Note di Matematica Vol. 15 - n. 1, 129-135 (1995)

ON A THEOREM OF GUPTA AND LEVIN

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Let $(R, +, \cdot)$ be an associative ring. For all $a, b \in R$ we set $a \circ b = ab - ba$ and $a * b = a + b + ab$. It is well-known that $(R, +, \circ)$ is a Lie ring and $(R, *)$ is a monoid. The ideals of $(R, +, \circ)$ are called the Lie ideals of $(R, +, \cdot)$. If the ring $(R, +, \cdot)$ has an identity then $(R, *)$ is isomorphic to the monoid $(R, \cdot)$. We denote by $Q(R)$ the set of invertible elements of the monoid $(R, *)$. Obviuosly, $(Q(R), *)$ is a group. For $a \in Q(R)$, the inverse element of a with respect to $*$ is denoted by a^-. In what follows, we denote by **N** the set of positive integers.

It is well-known that if the Lie ring $(R, +, \circ)$ is nilpotent, then the group $(Q(R), *)$ is nilpotent and the nilpotency class of $(Q(R), *)$ is not larger than that of $(R, +, \circ)$. This can be derived by the following theorem of Gupta and Levin [GL]. If $R^{[n]} (n \in \mathbf{N})$ is the two-sided ideal of $(R, +, \cdot)$ generated by the nth term $\gamma_n(R)$ of the lower central series of the Lie ring $(R, +, \circ)$, then

$$\gamma_n(Q(R)) \leq Q(R^{[n]}) \tag{1}$$

for all $n \in \mathbf{N}$. This result has been used in several papers on modular group algebras (see, for example [Sh]).

It was also proved by Laue [L1] that, for all $n \in \mathbf{N}$,

$$\gamma_n(Q(R)) \leq Q(\overline{\gamma_n(R)}) \tag{2}$$

where $\overline{\gamma_n(R)} := \{a | a \in R, a \circ R \subseteq \gamma_{n+1}(R)\}$.

Very little is known about the relationship between the ideals $R^{[n]}$ and the subrings $\overline{\gamma_n(R)}$ of $(R, +, \cdot)$. The main result of this note is Theorem 6 the line of reasoning of which simplifies and unifies the proofs known for (1) and (2).

We need a few preliminary facts. Inductively we set $x_1 \circ \ldots \circ x_n = (x_1 \circ \ldots \circ x_{n-1}) \circ x_n (n > 0)$. If A, B are subset of R, the additive subgroup generated by the elements $a \circ b$ $(a \in A, b \in B)$ is denoted by $A \circ B$. Moreover, we shall write $xy \circ z$ for $(xy) \circ z$ and, inductively, $x \circ_1 R = x \circ R$ and $x \circ_k R = (x \circ_{k-1} R) \circ R$ for all $x, y, z \in R$ and $k \in \mathbf{N}$, $k > 1$.

Let $(V, +)$ be a submonoid of $(R, +)$. We set $P_0(V) = V$ and for each $k \in \mathbf{N}$

$$P_k(V) = \{a | a \in R, \quad a \circ_k R \subseteq V\}$$

Remarks. *(1) For all $a \in R$ we have*

$$a \in P_k(V) \Longleftrightarrow a \circ R \subseteq P_{k-1}(V)$$

[1]The author aknowledge support from the Italian MURST (40%).

(2) If V is a Lie ideal of $(R, +, \cdot)$, then $P_k(V)$ is a Lie ideal of $(R, +, \cdot)$, for all $k \in \mathbf{N}$.
($P_k(V) / V$ is the kth centre of $(R / V, +, \circ)$ in this case).
The following two lemmas are of independent interest.

Lemma 1. *Let $(R, +, \cdot)$ be an associative ring and $(V, +)$ a submonoid of $(R, +)$. For all $k \in \mathbf{N}$ we have*
(1) $(P_k(V), +, \cdot)$ is a subring of $(R, +, \cdot)$,
(2) $Q(P_k(V)) = Q(R) \cap P_k(V)$
Proof. At first we show the case of $k = 1$. Of course $P_1(V)$ is not empty. Moreover, for all $r \in R$, $a, b \in P_1(V)$ we have

$$(a - b) \circ r = a \circ r + b \circ (-r) \in V$$
$$ab \circ r = ba \circ br + b \circ ra \in V$$

This proves *(1)*.
Since $(P_1(V), +, \cdot)$ is a subring of $(R, +, \cdot)$, we have $Q(P_1(V)) \subseteq Q(R) \cap P_1(V)$. Now let $a \in Q(R) \cap P_1(V)$. For all $r \in R$ we have

$$a^- \circ (-r) = a \circ (1 + a^-)r(1 + a^-)$$

Therefore $a^- \in P_1(V)$, proving *(2)*.
Finally, let $k \in \mathbf{N}$, $k > 1$, and put $W = \{b | b \in R, b \circ_{k-1} R \subseteq V\}$. Then $(W, +)$ is a submonoid of $(R, +)$ and $P_k(V) = \{a | a \in R, a \circ R \subseteq W\} = P_1(W)$. Thus, by the first part, the proof is complete.

An important step in the proof of the lemma below is the following equation:

$$\begin{aligned} a(b \circ c) \circ d &= c(b \circ d \circ a) + (b \circ d)(c \circ a) \\ &+ b \circ c \circ da - b \circ dc \circ a \\ &+ b \circ d \circ a \circ c - b \circ c \circ d \circ a \end{aligned} \tag{3}$$

for all $a, b, c, d \in R$ [L, Lemma 2].

Lemma 2. *Let $(R, +, \cdot)$ be an associative ring and $v_1, v_{-1} \in R$ such that $v_1 \circ v_{-1} = 0$. Let V be a Lie ideal of $(R, +, \cdot)$, and suppose that*

$$a \circ v_{(-1)^h} \in V \Longrightarrow v_{(-1)^{h+1}}(a \circ v_{(-1)^h}) \in V$$

for all $a \in R, h \in \{0, 1\}$.
Then, for all $k \in \mathbf{N}$, $a \in P_k(V)$ and $h \in \{0, 1\}$, we have

$$v_{(-1)^{h+1}}(a \circ v_{(-1)^h}) \in P_{k-1}(V)$$

Proof. The proof is by induction on k. Let $k = 1, h \in \{0, 1\}$ and $a \in P_1(V)$. Then, by Remark (1), $a \circ v_{(-1)^h} \in P_0(V) = V$. Hence, by our hypothesis on V, we have $v_{(-1)^{h+1}}(a \circ v_{(-1)^h}) \in V = P_0(V)$.

Let $k > 1$, $h \in \{0, 1\}$ and $a \in P_k(V)$. For all $x \in V$, by (3), we have

$$\begin{aligned} v_{(-1)^{h+1}}(a \circ v_{(-1)^h}) \circ x = {} & v_{(-1)^h}(a \circ x \circ v_{(-1)^{h+1}}) + (a \circ x)(v_{(-1)^h} \circ v_{(-1)^{h+1}}) \\ & + a \circ v_{(-1)^h} \circ xv_{(-1)^{h+1}} - a \circ xv_{(-1)^h} \circ v_{(-1)^{h+1}} \\ & + a \circ x \circ v_{(-1)^{h+1}} \circ v_{(-1)^h} - a \circ v_{(-1)^h} \circ x \circ v_{(-1)^{h+1}} \end{aligned}$$

Thus

$$v_{(-1)^{h+1}}(a \circ v_{(-1)^h}) \circ x \in v_{(-1)^h}(a \circ x \circ v_{(-1)^{h+1}}) + a \circ_2 R + a \circ_3 R$$

Now, $a \in P_k(V)$. By Remark (1), we have $a \circ x \in P_{k-1}(V)$. Hence, by our inductive hypothesis, $v_{(-1)^h}(a \circ x \circ v_{(-1)^{h+1}}) \in P_{k-2}(V)$. Moreover, by Lemma 1 and Remark (1), we have also $a \circ_2 R \subseteq P_{k-2}(V)$. Finally, since V is a Lie ideal of $(R, +, \cdot)$, we have $a \circ_3 R \subseteq P_{k-2}(V)$. Thus $v_{(-1)^{h+1}}(a \circ v_{(-1)^h} \circ x \in P_{k-2}(V)$ and, by Lemma 1 and Remark (1), the proof is complete.

We remark that if k is a non-negative integer and $V = \{0\}$, then $P_k(V)$ is the kth centre of the Lie ring $(R, +, \circ)$. Thus, by our lemma, we obtain Lemma 1 of [L] and the main step of the proof of Lemma 2 of [L].

Now, let $[U]$ be a descending chain of submonoids $U_1 \supseteq U_2 \supseteq \dots$ of $(R, +, \cdot)$. For each $n \in \mathbf{N}$, we set

$$\begin{aligned} P_n[U] &= \bigcap_{i \in \mathbf{N}} \{a | a \in R, \quad a \circ_i R \subseteq U_{n+i}\} \\ &= \bigcap_{i \in \mathbf{N}} P_i(U_{n+i}) \end{aligned}$$

Then $P_1[U] \supseteq P_2[U] \supseteq \dots$

By Lemma 1, we have immediately that

$$(P_n[U], +, \cdot) \textit{ is subring of } (R, +, \cdot), \tag{4}$$

$$Q(P_n[U]) = Q(R) \cap P_n[U] \tag{5}$$

for all $n \in \mathbf{N}$.

Moreover, by Remark (1), we have

$$a \in P_n[U] \Longleftrightarrow a \circ R \subseteq P_{n+1}[U] \cap U_{n+1} \tag{6}$$

for all $a \in R, n \in \mathbf{N}$. In particular, $P_n[U]$ is a Lie ideal of $(R, +, \cdot)$, for all $n \in \mathbf{N}$.

Despite of obvius similarities between Lemma 2 and the next lemma, the proofs show that subtle differences have to be noted.

Lemma 3. *Let $(R, +, \cdot)$ be an associative ring and $v_1, v_{-1} \in R$ such that $v_1 \circ v_{-1} = 0$. Let $[U]$ be a descending chain of submonoids $U_1 \supseteq U_2 \supseteq \dots$ of $(R, +)$ which satisfies the following condition:*

$$\begin{aligned} &\textit{For all} \quad j \in \mathbf{N}, \quad j > 1, \quad h \in \{0, 1\}, \quad u \in U_{j-1} \\ &\qquad u \circ v_{(-1)^h} \in U_j \Longrightarrow v_{(-1)^{h+1}}(u \circ v_{(-1)^h}) \in U_j. \end{aligned} \tag{7}$$

Then, for all $n \in \mathbf{N}$, $a \in P_n[U]$ and $h \in \{0,1\}$, we have

$$v_{(-1)^{h+1}}(a \circ v_{(-1)^h}) \in P_{n+1}[U]$$

Proof. At first we show, by induction on i ($i \in \mathbf{N}$), the following:

$$\text{For all } b, r_1, \dots r_i, w_1, w_{-1} \in R \text{ such that } w_1 \circ w_{-1} = 0 \text{ we have} \tag{8}$$

$$w_1(b \circ w_{-1}) \circ r_1 \circ \dots \circ r_1 \in w_{(-1)^i}(b \circ r_1 \circ \dots \circ r_1 \circ w_{(-1)^{i+1}}) + b \circ_{i+1} R + b \circ_{i+2} R.$$

For $i = 1$, (8) follows from the equation (3). For the inductive step we assume (8) for $i \in \mathbf{N}$ and we set $r = b \circ r_1 \circ \dots \circ r_i$. Then

$$\begin{aligned} w_1(b \circ w_{-1}) \circ r_1 \circ \dots \circ r_i \circ r_{i+1} &\in w_{(-1)^i}(r \circ w_{(-1)^{i+1}}) \circ r_{i+1} + b \circ_{i+2} R + b \circ_{i+3} R \\ &\subseteq w_{(-1)^{i+1}}(r \circ r_{i+1} \circ w_{(-1)^i}) + r \circ_2 R + r \circ_3 R + b \circ_{i+1} R + b \circ_{i+3} R \\ &\subseteq w_{(-1)^{i+1}}(b \circ r_1 \circ \dots \circ r_{i+1} \circ w_{(-1)^{i+2}}) + b \circ_{i+2} R + b \circ_{i+3} R \end{aligned}$$

Now, let $h \in \{0,1\}$, $n \in \mathbf{N}$ and $a \in P_n[U]$. For all $i \in \mathbf{N}$, $r_1, \dots r_i \in R$ we have

$$a \circ r_1 \circ \dots \circ r_i \in U_{n+i}, \quad a \circ r_1 \circ \dots \circ r_i \circ v_{(-1)^{h+i+1}} \in U_{n+i+1}$$

Then, by our hypothesis on $[U]$, we obtain

$$v_{(-1)^{h+i}}(a \circ r_1 \circ \dots \circ r_i \circ v_{(-1)^{h+i+1}} \in U_{n+i+1}$$

Moreover, also $a \circ_{i+1} R \subseteq U_{n+i+1}$ and $a \circ_{i+2} R \subseteq U_{n+i+2} \subseteq U_{n+i+1}$.
Thus, by (8), we have

$$v_{(-1)^{h+1}}(a \circ v_{(-1)^h}) \circ r_1 \circ \dots \circ r_i \in U_{n+i+1}$$

The proof is complete.

From Lemma 3 and (6) we conclude immediately the following

Corollary 4. *Let $(R, +, \cdot)$ be an associative ring, $v \in Q(R)$ and $[U]$ a descending chain of submonoids $U_1 \supseteq U_2 \supseteq \dots$ of $(R, +)$ which satisfies (7) with $v_1 := v$ and $v_{-1} := v^-$. Then $(P_n[U] / P_{n+1}[U], +)$ is centralized by v (acting by conjugation), for all $n \in \mathbf{N}$ (i.e., in this case: for all $u \in U_{j-1}$, $u \circ v \in U_j$ implies that $v^- * u * v - u \in U_j$, and $u \circ v^- \in U_j$ implies that $v * u * v^- - u \in U_j$).*

We remark that evidently any descending chain of left ideals of $(R, +, \cdot)$ satisfies (7), for all $v_1, v_{-1} \in R$ such that $v_1 \circ v_{-1} = 0$. Moreover, we remark that also the lower central series of the Lie ring $(R, +, \circ)$ satisfies (7), for all $v_1, v_{-1} \in R$ such that $v_1 \circ v_{-1} = 0$. This is an immediate consequence of following proposition.

Proposition 5. *[L1, Prop.2(ii)] Let $(R, +, \cdot)$ be an associative ring. If $v, v' \in R$ such that $\gamma_n(R)(v \circ v') \subseteq \gamma_{n+1}(R)$ for all $n \in \mathbf{N}$, then $v'(\gamma_n(R) \circ v) \subseteq \gamma_{n+1}(R)$ for all $n \in \mathbf{N}$,*

Proof. We observe that, for all $v, v', w \in R$,

$$\begin{aligned} v'(w \circ v) &= v'w \circ v + (v \circ v')w \\ &= v'w \circ v + w(v \circ v') + v \circ v' \circ w \end{aligned}$$

which settles the case of $n = 1$. Now let $n > 1$ and $w \in \gamma_n(R)$. We have to show that $v'(w \circ v) \in \gamma_{n+1}(R)$ and may assume that $w = w^* \circ z$ for some $w^* \in \gamma_{n-1}(R)$, $z \in R$. By (6), we have

$$\begin{aligned} &v'(w^* \circ z \circ v) + v(w^* \circ v') \circ z + (w^* \circ z)(v \circ v') \\ &\quad + w^* \circ zv' \circ v - w^* \circ v' \circ zv + w^* \circ v' \circ z \circ v - w^* \circ z \circ v \circ v'. \end{aligned}$$

Inductively, we know that the first term on the right-hand side is contained in $\gamma_{n+1}(R)$. By our hypothesis, this holds also for the second term, and trivially for the remaing ones. The proof is complete.

Our principal result is the following:

Theorem 6. *Let $(R, +, \cdot)$ be an associative ring and $[U]$ a chain of submonoids of $(R, +)$ satisfying (7), for all $v_1, v_{-1} \in R$ such that $v_1 * v_{-1} = 0$.*
If $Q(R) \subseteq P_1[U]$, we have

$$\gamma_n(Q(R)) \geq Q(P_n[U])$$

for all $n \in \mathbf{N}$.

Proof. The proof is by induction on n. The case $n = 1$ is trivial. Let $n \in \mathbf{N}$, and assume that $\gamma_n(Q(R)) \leq Q(P_n[U])$. In order to prove that $\gamma_{n+1}(Q(R)) \leq Q(P_{n+1}[U])$, it sufficies to show, in view of Lemma 1, that $[a, b] \in P_{n+1}[U]$ for all $a \in \gamma_n(Q(R))$, $b \in Q(R)$. Now, for all $d \in R$, we have

$$\begin{aligned} [a, b] \circ d &= (1 + a^-)(1 + b^-)(a \circ b) \circ d \\ &= (1 + b^-)(a \circ b) \circ (1 + a^-)d \\ &+ a^- \circ (1 + b^-)(a \circ b) \circ d \\ &+ a \circ d(1 + a^-) \circ d(1 + b^-) - a \circ b \circ (1 - a^-)d(1 + b^-) \end{aligned}$$

[Du, Lemma 3]
By the inductive hypothesis, we have $a \in P_n[U]$. Thus, by Corollary 4, $(1+b^-)(a \circ b) \in P_{n+1}[U]$ and, by (6), $(1 + b^-)(a \circ b) \circ (1 + a^-)d \in U_{n+2} \cap P_{n+2}[U]$.
By our inductive hypothesis, this holds also for the remaining terms. Thus, by (6), the proof is complete.

Corollary 7. *Let $(R, +, \cdot)$ be an associative ring. For all $n \in \mathbf{N}$ we have*

$$\gamma_n(Q(R)) \leq Q(\overline{\gamma_n(R)})$$

Proof. Let $[U]$ be the descending chain of submonoids $U_1 \supseteq U_2 \supseteq \dots$ of $(R, +)$ defined by $U_j := \gamma_j(R)$, for all $j \in \mathbf{N}$. By Proposition 5, the chain $[U]$ satisfies (7), for all $v_1, v_{-1} \in R$

such that $v_1 * v_{-1} = 0$. Moreover, $P_n[U] = \overline{\gamma_n(R)}$, for all $n \in \mathbf{N}$. Thus, by Theorem 6, the proof is complete.

Corollary 8. *Let $(R, +, \cdot)$ be an associative ring. For all $n \in \mathbf{N}$ we have*

$$\gamma_n(Q(R)) \leq Q(R^{[n]})$$

Proof. Let $[U]$ be the descending chain of submonoids $U_1 \supseteq U_2 \supseteq \dots$ of $(R, +)$ defined by $U_j =: R^{[j]}$, for all $j \in \mathbf{N}$. Evidently the chain $[U]$ satisfies (7), for all $v_1, v_{-1} \in R$ such that $v_1 * v_{-1} = 0$. We show that $\gamma_n(Q(R)) \subseteq U_n$, for all $n \in \mathbf{N}$. It suffices to show that $[a, b] \in U_n$, for all $a \in \gamma_{n-1}(Q(R))$ $(n > 1)$, $b \in Q(R)$. Now, since $a \in \gamma_{n-1}(Q(R))$, Theorem 6 implies that $a \in P_{n-1}[U]$. Hence $a \circ b \in U_n$. Therefore $[a, b] = (1 + a^-)(1 + b^-)(a \circ b) \in U_n$. Thus the proof is complete.

REFERENCES

[B] Y.A. BAHTURIN, *Identical Relations in Lie Algebras*, VNU Science Press, Utrecht, 1987.
[Du] X. DU, *The centers of a radical ring*, Canad. Math. Bull. 35 (1992), 174-179.
[GL] N.D. GUPTA, F. LEVIN, *On the Lie ideals of a ring*, J. Algebra 81 (1983), 225-231.
[L] H. LAUE, *On the associated Lie ring and the adjoint group of a radical ring*, Canad. Math. Bull. 27 (1984), 215-222.
[L1] H. LAUE, *On central chains in rings*, Preprint no. 6, University of Lecce, 1990.
[Sh] A. SHALEV, *The nilpotency class of the unit group of a modular group algebra III*, Arch. Math. (Basel) 60 (1993), 136-145.

Received October 10, 1995
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