf{v}]] = 0$ gives us the relation

$$<f(\mathbf{u}),\mathbf{v}><\mathbf{a},\mathbf{w}> + <f(\mathbf{v}),\mathbf{w}><\mathbf{u},\mathbf{a}> + <f(\mathbf{w}),\mathbf{u}><\mathbf{v},\mathbf{a}> = 0.$$

From here it follows

$$-f(\mathbf{u}) < \mathbf{v}, \mathbf{a} > + f(\mathbf{v}) < \mathbf{u}, \mathbf{a} > + < f(\mathbf{u}), \mathbf{v} > \mathbf{a} = 0. \tag{7}$$

We denote $\mathbf{b} = f(\mathbf{a})$, then it follows from (7) for $\mathbf{v} = \mathbf{a}$, that the linear transformation f is completely defined by its value $\mathbf{b} = f(\mathbf{a})$, so that it holds:

$$f(\mathbf{u}) = \frac{1}{||\mathbf{a}||^2}\{< \mathbf{u}, \mathbf{a} > \mathbf{b} - < \mathbf{u}, \mathbf{b} > \mathbf{a}\}. \tag{8}$$

If holds $\mathbf{b} = 0$, then $f = 0$ holds also. We denote $\mathcal{K}$ the orthogonal complement of the vector a in the subspace Γ. The equations $[\mathbf{a}, \mathbf{e}] = ||\mathbf{a}||^2\mathbf{e}$, $[\mathbf{q}, \mathbf{a}] = 0$, $[\mathbf{q}, \mathbf{e}] = 0$, $[\mathbf{q}, \mathbf{p}] = 0$ follows from (1) for each $\mathbf{p}, \mathbf{q} \in \mathcal{K}$, so that the decomposition $\mathcal{G} = \mathcal{K} \oplus \mathcal{H}$ presents the direct orthogonal sum of Lie algebras, where $\mathcal{H} = Spann\ \{\mathbf{a}, \mathbf{b}\}$ is the 2-dimensional non-commutative Lie algebra. From (5) it follows now, that the relations $\hat{r}(\mathbf{e}) = -||\mathbf{a}||^2\mathbf{e}$, $\hat{r}(\mathbf{a}) = -||\mathbf{a}||^2\mathbf{a}$, $\hat{r}(\mathbf{u}) = 0$ hold for each $\mathbf{u} \in \mathcal{K}$, and that the Ricci form has the signature $(-, -, 0, 0, \ldots, 0)$. By this the Proposition 1 and the Theorem 2 are proved for this case ($\mathbf{b} = 0$).

In order to continue the proofs of the proposition 1 and the Theorem 2 we assume, that $\mathbf{b} \neq 0$ holds. Then the vectors $\mathbf{a}$ and $\mathbf{b}$ are orthogonal, because the linear transformation f is skew-adjoint. It follows from (8), that the relations

$$f(\mathbf{a}) = \mathbf{b}, f(\mathbf{b}) = -\lambda^2\mathbf{a}, f(\mathbf{u}) = 0, \mathbf{u} \in \mathcal{Q}, \tag{9}$$

hold, where $\lambda = ||b| \,/\, ||a||$, and $\mathcal{Q}$ is the orthogonal complement of the subspace *Spann* $\{\mathbf{a}, \mathbf{b}\}$ in the hyperplane Γ. The relations

$$\begin{aligned} [\mathbf{a}, \mathbf{e}] &= ||\mathbf{a}||^2\mathbf{e}, [\mathbf{a}, \mathbf{b}] = ||\mathbf{b}||^2\mathbf{e}, [\mathbf{b}, \mathbf{e}] = 0, \\ [\mathbf{u}, \mathbf{e}] &= [\mathbf{u}, \mathbf{a}] = [\mathbf{u}, \mathbf{b}] = 0, \mathbf{u} \in \mathcal{Q} \end{aligned} \tag{10}$$

follow from (9), (1) and (2). From (9) we obtain

$$f^2(\mathbf{a}) = -\lambda^2\mathbf{a}, f^2(\mathbf{b}) = -\lambda^2\mathbf{b}, f^2(\mathbf{u}) = 0, \mathbf{u} \in \mathcal{Q}, \tag{11}$$

so that it holds: trace $(f^2) = -2\lambda^2$. The equations

$$\begin{aligned} r(\mathbf{e}) &= -\mathbf{b} - (||a||^2 - \frac{1}{2}\lambda^2)\mathbf{e}, \\ r(\mathbf{a}) &= -(||a||^2 + \frac{1}{2}\lambda^2)\mathbf{a}, \\ r(\mathbf{b}) &= -\frac{1}{2}\lambda^2\mathbf{b} - ||\mathbf{b}||^2\mathbf{e}, \\ r(\mathbf{u}) &= 0, \mathbf{u} \in \mathcal{Q}. \end{aligned} \tag{12}$$

follow from (10), (11) and (5). It follows from (10), that the unit vector $\mathbf{e}_3 = \xi \cdot \eta\ (-||\mathbf{b}||^2\mathbf{e} + ||\mathbf{a}||^2\mathbf{b})$ (with $\xi = (||\mathbf{b}||^2 + ||\mathbf{a}||^4)^{-2}, \eta = 1 \,/\, ||\mathbf{b}||$) defines a central direction of the Lie algebra $\mathcal{G}$, which lies not in the commutative ideal $\mathcal{Q}$. Let $\mathbf{e}_2 = \xi(||\mathbf{a}||^2\ e + \mathbf{b})$ be the unit vector, which is orthogonal to this direction in subspace *Spann* $\{\mathbf{b}, \mathbf{e}\}$ and let be also $\mathbf{e}_1 = \mathbf{a}$. Let $\{\mathbf{e}_4, \mathbf{e}_5, \dots, \mathbf{e}_N\}$ be a orthonormal basis of the ideal $\mathcal{Q}$. A simple calculation shows, that the choosen basis $\mathbf{e}_1, \mathbf{e}_2, \dots, \mathbf{e}_N$ diagonalize the quadratic Ricci form, so that the relations

$$\begin{aligned}
r(\mathbf{e}_1) &= -(\|\mathbf{a}\|^2 + \frac{1}{2}\lambda^2)\mathbf{e}_1, \\
r(\mathbf{e}_2) &= -(\|\mathbf{a}\|^2 + \frac{1}{2}\lambda^2)\mathbf{e}_2, \\
r(\mathbf{e}_3) &= \frac{1}{2}\lambda^2\mathbf{e}_3, \\
r(\mathbf{e}_i) &= 0, (i = 4, 5, \dots, N)
\end{aligned}$$

hold. Therefore the relations $p = 1, n = 2, z = N - 3$ hold for the signature (p, n, z) of the Ricci form.

It follows now from (10), that $[\mathbf{e}_1, \mathbf{e}_2] = \|\mathbf{a}\|\mathbf{e}_2 - \lambda\mathbf{e}_3$ holds and that the equations $[\mathbf{e}_i, \mathbf{e}_j] = 0$ hold for all other vectors of the choosen basis. We denote $\mathbf{x} = \|\mathbf{a}\|\mathbf{e}_2 - \lambda\mathbf{e}_3, \mathbf{y} = \frac{1}{\|\mathbf{a}\|}\mathbf{e}_1$, and $\mathcal{K} = Spann\ \{\mathcal{Q}, \mathbf{e}_3\}$. Then the equations $[\mathbf{y}, \mathbf{x}] = \mathbf{x}$, $[\mathbf{x}, \mathcal{K}] = 0, [\mathbf{y}, \mathcal{K}] = 0$ and $[\mathcal{K}, \mathcal{K}] = 0$ hold, which means, that $\mathcal{G} = \mathcal{H} \oplus \mathcal{K}$ holds (direct, but non-orthogonal sum of ideals), where $\mathcal{H} = Spann\ \{\mathbf{x}, \mathbf{y}\}$ is isomorphic to the 2-dimensional non-commutative Lie algebra and $\mathcal{K}$ is the commutative factor of the algebra $\mathcal{G}$. This finishes the proof of Proposition 1 and also of Theorem 2. □

REFERENCES

[1] J. MILNOR, *Curvatures of left invariant metrics on Lie groups*, Advances in Math. 21, 1976, 293-329.

[2] I. DOTTI MIATELLO, *Ricci curvature of left invariant metrics on solvable unimodular Lie groups*, Math. Z. 180, 1982, 257-263.

[3] V.V. KAISER, *Ricci-Krümmungen linksinvarianter Metriken auf 4-dimensionalen nichtunimodularen Lie'schen Gruppen mit kommutativen Faktoren*, Seminar Sophis Lie, 3, Number 2 (1993), 137-145.

[4] P. ELBERLEIN, *Geometry of 2-step nilpotent groups with a left invariant metric*, Univ. North. Carolina, USA, 1990, 1-81.

[5] J. SCHOUTEN, *Der Ricci-Kalkül*, Berlin, 1924.

Received October 1, 1995
V.V. Kaiser
Mathematisches Institut
Universität Erlangen - Nürnberg
Bismarckstraße $1^{1/2}$
D-91054 Erlangen - GERMANY