# ON MALLIOS $\mathcal{A}$-CONNECTIONS AS CONNECTIONS ON PRINCIPAL SHEAVES

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**Abstract.** *Motivated by [5], we associate a* vector sheaf $\mathcal{E}$ *with a* principal sheaf $\mathcal{P}(\mathcal{E})$, *called* sheaf of frames *of* $\mathcal{E}$. *We show that* $\mathcal{A}$*-connections on* $\mathcal{E}$ *correspond to connections on* $\mathcal{P}(\mathcal{E})$. *The latter are defined by an appropriate family of local matrices or, equivalently, by a morphism acting on* $\mathcal{P}(\mathcal{E})$, *analogously to the operator of an* $\mathcal{A}$*-collection.*

## 1. INTRODUCTION

In his abstract version of Weil's integrality theorem, published in this journal ([5]), A. Mallios introduced the notion of an $\mathcal{A}$-connection. Roughly speaking, this is a connection defined on a vector sheaf, that is on a locally free $\mathcal{A}$-module of finite rank, over a topological space, where $\mathcal{A}$ is sheaf of commutative, unital and associative $\mathbb{C}$-algebras.

This is a remarkable extension of the classical theory of linear connections within a purely algebraic and topological context, *without any notion of differentiability*, and constitutes the hard core of the forthcoming book [6]. In this framework one is freed from many constraints and is able to extend a great part of the traditional differential geometry to spaces which are *non-smooth* at all, a fact that seems to be particularly interesting for modern physics (in this respect see, for instance, [4] and [5]). Yet, the "structure sheaves" involved in [5] are not necessarily "functional" ones (viz. sheaves of germs of functions), as was the case so far (see e.g. [4]), but quite abstract algebra sheaves, a fact that broadens the applicability of the technique in question (see also the relevant comments in [4; p. 358]).

On the other hand, going back to the classical context, it is customary, as well as convenient, to associate linear connections on vector bundles with connections on the corresponding principal bundle of frames.

The purpose of the present note is to apply the last idea within the aforementioned abstract framework. More precisely, for a given vector sheaf $\mathcal{E}$, we construct a principal sheaf $\mathcal{P}(\mathcal{E})$ with structure sheaf $\mathcal{GL}(\mathcal{A}, n)$ (in the sense of [3]) and we show that $\mathcal{E}$ is associated with $\mathcal{P}(\mathcal{E})$ (Propositions 3.2 and 3.3). Analogously to the classical case, $\mathcal{P}(\mathcal{E})$ is defined to be the sheaf of frames of $\mathcal{E}$. Moreover, we prove that the local connection matrices of an $\mathcal{A}$-connection (corresponding to the classical local connection forms), though they cannot be globalized to a form on $\mathcal{P}(\mathcal{E})$, give rise to an appropriate sheaf morphism $D$ on the latter. This is the principal sheaf analogue of the operator of $\mathcal{A}$-connection and its classical counterpart of covariant differentiation. As a result, we establish a bijection between $\mathcal{A}$-connections and morphisms $D$ (Theorem 5.5 in conjunction with Theorem 5.4).

The main results, outlined above, are given in Sections 4 and 5. Sections 2 and 3 are preparatory and mostly review what we need from the theory of vector sheaves and $\mathcal{A}$-connections. Of particular importance is Theorem 3.4, describing such connections by means of local matrices. The final section contains a brief discussion on the curvature. It is included here as an example of the effectiveness of the proposed study of $\mathcal{A}$-connections within the framework of principal sheaves.

The present approach and [7] motivate the study of connections on certain arbitrary principal

sheaves (not necessarily related with vector sheaves), as it is fully explained in ([8]).

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## 2. PRELIMINARIES

Although we are motivated by [5], to which we refer for the main terminology and notations, for the reader's convenience we review some of the basic material needed in the sequel.

We start with a fixed *algebraized space* $(X, \mathcal{A})$, where $X$ is a topological space and $\mathcal{A}$ a sheaf of commutative, associative and unital (linear) $\mathbb{C}$-algebras over $X$. We also assume the existence of a *differential triad* $(A, \partial, \Omega^1)$, where $\Omega^1$ is an $\mathcal{A}$-*module* (that is, a sheaf of $\mathcal{A}$-modules) over $X$ and $\partial : A \to \Omega^1$ is a $\mathbb{C}$-linear morphism satisfying the *Leibniz condition*

$$\partial(s.t) = s.\partial(t) + t.\partial(s), \tag{2.1}$$

for every $s, t \in \mathcal{A}(U)$ and $U \subset X$ open.

**Remarks.** 1) The term differential triad, as well as any similar terminology involving the adjective "differential", is used in order to connect the classical differential geometry with the present non-smooth context.

2) $\partial$ figuring in (2.1) is in fact the induced morphism between sections. The customary identification of a sheaf with the sheaf of germs of its sections, as well as the identification of a sheaf morphism with the one between the corresponding presheaves of sections, will be applied without any particular mention. Our main references concerning sheaf theory are [1] and [2].

We denote by $\mathcal{A}^\bullet$ the *sheaf of units* (: invertible elements) of $\mathcal{A}$, i.e. $\mathcal{A}^\bullet$ is generated by the (complete) presheaf of abelian groups $U \to (A(U))^\bullet$, for every open $U \subseteq X$. Therefore,

$$\mathcal{A}^\bullet(U) \cong (A(U))^\bullet.$$

We define now the morphism of sheaves of (abelian) groups $\tilde{\partial} : \mathcal{A}^\bullet \to \Omega^1$ given by

$$\tilde{\partial}(s) := s^{-1}.\partial(s), s \in \mathcal{A}^\bullet(U), \tag{2.2}$$

for every open $U \subseteq X$. In analogy to the classical case, $\tilde{\partial}$ is the *logarithmic or total differential derivation* of $\mathcal{A}$, induced by $\partial$. It is straightforward that

$$\tilde{\delta}(s.t) = \tilde{\delta}(s) + \tilde{\partial}(t), \qquad s, t \in \mathcal{A}^\bullet(U). \tag{2.3}$$

We consider also the complete presheaf of matrices

$$U \to \mathcal{M}_n(\mathcal{A}(U)), \qquad U \subseteq X \text{ open.}$$

It generates the *matrix algebra sheaf* $\mathcal{M}_n(\mathcal{A})(n \geq 1)$ with

$$\mathcal{M}_n(\mathcal{A})(U) \cong M_n(\mathcal{A}(U)).$$

Thus we may define the $\mathcal{A}$-module

$$\mathcal{M}_n(\Omega^1) := \mathcal{M}_n(\mathcal{A}) \otimes_{\mathcal{A}} \Omega^1 \cong (\Omega^1)n^2,$$

on which $\partial$ induces the morphism (using the same symbol)

$$\partial : \mathcal{M}_n(\mathcal{A}) \to \mathcal{M}_n(\Omega^1)$$

given by

$$\partial(a) := (\partial(a_{ij})), \tag{2.4}$$

if $a = (a_{ij}) \in \mathcal{M}_n(\mathcal{A})(U) \cong \mathcal{M}_n(\mathcal{A}(U)), U \subseteq X$ open.

Analogously to $\mathcal{A}^\bullet$, the *general linear group sheaf* of $\mathcal{A}$ (of order $n$), denoted by $\mathcal{GL}(n, \mathcal{A})$, is the sheaf of units of $\mathcal{M}_n(\mathcal{A})$, defined by the (complete) presheaf

$$U \to GL(n, A(U)) = M_n(A(U))^\bullet, \quad U \subseteq X \text{ open.}$$

Hence, $\mathcal{GL}(n, \mathcal{A})(U) \cong GL(n, \mathcal{A}(U))$ and $\mathcal{GL}(n, \mathcal{A}) = \mathcal{M}_n(\mathcal{A})^\bullet$, $n \geq 1$. We define a logarithmic differential on $\mathcal{GL}(n, \mathcal{A})$ by setting

$$\tilde{\partial}(a) := a^{-1}.\partial(a), \quad a = (a_{ij}) \in \mathcal{GL}(n, \mathcal{A})(U) \cong GL(n, \mathcal{A}(U)). \tag{2.5}$$

Finally, the *adjoint representation*

$$\mathcal{A}d : \mathcal{GL}(n, \mathcal{A}) \to \mathcal{E}nd(\mathcal{M}_n(\mathcal{A}))^\bullet,$$

given by $[\mathcal{A}d(g)](a) := g.a.g^{-1}$, induces the representation $\mathcal{A}d : \mathcal{GL}(n, \mathcal{A}) \to \mathcal{A}ut(\mathcal{M}_n(\Omega^1))$ $:= End(M_n(\Omega^1))^\bullet$ with

$$\mathcal{A}d(g)(a \otimes \theta) := (g.a.g^{-1}) \otimes \theta, \tag{2.6}$$

for any $g \in \mathcal{GL}(n, \mathcal{A})(U), a \in \mathcal{M}_n(\mathcal{A})(U), \theta \in \Omega^1(U)$ and $U \subseteq X$ open.

Combing (2.5) and (2.6) we readily check that the following formulas hold true:

$$\tilde{\partial}(a.b) = \mathcal{A}d(b^{-1}).\tilde{\partial}(a) + \tilde{\partial}(b) \tag{2.7}$$

$$\tilde{\partial}(a^{-1}) = -\mathcal{A}d(a^{-1}).\tilde{\partial}(a) \tag{2.8}$$

$$\mathcal{A}d(a.b).\omega = \mathcal{A}d(a).(\mathcal{A}d(b).\omega), \tag{2.9}$$

for every $a, b \in \mathcal{GL}(n, A)(U), \omega \in \mathcal{M}_n(\Omega^1)(U)$ and $U \subseteq X$ open.

## 3. VECTOR SHEAF CONNECTIONS

In the sequel $\mathcal{E} \equiv (\mathcal{E}, X, \pi_E)$ denotes a *locally free $\mathcal{A}$-module* of *finite rank*, say $n$, over $X$. By definition, this means that $\mathcal{E}$ is an $\mathcal{A}$-module such that there exists an open cover $\mathbb{C} = \{U_\alpha | \alpha \in I\}$ of $X$ and $\mathcal{A}$-isomorphism

$$\phi_\alpha : \mathcal{E}|_{U_\alpha} \xrightarrow{\cong} A^n|_{U_\alpha} \cong (A|_{U_\alpha})^n. \tag{3.1}$$

For simplicity, such an $\mathcal{E}$ is called a *vector sheaf* of rank $n$ (see also [5], [6]).

**Definition 3.1.** *An $\mathcal{A}$-connection on $\mathcal{E}$ is a $\mathbb{C}$-linear morphism*

$$\nabla : \mathcal{E} \to \mathcal{E} \otimes_{\mathcal{A}} \Omega^1 =: \Omega^1(\mathcal{E})$$

*satisfying the Leibniz (or Koszul) condition*

$$\nabla(\alpha . s) = \alpha . \nabla s + s \otimes \tilde{\partial}\alpha, \tag{3.2}$$

*for every $\alpha \in \mathcal{A}(U), s \in \mathcal{E}(U)$ and $U \subseteq X$ open.*

For the existence of $\mathcal{A}$-connections, examples and related topics we refer to [5] and [6], where $\nabla$ is denoted by $D$ (the latter is used here for the principal connections of Section 4).

In the sequel we shall express $\nabla$, locally, by appropriate matrices. First we observe that, in a standard way, the isomorphisms (3.1) determine a coordinate cocycle $(g_{\alpha\beta}) \in Z^1(\mathbb{C}, \mathcal{GL}(n, \mathcal{A}))$, classifying, up to isomorphism, $\mathcal{E}$. Thus, for each $g_{\alpha\beta}$, we may write

$$g_{\alpha\beta} = (g_{ij}^{\alpha\beta})_{1 \le i,j \le n} \in \mathcal{GL}(n, \mathcal{A})(U_{\alpha\beta}) \cong GL(n, \mathcal{A}(U_{\alpha\beta})),$$

if $U_{\alpha\beta} := U_\alpha \cap U_\beta \neq \emptyset$. For simplicity, we shall drop the superscript $\alpha\beta$ in the elements of the matrix, whenever it is clear that we are restricted over $U_{\alpha\beta}$. Similarly, (3.1) determines a natural basis $e^\alpha := (e_1^\alpha, \ldots, e_n^\alpha)$ of $\mathcal{E}(U_\alpha)$ by

$$e_i^\alpha(x) := \phi_\alpha^{-1}(0_x, \ldots, 1_x, \ldots, 0_x), \quad x \in U_\alpha,$$

if $0_x$ and $1_x$ (in the i-th entry) are the zero and unit elements of the fibre $\mathcal{A}_x$ respectively. Therefore, any $s \in \mathcal{E}(U_\alpha)$ takes the form

$$s = \sum_{i=1}^{n} s_i^\alpha \cdot e_i^\alpha, \quad s_i^\alpha \in \mathcal{A}(U_\alpha). \tag{3.3}$$

Since now $\mathcal{E}(U_\alpha)$ is a free module of rank $n$, we can show (see also [6; Vol. II, p. 100, formulas (1.9) and (1.10)]) that

$$(\mathcal{E} \otimes_A \Omega^1)(U_\alpha) \cong \mathcal{E}(U_\alpha) \otimes_{\mathcal{A}(U_\alpha)} \Omega^1(U_\alpha), \tag{3.4}$$

implying (as in the ordinary case) that

$$\nabla(e_j^\alpha) = \sum_{i=1}^{n} e_i^\alpha \otimes \omega_{ij}^\alpha, \quad 1 \le j \le n, \tag{3.5}$$

where each $\omega_{ij}^\alpha \in \Omega^1(U_\alpha)$. Hence, we obtain the $n \times n$ matrix

$$\omega^\alpha := (\omega_{ij}^\alpha) \in M_n(\Omega^1(U_\alpha)) \cong M_n(\mathcal{A}(U_\alpha)) \otimes_{\mathcal{A}(U_\alpha)} \Omega^1(U_\alpha), \tag{3.6}$$

the last identification being proved analogously to (3.4). The matrix $\omega^\alpha$, called *local connection matrix* (with respect to $U_\alpha$), completely determines the restriction of $\nabla$ on $\mathcal{E}|_{U_\alpha}$. Indeed, an easy computation, based on (3.2) and (3.5), shows that

$$\nabla(s) = \sum_{i=1}^{n}(s_i^\alpha \cdot \nabla(e_i^\alpha) + e_i \otimes \partial\, s_i^\alpha) = \sum_{i=1}^{n} e_i^\alpha \otimes (\partial\, s_i^\alpha + \sum_{j=1}^{n} s_j^\alpha \cdot \omega_{ij}^\alpha), \qquad (3.7)$$

for any section of the form (3.3).

**Lemma 3.2.** *The local connection matrices $(\omega^\alpha)_{\alpha\in I}$, with respect to $\mathbb{C}$, satisfy the compatibility condition (viz. local gauge equivalence)*

$$\omega^\beta = \mathcal{A}d(g_{\alpha\beta}^{-1}) \cdot \omega^\alpha + \tilde{\partial} g_{\alpha\beta}. \qquad (3.8)$$

**Proof.** Considering also the pair $(U_\beta, \phi_\beta)$, $U_\beta \in \mathbb{C}$, we have a natural basis $e^\beta$ of $\mathcal{E}(U_\beta)$ and the matrix $\omega^\beta = (\omega_{ij}^\beta)$ determined by

$$\nabla(e_j^\beta) = \sum_{i=1}^{n} e_i^\beta \otimes \omega_{ij}^\beta, \quad 1 \le j \le n. \qquad (3.5')$$

To find the relation between $\omega^\alpha$ and $\omega^\beta$, in case $U_{\alpha\beta} \neq \emptyset$, we first observe that, on the overlappings,

$$e_j^\beta = \sum_{i=1}^{n} g_{ij} \cdot e_i^\alpha, \quad 1 \le j \le n \qquad (3.9)$$

(recall the convention about the matrix $g_{\alpha\beta}$ above). Substituting (3.9) in (3.5'), we get

$$\nabla(e_j^\beta) = \sum_{i=1}^{n}\left(\sum_{k=1}^{n} g_{kj} \cdot e_K^\alpha\right) \otimes \omega_{ij}^\beta = \sum_{k=1}^{n} e_k^\alpha \otimes \left(\sum_{i=1}^{n} g_{ki} \cdot \omega_{ij}^\beta\right). \qquad (3.10)$$

Also, applying $\nabla$ on (3.9) and using (3.5), we obtain

$$\nabla(e_j^\beta) = \nabla\left(\sum_{i=1}^{n} g_{ij} \cdot e_i^\alpha\right) = \sum_{i=1}^{n}\left(\sum_{k=1}^{n} e_k^\alpha \otimes \omega_{ki}^\alpha\right) \cdot g_{ij} + e_i^\alpha \otimes \partial\, g_{ij}) =$$

$$\sum_{k=1}^{n} e_k^\alpha \otimes \left(\sum_{i=1}^{n} \omega_{ki}^\alpha \cdot g_{ij}\right) + \sum_{k=1}^{n} e_k^\alpha \otimes \partial\, g_{kj} = \sum_{i=1}^{n} \omega_{ki}^\alpha \cdot g_{ij} + \partial\, g_{kj},$$

which compared with (3.10) implies that

$$\sum_{i=1}^{n} g_{ki} \cdot \omega_{ij}^\beta = \sum_{i=1}^{n} \omega_{ki}^\alpha \cdot g_{ij} + \partial\, g_{kj} \qquad (3.11)$$

and, in turn,

$$g_{\alpha\beta} \cdot \omega^{\beta} = \omega^{\alpha} \cdot g_{\alpha\beta} + \partial\, g_{\alpha\beta}.$$

Taking into account the definition of the adjoint representation $\mathcal{A}d$ and that of the logarithmic differential (see Section 1), we conclude the proof.

Conversely, we prove

**Lemma 3.3.** *Assume that we are given a family*

$$\omega^{\alpha} \in \mathcal{M}_n(\Omega^1)(U_\alpha) \equiv (\mathcal{M}_n(\mathcal{A}) \otimes_{\mathcal{A}} \Omega^1)(U_\alpha) \equiv M_n(\Omega^1(U_\alpha)), \quad a \in I,$$

*(with respect to $\mathcal{C}$), satisfying (3.8). Then there exists a unique $\mathcal{A}$-connection $\nabla$ with local connection matrices the given $\omega^{\alpha}$ 's.*

**Proof.** Motivated by (3.7), we set

$$\nabla^{\alpha}(s) := \sum_{i=1}^{n} e_i^{\alpha} \otimes (\partial\, s_i^{\alpha} + \sum_{j=1}^{n} s_j^{\alpha} \cdot \omega_{ij}^{\alpha}), \quad a \in I, \tag{3.12}$$

for every section of $\mathcal{E}$, expressed locally (over $U_\alpha$) by (3.3). Since, over $U_{\alpha\beta}$,

$$s_i^{\alpha} = \sum_{j=1}^{n} g_{ij} \cdot s_j^{\beta},$$

taking into account (3.8) in its equivalent form (3.11), and formula (3.8), the analogue of (3.12) for $s|U_\beta$ is transformed in the following way:

$$\nabla^{\beta}(s) := \sum_{i=1}^{n} e_i^{\beta} \otimes (\partial\, s_i^{\beta} + \sum_{j=1}^{n} s_j^{\beta} \cdot \omega_{ij}^{\beta}) =$$

$$\sum_{i=1}^{n} [(\sum_{k=1}^{n} g_{ki} \cdot e_k^{\alpha}) \otimes (\partial\, s_i^{\beta} + \sum_{j=1}^{n} s_j^{\beta} \cdot \omega_{ij}^{\beta})] =$$

$$\sum_{k=1}^{n} e_k^{\alpha} \otimes (\sum_{i=1}^{n} g_{ki} \cdot \partial\, s_i^{\beta} + \sum_{i=1}^{n} \sum_{j=1}^{n} s_j^{\beta} \cdot g_{ki} \cdot \omega_{ij}^{\beta}) =$$

$$\sum_{k=1}^{n} e_k^{\alpha} \otimes [\sum_{i=1}^{n} g_{ki} \cdot \partial\, s_i^{\beta} + \sum_{j=1}^{n} s_j^{\beta} \cdot (\sum_{i=1}^{n} \omega_{ki}^{\alpha} \cdot g_{ij} + \partial\, g_{kj})] =$$

$$\sum_{k=1}^{n} e_k^{\alpha} \otimes [(\sum_{i=1}^{n} g_{ki} \cdot \partial\, s_i^{\beta} + \sum_{i=1}^{n} s_i^{\beta} \cdot \partial\, g_{ki}) + \sum_{i=1}^{n} (\sum_{i=1}^{n} g_{ij} \cdot s_j^{\beta}) \cdot \omega_{ki}^{\alpha}] =$$

$$\sum_{i=1}^{n} e_i^{\alpha} \otimes (\partial\, s_i^{\alpha} + \sum_{j=1}^{n} \cdot s_j^{\alpha} \cdot \omega_{ij}^{\alpha}) = \nabla^{\alpha}(s).$$

The previous equalities show that the $\nabla^{\alpha}$ 's determine a connection $\nabla$ on $\mathcal{E}$. The rest of the proof is clear.

Summarizing we obtain the following basic result

**Theorem 3.4.** *Let $(\mathcal{E}, X, \pi_E)$ be a vector sheaf of finite rank n. Then, an $\mathcal{A}$-connection on $\mathcal{E}$ corresponds bijectively to a family of local matrices $\{\omega^{\alpha} \in \mathcal{M}_n(\Omega^1)(U_\alpha) | \alpha \in I\}$ satisfying compatibility condition (3.8), relative to an open cover $\mathcal{C} = \{U_\alpha | \alpha \in I\}$ of X, over which $\mathcal{E}$ is isomorphic to $\mathcal{A}^n$.*

**Remarks.** 1) An analogous proof is given in [6, Vol. 2; Theorem 4.1, p. 116]. An open cover $\mathbb{C}$, as above, is called therein *local frame* (see also [5]).

2) in a more sophisticated way, the family $(\omega^{\alpha})_{\alpha \in I}$ is a zero-cochain $(\omega^{\alpha})_{\alpha \in I} \in C^0(\mathcal{C}, \mathcal{M}_n(\Omega^1))$. Influenced by the classical case, it could legitimately be called the 0-cochain of local connection forms of $\nabla$, if we think of $\Omega^1$ as representing the analogue of differential 1-forms of the classical case.

## 4. THE SHEAF OF FRAMES

In this section we fix a vector sheaf $(\mathcal{E}, X, \pi)$ of rank $n$. We denote by $\mathcal{B}$ the basis of topology on $X$ consisting of all open $V \subseteq X$ such that $V \subseteq U_\alpha$, for some $\alpha \in I$.

If $Iso_{\mathcal{A}|U_\alpha}(\mathcal{A}^n|U_\alpha, \mathcal{E}|U_\alpha)$ is the group of all $\mathcal{A}|U_\alpha$-module isomorphisms, then

$$U \longmapsto Iso_{\mathcal{A}|U}(\mathcal{A}^n|U, \mathcal{E}|U), \tag{4.1}$$

$U$ running in $\mathcal{B}$, is a presheaf.

**Definition 4.1.** *The sheaf $\mathcal{P}(\mathcal{E})$, generated by (4.1), is the* sheaf of frames *of the given vector sheaf $\mathcal{E}$.*

By the very construction, since (4.1) is a complete presheaf, we have the identification (viz. bijection)

$$\mathcal{P}(\mathcal{E})(U_\alpha) \cong Iso_{\mathcal{A}|U_\alpha}(\mathcal{A}^n|U_\alpha, \mathcal{E}|U_\alpha), \tag{4.2}$$

for every $U_\alpha \in \mathcal{C}$. The same identification clearly holds for every $U \in \mathcal{B}$.

On the other hand, for every $U \in \mathcal{B}$, the mapping

$$\delta_U : Iso_{\mathcal{A}|U}(\mathcal{A}^n|U, \mathcal{E}|U) \times \mathcal{GL}(n, \mathcal{A}|U) \to Iso_{\mathcal{A}|U}(\mathcal{A}^n|U, \mathcal{E}|U),$$

given by

$$\delta_U(f, g) \equiv f.g := f \circ g, \tag{4.3}$$

determines an action. Thus, in turn, we get an action of $\mathcal{GL}(n, \mathcal{A})$ on $\mathcal{P}(\mathcal{E})$. Moreover, each isomorphism (4.1), in conjunction with the action $\delta_U$, induces a $\mathcal{GL}(n, \mathcal{A})(U_\alpha)$-equivariant isomorphism

$$\Phi_\alpha : \mathcal{P}(\mathcal{E})(U_\alpha) \to \mathcal{GL}(n, \mathcal{A})(U_\alpha) : f \mapsto \Phi_\alpha(f) := \phi_\alpha \circ f, \tag{4.4}$$

of course after (4.2). Since an analogous isomorphism holds for every $V \subseteq U_\alpha$, we get a $\mathcal{GL}(n, \mathcal{A})|U_\alpha$-equivariant sheaf isomorphism

$$\mathcal{P}(\mathcal{E})|U_\alpha \cong \mathcal{GL}(n, A)|U_\alpha, \quad U_\alpha \in \mathcal{C}. \tag{4.5}$$

The previous considerations in fact prove

**Proposition 4.2.** *The sheaf of frames $\mathcal{P}(\mathcal{E})$ of $\mathcal{E}$ is a principal sheaf (it the sense of [3]), with structure sheaf $\mathcal{GL}(n,\mathcal{A})$. Moreover, it is of type $\mathcal{GL}(\mathcal{A})$, i.e. Condition (4.5) holds.*

It is quite standard to show that $\mathcal{P}(\mathcal{E})$ is fully determined by the 1-cocycle $(g_{\alpha\beta}) \in Z^1$ $\mathcal{C}, \mathcal{GL}(n,\mathcal{A}))$, given by

$$g_{\alpha\beta} = (\Phi_\alpha \circ \Phi_\beta^{-1})(1_{\alpha\beta}), \tag{4.6}$$

where $1_{\alpha\beta}$ is the unit element of

$$\mathcal{GL}(n,\mathcal{A})(U_\alpha\beta) \cong GL(n,\mathcal{A}(U_{\alpha\beta})) \cong Iso_{\mathcal{A}|U_{\alpha\beta}}(\mathcal{A}^n|U_{\alpha\beta}, \mathcal{A}^n|U_{\alpha\beta})$$

(for the last relations cf. also e.g. [5; Vol. II, p. 91]).

**Remark.** Equality (4.6) implies that $(g_{\alpha\beta})$ coincides with the coordinate cocycle of $\mathcal{E}$. Thus, starting precisely with the latter, one can construct (by the general methods of e.g. [3]) a principal sheaf $\mathcal{P}$ of type $\mathcal{GL}(n,\mathcal{A})$ and with structure group $\mathcal{GL}(n,\mathcal{A})$. It can be proved that $\mathcal{P}$ and $\mathcal{P}(\mathcal{E})$ are isomorphic.

For the sake of completeness we also prove that $\mathcal{E}$ is associated with $\mathcal{P}(\mathcal{E})$ in a natural way. To this end, for every $U \in \mathcal{B}$, we define an action of $\mathcal{GL}(n,\mathcal{A})(U)$ on (the right of) $\mathcal{P}(\mathcal{E})(U) \times \mathcal{A}^n(U)$, given by

$$(f,a).g := (f \circ g, g^{-1} \circ a), \quad (f,a) \in \mathcal{P}(\mathcal{E})(U) \times \mathcal{A}^n(U), g \in \mathcal{GL}(n,\mathcal{A})(U).$$

This induces an equivalence relation on $\mathcal{P}(\mathcal{E})(U) \times \mathcal{A}^n(U)$ and determines the quotient

$$Q(U) := \mathcal{P}(\mathcal{E})(U) \times_{\mathcal{G}(U)} \mathcal{A}^n(U),$$

where, for the sake of simplicity, we have set

$$\mathcal{G}(U) := \mathcal{GL}(n,\mathcal{A})(U) \equiv \mathcal{GL}(n,\mathcal{A}|U).$$

Therefore, we obtain the (not necessarily complete) presheaf

$$U \to Q(U), \quad U \in \mathcal{B},$$

which, in turn, generates the quotient sheaf

$$\mathcal{Q} = (\mathcal{P}(\mathcal{E}) \times_X \mathcal{A}^n) \,/\, \mathcal{GL}(n,\mathcal{A}).$$

**Proposition 4.3.** *With the previous notations, $\mathcal{E}$ is isomorphic with the quotient sheaf $Q$.*

**Proof.** For each $U \in \mathcal{B}$, we define the mapping

$$\Psi_U : Q(U) \to \mathcal{E}(U) : [f,a] \longmapsto f \circ a.$$

It is straightforward that $\Psi_U$ is well defined and 1-1. On the other hand, for its surjectivity it suffices to observe that a given section $s \in \mathcal{E}(U)$ coincides with $\Psi_U([(\phi_\alpha|U)^{-1}, (\phi_\alpha|U) \circ s])$, if $U \subseteq U_\alpha$ and $\phi_\alpha$ is the isomorphism (3.1). Therefore $(\Psi_U)_{U \in \mathcal{B}}$ induces the desired isomorphism.

Finally, for later use we define the *natural section* of $\mathcal{P}(\mathcal{E})$, with respect to $\mathcal{C}$,

$$\sigma_\alpha := \Phi_\alpha^{-1}(1_\alpha), \tag{4.7}$$

where $1_\alpha := id|\mathcal{A}^n(U_\alpha)$ and (4.2) is still employed.

## 5. CONNECTIONS ON THE SHEAF OF FRAMES

Recalling from Section 3 that

$$\mathcal{M}_n(\Omega^1) := \mathcal{M}_n(\mathcal{A}) \otimes_{\mathcal{A}} \Sigma^1$$

as well as the action (4.3), we give the following

**Definition 5.1.** *A connection on $\mathcal{P}(\mathcal{E})$ is a morphism of sheaves of sets $D : \mathcal{P}(\mathcal{E}) \to \mathcal{M}_n(\Omega^1)$ such that*

$$D(\sigma \cdot g) = \mathcal{A}d(g^{-1}).D(\sigma) + \tilde{\partial} g, \tag{5.1}$$

*for every $\sigma \in \mathcal{P}(\mathcal{E})(U), g \in \mathcal{GL}(n, \mathcal{A})(U)$ and any $U \subseteq X$ open.*

Evaluating (the induced morphism) $D$ at the natural sections (4.7), we obtain the *local connection matrices* of $D$

$$\theta^\alpha = D(\sigma_\alpha), \quad \alpha \in I. \tag{5.2}$$

**Lemma 5.2.** *The local connection matrices of $D$ satisfy the compatibility condition*

$$\theta^\beta = \mathcal{A}d(g_{\alpha\beta}^{-1}) \cdot \theta^\alpha + \tilde{\partial} g_{\alpha\beta}. \tag{5.3}$$

**Proof.** Direct consequence of equalities (5.1) and $\sigma_\beta = \sigma_\alpha \cdot g_{\alpha\beta}$, under appropriate restrictions on $U_{\alpha\beta} \neq \emptyset$.

Conversely we get

**Lemma 5.3.** *A family of local matrices $\{\theta^\alpha \in \mathcal{M}_n(\Omega^1)(U_\alpha) | \alpha \in I\}$, satisfying (5.3), determines a unique connection $D$ on $\mathcal{P}(\mathcal{E})$ with corresponding local connection matrices precisely the given $\theta^\alpha$'s.*

**Proof.** For any open $U \subseteq X$ and $\sigma \in \mathcal{P}(\mathcal{E})(U)$, there is a unique $g_\alpha \in \mathcal{GL}(n, \mathcal{A})$ $(\mathcal{U} \cap \mathcal{U}_\alpha)$ such that $\sigma = \sigma_\alpha \cdot g_\alpha$ on $\mathcal{U} \cap \mathcal{U}_\alpha$. Since $U = \bigcup_{\alpha \in I}(\mathcal{U} \cap \mathcal{U}_\alpha)$, we define the mapping

$$D_U : \mathcal{P}(\mathcal{E})(U) \to \mathcal{M}_n(\Omega^1)(U)$$

by requiring that (if $\mathcal{U} \cap \mathcal{U}_\alpha \neq \emptyset$)

$$D_U(\sigma)|_{\mathcal{U} \cap \mathcal{U}_\alpha} := \mathcal{A}d(g_\alpha^{-1})|_{\mathcal{U} \cap \mathcal{U}_\alpha} + \tilde{\partial}(g_\alpha|_{\mathcal{U} \cap \mathcal{U}_\alpha}). \tag{5.4}$$

We prove that $D_U$ is indeed well defined by showing that (5.4) coincides on $\mathcal{U} \cap \mathcal{U}_{\alpha\beta}$ with its analogue

$$D_U(\sigma)|_{\mathcal{U}\cap\mathcal{U}_\beta} := \mathcal{A}d(g_\beta^{-1}) \cdot \theta^\alpha|_{\mathcal{U}\cap\mathcal{U}_\beta} + \tilde{\partial}(g_\beta|_{\mathcal{U}\cap\mathcal{U}_\beta}),$$

for any $U_\beta \in \mathcal{C}$ with $\mathcal{U} \cap \mathcal{U}_{\alpha\beta} \neq \emptyset$. This is the case, since $g_\alpha = g_{\alpha\beta} \cdot g_\beta$ on $\mathcal{U} \cap \mathcal{U}_{\alpha\beta}$. Hence, taking into account equalities (2.7) - (2.9) and omitting the restrictions,

$$\mathcal{A}d(g_\beta^{-1}) \cdot \theta^\alpha + \tilde{\partial} g_\beta =$$

$$\mathcal{A}d(g_\alpha^{-1} \cdot g_{\alpha\beta}) \cdot (\mathcal{A}d(g_{\alpha\beta}^{-1}) \cdot \theta^\alpha + \tilde{\partial} g_{\alpha\beta}) + \tilde{\partial}(g_{\alpha\beta}^{-1} \cdot g_\alpha) =$$

$$\mathcal{A}d(g_\alpha^{-1}) \cdot \theta^\alpha + \mathcal{A}d(g_\alpha^{-1}) \cdot \tilde{\partial} g_{\alpha\beta} + \mathcal{A}d(g^{-1}\alpha) \cdot \tilde{\partial} g_{\alpha\beta}^{-1} =$$

$$\mathcal{A}d(g_\alpha^{-1}) \cdot \theta^\alpha + \tilde{\partial} g_\alpha,$$

which proves the assertion.

It is clear that $(D_U)_{U\in\mathcal{C}}$ is a preashef morphism which determines a morphism $D$. It is a connection since, for any $\sigma \in \mathcal{P}(\mathcal{E})(U)$ and $g \in \mathcal{GL}(n, \mathcal{A})(U)$, $\sigma \cdot g = \sigma_\alpha \cdot (g_\alpha \cdot g)$ on $\mathcal{U} \cap \mathcal{U}_\alpha$; thus (5.4) gives (by omitting again the restrictions)

$$D_U(\sigma \cdot g) = \mathcal{A}d(g^{-1} \cdot g_\alpha^{-1}) \cdot \theta^\alpha + \tilde{\partial}(g_\alpha \cdot g) =$$

$$\mathcal{A}d(g^{-1}) \cdot (\mathcal{A}d(g_\alpha^{-1}) \cdot \theta^\alpha + \tilde{\partial} g_\alpha) + \tilde{\partial}(g) =$$

$$\mathcal{A}d(g^{-1}) \cdot D_U(\sigma) + \tilde{\partial}(g).$$

Finally, assume there exists another connection $D'$ such that $D'(\sigma_\alpha) \equiv D'_{U_\alpha}(\sigma_\alpha) = \theta^\alpha$. Then, for each $D'_U$ and $\sigma$ as before,

$$D'_U(\sigma)|_{\mathcal{U}\cap\mathcal{U}_\alpha} = D'|_{\mathcal{U}\cap\mathcal{U}_\alpha}(\sigma_\alpha|_{\mathcal{U}\cap\mathcal{U}_\alpha} \cdot g_\alpha) =$$

$$\mathcal{A}d(g_\alpha^{-1}) \cdot D'|_{\mathcal{U}\cap\mathcal{U}_\alpha}(\sigma_\alpha|_{\mathcal{U}\cap\mathcal{U}_\alpha}) + \tilde{\partial} g_\alpha =$$

$$\mathcal{A}d(g_\alpha^{-1}) \cdot \theta^\alpha|_{\mathcal{U}\cap\mathcal{U}_\alpha} + \tilde{\partial} g_\alpha = D_U(\sigma)|_{\mathcal{U}\cap\mathcal{U}_\alpha}$$

by which we conclude the proof.

Summarizing the above lemmata we state the following principal sheaf counterpart of Theorem 3.4.

**Theorem 5.4.** *Let $\mathcal{P}(\mathcal{E})$ be the sheaf of frames of a vector sheaf $\mathcal{E}$. Then, a connection $D$ on $\mathcal{P}(\mathcal{E})$ corresponds bijectively to a family of local matrices $\{\theta^\alpha \in \mathcal{M}_n(\Omega^1)(U_\alpha) \,|\alpha \in I\}$ satisfying the compatibility condition (5.3), with respect to the open cover $\mathcal{C}$, over which $\mathcal{P}(\mathcal{E})$ is isomorphic to $\mathcal{GL}(n, \mathcal{A})$.*

As a consequence, we obtain the following main result.

**Theorem 5.5.** *There exists a bijection between $\mathcal{A}$-connections on $\mathcal{E}$ and connections $D$ on $\mathcal{P}(\mathcal{E})$.*

**Proof.** It suffices to observe that a family of local matrices satisfying the compatibility condition (3.8) $\equiv$ (5.3) determines, by Theorems 3.3 and 5.4, a connection $\nabla$ on $\mathcal{E}$ and a connection $D$ on $\mathcal{P}(\mathcal{E})$ respectively.

Applying the definition of the local matrices of $\nabla$ (cf. equality (3.5)) and that of $D$ (cf. (5.2)), Theorem 5.5 implies

**Corollary 5.6.** *Over each $U_\alpha \in \mathcal{C}$, the following formula holds:*

$$\nabla(e_j^\alpha) = \sum_{i=1}^{n} e_i^\alpha \otimes (D\sigma_\alpha)_{ij}\ ,\ 1 \leq j \leq n.$$

*where $(D\sigma_\alpha)_{ij}$ is the ij-entry of the matrix $D\sigma_\alpha$.*

## 6. THE CURVATURE IN BRIEF

We shall close this note by a brief discussion on the curvature of a connection $D$, as another example of the effectiveness of the present approach of $\mathcal{A}$-connections. More details can be found in [7] and [9], whereas in [5] and [6] the same subject is treated directly within the vector sheaf context.

In order to be able to define the notion of curvature, both for connections on $\mathcal{E}$ and $\mathcal{P}(\mathcal{E})$, we should extend the initial differential triad to the *curvature datum* (cf. [5], [6])

$$(A, \partial, \Omega^1, d^1, \Omega^2),$$

where $\Omega^2 := \Omega^1 \wedge \Omega^2$ and $d^1 : \Omega^1 \to \Omega^2$ is a $\mathbb{C}$-linear morphism satisfying the following conditions:

$$d^1(s \cdot \theta) = s \cdot d^1(\theta) - \theta \wedge \partial(s), \tag{6.1}$$

for every $s \in A(U), \theta \in \Omega^1(U)$ and open $U \subseteq X$,

$$d^1 \circ \partial = 0. \tag{6.2}$$

**Definition 6.1.** *The curvature of a connection $D$ on $\mathcal{P}(\mathcal{E})$ is the morphism of sheaves of sets $R^D : \mathcal{P}(\mathcal{E}) \to \mathcal{M}_n(\Omega^2) := \mathcal{M}_n(\mathcal{A}) \otimes_{\mathcal{A}} \Omega^2$ given by $R^D := \mathcal{D} \circ D$, if $\mathcal{D} : \mathcal{M}_n(\Omega^1) \to \mathcal{M}_n(\Omega^2)$ is the morphism defined by*

$$\mathcal{D}(\theta) := d^1\theta + \theta \wedge \theta = d^1\theta + 1/2 \cdot [\theta, \theta], \tag{6.3}$$

*for every $\theta \in \mathcal{M}_n(\Omega^1)(U)$ and $U \subseteq X$ open.*

The corresponding *local curvature matrices* are determined by

$$\Theta^\alpha := R^D(\sigma_\alpha) \in \mathcal{M}_n(\Omega^2)(U_\alpha).$$

By the definition of $R^D$ we get the *structural equation*

$$\Theta^\alpha = d^1\theta^\alpha + \theta^\alpha \wedge^\alpha = d^1\theta^\alpha + 1/2.[\theta^\alpha, \theta^\alpha]. \tag{6.4}$$

Furthermore, it is easy to show that

$$\Theta^{\beta} = \mathcal{A}d(g_{\alpha\beta}^{-1}) \cdot \Theta^{\alpha}, \tag{6.5}$$

which is the compatibility condition of the local curvature matrices on the overlappings.

**Lemma 6.2.** *If $D \equiv (\theta^{\alpha})_{\alpha \in I}$ is a connection on $\mathcal{P}(\mathcal{E})$, then its curvature $R^D$ is completely determined by (6.4) and (6.5).*

**Proof.** Modifying the proof of Lemma 5.2, we define the morphism $R : \mathcal{P}(\mathcal{E}) \to \mathcal{M}_n(\Omega^2)$, given by

$$R(\sigma) := \mathcal{A}d(g_{\alpha}^{-1}) \cdot \theta^{\alpha} \text{ on } \mathcal{U} \cap \mathcal{U}_{\alpha},$$

if $\sigma \in \mathcal{P}(\epsilon)(U)$ and $g_{\alpha} \in \mathcal{GL}(n, \mathcal{A})(U_{\alpha})$ with $\sigma = \sigma^{\alpha} \cdot g_{\alpha}$ on $\mathcal{U} \cap \mathcal{U}_{\alpha}$. We readily check that $R = R^D$.

If we interpret now the $\theta^{\alpha}$'s as the local matrices of the connection $\nabla$ corresponding to $D$ (by Theorem 5.5), i.e.

$$\omega^{\alpha} = (\theta_{ij}^{\alpha})_{1 \leq i,j \leq n}, \quad \alpha \in I,$$

and the $\Theta^{\alpha}$'s as matrices of the form

$$\Theta^{\alpha} = (R_{ij}^{\alpha})_{1 \leq i,j \leq n}, \quad R_{ij}^{\alpha} \in \Omega^2(U_{\alpha}),$$

then (6.4) implies that

$$R_{ij}^{\alpha} = d^1 \omega_{ij}^{\alpha} + \sum_{k=1}^{n} (\omega_{ik}^{\alpha} \wedge \omega_{kj}^{\alpha})$$

or, in virtue of (3.6),

$$(R_{ij}^{\alpha}) = d^1 \omega^{\alpha} + \omega^{\alpha} \wedge \omega^{\alpha}. \tag{6.6}$$

Similarly, (6.5) leads to

$$(R_{ij}^{\beta}) = Ad(g_{\alpha\beta}^{-1})((R_{ij}^{\alpha})) := g_{\alpha\beta}^{-1} \cdot (R_{ij}^{\alpha}) \cdot g_{\alpha\beta} \tag{6.7}$$

(note the difference between the adjoint representation $Ad$, acting on matrices as above, and $\mathcal{A}d$ acting on $\mathcal{M}_n(\Omega^1)(U_{\alpha})$ by (2.6)).

Therefore, similarly to the proof of Lemma 3.3, (6.6) and (6.7) determine an $\mathcal{A}$-morphism $R : \mathcal{E} \to \Omega^2(\mathcal{E})$ such that

$$\mathcal{R}(e_j^{\alpha}) = \sum_{i=1}^{n} e_i^{\alpha} \otimes R_{ij}^{\alpha}.$$

It is not hard to show that $\mathcal{R}$ coincides with the curvature $R^{\nabla}$ of $\nabla$, defined otherwise by the *1st prolongation of* $\nabla$ (for details see [5] and [6, Vol. 2; Chapter VIII, Sections 3-4]).

Hence we obtain

**Theorem 6.3.** *Under the bijection of Theorem 5.5, the corresponding curvature operators $R^D$ and $R^{\nabla}$ have the same local curvature matrices, over $\mathcal{C}$, from which they are fully determined by means of (6.4) and (6.5), or their equivalent (6.6) and (6.7).*

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