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ON THE ESTIMATE THE L_2 AND L_∞ NORM OF THE ERROR FOR THE INITIAL VALUE PROBLEMS

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Abstract. *Our objective in this paper is to examine conditions under which the variational principle has a global extremal properties and to complete the error estimate procedure of [5]. We will use the value of the functional to estimate the L_2 and L_∞ norm of the error for the initial value problems, described by a non-linear second order differential equation belonging to the class treated in [5]. In our analysis integral inequalities given in [1-4] are of central importance. The analysis will be illustrated by an concrete example.*

1. INTRODUCTION

In our earlier paper [5] we derived an extremum variational principle for a class of boundary value problems described by a second order non-linear differential equation. The main characteristics of this variational principle are: a) the value of the functional on the exact solution is equal to zero; b) its stationary and extremal properties are independent of the boundary conditions and therefore are unchanged if the initial instead of boundary conditions are prescribed. Also, in [5] an error estimate procedure (for estimating the L_2 norm of the error) for boundary value problems was presented. All considerations in [5] are based on local extremal properties of the variational principle.

In this paper, the conditions under which the variational principle has a global extremal properties are examined and the error estimate procedure of [5] is completed. We will use the value of the functional to estimate the L_2 and L_∞ norm of the error for the initial value problems described by a non-linear second order differential equation belonging to the class treated in [5]. The analysis will be illustrated by an concrete example.

2. THE VARIATIONAL PRINCIPLE

Let us consider the following second order nonlinear differential equation

$$\ddot{q} - F(q,t) = 0, \quad t \in (a,b) \tag{1}$$

subject to the initial conditions

$$q(a) = c_1, \quad \dot{q}(a) = c_2. \tag{2}$$

In (1) and (2) a, b, c_1 and c_2 are arbitrary constants. We assume that the function F and the initial conditions (2) are such that the solution of the problem exists and is unique. For the problem described by the differential equation (1) a variational principle was constructed in [5]. If we set

$$f(q,t) = \int^q F(y,t)dy \tag{3}$$

and use $F^{-1}(z,t)$ to denote the solution of the equation $F(q,t) = z$ with respect to q, then from [5] we conclude that the following functional

$$I(q) = \int_a^b [\dot{q}^2 + f(q,t) + \ddot{q}F^{-1}(\ddot{q},t) - f(F^{-1}(\ddot{q},t),t]dt - (\dot{q}q)_a^b, \tag{4}$$

is stationary on the exact solution q of equation (1) with initial conditions (2). Moreover $I(q) = 0$. The above statement could be easily checked by calculating the first δI variation of (4).

3. ERROR ESTIMATE PROCEDURE

Let $Q = q + \delta q$ be an approximate solution to the problem (1), (2). Here, we must underline a very important fact that in our analysis δq can be a quantity of any finite magnitude. It is not necessary that δq be a small quantity. We assume that Q satisfies exactly the initial conditions (2) of the problem. This implies that

$$\delta q(a) = 0, \quad \delta\dot{q}(a) = 0, \tag{5}$$

is satisfied. The value of the functional (4) on Q could be written as

$$I(Q) = I(q) + \delta I(q, \delta q) + \delta^2 I(\Psi, \delta q), \tag{6}$$

where

$$\Psi = q + \varepsilon(Q - q), \quad 0 \le \varepsilon \le 1. \tag{7}$$

Using the fact that $I(q) = 0$ and that I is stationary at q, i.e., $\delta I(q, \delta q) = 0$ and calculating the second variation of (4) we get

$$I(Q) = \frac{1}{2}\int_a^b [A(\Psi)\delta\ddot{q}^2 - 2\delta q\delta\ddot{q} + C(\Psi)\delta q^2]dt. \tag{8}$$

In (8) the form of functions $A(\Psi)$ and $C(\Psi)$ depends on F and F^{-1} and could be determinated in each specified case.

i) Case 1. Suppose that $A(\Psi) > 0$. For Ψ of the form (7) suppose also that there exist numbers D_1, D_2 (which depends on Q so we sometimes denote them by $D_1 = D_1(Q)$, $D_2 = D_2(Q)$) satisfying

$$A(\Psi) \ge D_1(Q) > 0, \tag{9}$$

$$C(\Psi) \ge D_2(Q). \tag{10}$$

We transform now the second term in (8) by using the following inequality [3]

$$\int_a^b |\delta_q \delta_{\ddot{q}}| dt \le \frac{(b-a)^2}{(24)^{1/2}} \int_a^b \delta\ddot{q}^2 dt. \tag{11}$$

With (9) - (11) and equation (8) becomes

$$2I(Q) \geq \left[D_1 - \frac{1}{\sqrt{6}}(b-a)^2\right] \|\delta\ddot{q}\|_{L_2}^2 + D_2\|\delta q\|_{L_2}^2. \tag{12}$$

To obtain the relation between the L_2 norm of δq and $\delta\ddot{q}$ we use the inequality of Troesch [2, (1.1)]

$$\int_0^1 h(t)\dot{f}^2 dt \geq \frac{\pi^2}{4}\int_0^1 h dt \int_0^1 f^2 dt, \tag{13}$$

which is applicable because of (5). By substituting $h = 1, f = \delta\dot{q}$ and $h = 1, f = \delta q$ in (13) it follows that

$$\|d\ddot{q}\|_{L_2} \geq \frac{\pi}{2(b-a)}\|d\dot{q}\|_{L_2}, \tag{14}$$

$$\|d\dot{q}\|_{L_2} \geq \frac{\pi}{2(b-a)}\|dq\|_{L_2}. \tag{15}$$

Using (14) and (15) in (12) the bound on $\|\delta q\|_{L_2}$ becomes

$$\|\delta q\|_{L_2} \leq \left\{\frac{2I(Q)}{D_2 + \left[D_1 - \frac{(b-a)^2}{\sqrt{6}}\right]\left[\frac{\pi}{2(b-a)}\right]^4}\right\}^{1/2}, \tag{16}$$

which is valid if the denominator is positive.

To estimate the L_∞ norm of δq

$$\|\delta q\|_{L_\infty} = \sup_{t\varepsilon(a,b)} |\delta q(t)| \tag{17}$$

we use first the following inequality [3]

$$\|\delta\dot{q}\|_{L_2}^2 \leq -\varepsilon\|\delta\ddot{q}\|_{L_2}^2 + K(\varepsilon)\|\delta q\|_{L_2}^2, \tag{18}$$

where $\varepsilon > 0$ is an arbitrary constant and $K(\varepsilon) = (1/\varepsilon) + 12/(b-a)^2$. From (18) it follows that

$$\|\delta q\|_{L_2}^2 \geq \frac{1}{K(\varepsilon)}[\|d\dot{q}\|_{L_2}^2 - \varepsilon\|d\ddot{q}\|_{L_2}^2]. \tag{19}$$

Suppose now that, instead of (9) and (10) we have the following estimates

$$A(\Psi) \geq \hat{D}_1(Q, m) > 0, \tag{20}$$

$$C(\Psi) \geq \hat{D}_2(Q, m). \tag{21}$$

In (20) and (21) $\hat{D}_1$ and $\hat{D}_2$ are constants depending on Q, its derivatives and $m = \|\delta q\|_{L_\infty}$. Using (11), (14) and (19) - (21), relation (8) becomes

$$2I(Q) \geq \left\{\left[\frac{\pi}{2(b-a)}\right]^2\left[\hat{D}_1 - \frac{1}{\sqrt{6}}(b-a)^2\right] + \right.$$

$$+\frac{\hat{D}_2}{K(\varepsilon)}\left[1-\varepsilon\left(\frac{\pi}{2b-2a}\right)^2\right]\Bigg\}\|d\dot{q}\|^2_{L_2} \tag{22}$$

which is valid for further application if the right hand side is positive.

However, by Cauchy inequality and (5)

$$\|\delta q\|_{L_\infty} \geq (b-a)^{1/2}\|d\dot{q}\|_{L_2}. \tag{23}$$

Therefore from (22) and (23) we have

$$m^2\left\{\left[\frac{\pi}{2(b-a)}\right]^2\left[\hat{D}_1(m)-\frac{(b-a)^2}{\sqrt{6}}\right]+\right.$$

$$\left.+\frac{\hat{D}_2(m)}{K(\varepsilon)}\left[1-\varepsilon\left(\frac{\pi}{2b-2a}\right)^2\right]\right\} \leq 2I(Q)(b-a),$$

$$m = \|\delta q\|_{L_\infty}. \tag{24}$$

Remark 1. In principle, for a particular problem, (24) can furnish us with: a) an upper and a lower bound on $\|\delta q\|_{L_\infty}$; b) only an upper bound; c) only a lower bound. Also, it is possible that no bound on $\|\delta q\|_{L_\infty}$ follows from (24). If $\hat{D}_1$ and $\hat{D}_2$ in (20) and (21) do not depend on m, then (24) explicity gives an upper bound on $\|\delta q\|_{L_\infty}$.

Remark 2. Since (18) is valid for any $\varepsilon > 0$, we may choose it in the optimal way, i.e., so as to make the upper bound on $\|\delta q\|_{L_\infty}$, that follows from (24), minimal.

ii) Case 2. Suppose that $A(\Psi) < 0$ and there exist two constants (depending on Q and its derivatives) such that

$$-A(\Psi) \geq D_3(Q) > 0, \tag{25}$$

$$-C(\Psi) \geq D_4(Q). \tag{26}$$

Using the same procedure as in Case 1 we get

$$\|\delta q\|_{L_2} \leq \left\{\frac{-2I(Q)}{D_4+\left[D_3-\frac{(b-a)^2}{\sqrt{6}}\right]\left[\frac{\pi}{2(b-a)}\right]^4}\right\}^{1/2}. \tag{27}$$

Also, if instead of (20) and (21), we have

$$-A(\Psi) \geq \hat{D}_3(Q,m) > 0, \tag{28}$$

$$-C(\Psi) \geq \hat{D}_4(Q,m), \tag{29}$$

then

$$m^2\left\{\left[\hat{D}_3(m)-\frac{(b-a)^2}{6}\left[\frac{\pi}{2(b-a)}\right]^2+\right.\right.$$

$$+\frac{\hat{D}_4(m)}{K(\varepsilon)}\left[1-\varepsilon\left(\frac{\pi}{2b-2a}\right)^2\right]\Bigg\}\leq -2I(Q)(b-a),$$

$$m=\|\delta q\|_{L_\infty}. \tag{30}$$

Estimates (27) and (30) are useful if the denominator in (27) and the left hand side in (30) are positive.

The inequality (30) can be used for error estimates in the same way as it is indicated by Remarks 1 and 2. Only, (24), $\hat{D}_1$ and $\hat{D}_2$ must be replaced by (30), $\hat{D}_3$ and $\hat{D}_4$, respectively.

4. GLOBAL EXTREMAL PROPERTIES

From the previous analysis, it is obvious that the inequalities (16) and (24) can be written as

$$I(Q)\geq 0 \tag{31}$$

while (27) and (30) yields

$$I(Q)\leq 0. \tag{32}$$

Therefore, and remembering that: a) the first variation of the variational principle (4) is zero for exact solution q of the equations (1) and (2); b) $I(Q)$ is equal to zero; c) the trial function Q is arbitrary and it is not necessary that the Q is close to the exact solution q; we can formulate following theorems:

Theorem A. *If conditions (9) and (10) are satisfied and the denominator in (16) is positive, or if conditions (20) and (21) are satisfied and the left hand side in (24) is positive then the varitational principle (4) has a global minimum at $Q=q$.*

Theorem B. *If conditions (25) and (26) are satisfied and the denominator in (27) is positive, or if conditions (28), (29) are satisfied and the left hand side in (30) is positive then the variational principle (4) has a global maximum at $Q=q$.*

Remark 3. Sometimes conditions of the Theorem A or B are not satisfied for an arbitrary class of functions Q, but for some subclass of functions Q (for example: bounded functions, monotonically decreasing functions, etc.). In such cases we have the global extremal properties of the variational principle (4) in the appropriate subclass of trial functions Q. The subclass is defined by some general property of solution to the equations (1) and (2).

5. AN EXAMPLE

In that follows we consider the nonlinear electric oscillator

$$\ddot{q}+kq^3=0,\quad t\in(0,b), \tag{33}$$

$$q(0)=q_0,\quad \dot{q}(0)=0, \tag{34}$$

where $k>0$, q_0 and b are constants. The value for b we will choose later. The functional (4) in this case is

$$I=\int_0^b\left[\dot{q}^2-\frac{k}{4}q^4-\frac{3}{4}k^{-1/3}(-\ddot{q})^{4/3}\right]dt-(q\dot{q})_0^b. \tag{35}$$

The functions $A(\Psi)$ and $C(\Psi)$ in the present case are

$$A(\Psi) = -\frac{1}{3k^{1/3}}(-\ddot{\Psi})^{-2/3}, \quad C(\Psi) = -3k\Psi^2. \tag{36}$$

We take now the approximate solution in the subclass of monotonically decreasing functions in our interval of consideration as

$$Q = q_0 \cos \omega t, \tag{37}$$

where ω is constant to be determined and choose $b = \pi/2\omega$. With this value for b and (37), equation (35) becomes

$$I(\omega) = \frac{1}{\omega}\left[q_0^2\omega^2\frac{\pi}{4} - \frac{k}{4}q_0^4\frac{3\pi}{16} - \frac{3\cdot 2^{1/3}}{4k^{1/3}}q_0^{4/3}\cdot\omega^{8/3}B(7/6, 7/6)\right] \tag{38}$$

where $B(x, y) = \Gamma(x)\Gamma(y)/\Gamma(x + y)$, Γ being the Euler Gamma function. Extremizing (38) with respect to ω, we get

$$\omega = 0.82527 \cdot q_0 k^{1/2}, \quad I(\omega) = -2.624 \cdot 10^{-2} q_0^3 k^{1/2}. \tag{39}$$

Using the fact (that follows from the following first integral $kq^4 = kq_0^4 - 2\dot{q}^2$ of the equation (33)) that $q(t) \leq q_0$, assuming that $Q(t) \leq q_0$, and (37), we get the following estimates for $A(\Psi)$ and $C(\Psi)$

$$-A(\Psi) \geq \hat{D}_3 = \frac{1}{3kq_0^2}, \quad C(\Psi) \geq \hat{D}_4 = 0. \tag{40}$$

In this case, we see that the inequality (30) is useful for

$$b = \lambda/\omega, \quad \textit{where} \quad \lambda < 0.7457. \tag{41}$$

Then the error estimate follows from (30)

$$\|\delta q\|_{L_\infty} \leq \left\{\frac{-24\lambda^3\bar{I}(Q,\lambda)}{q_0 k^{1/2}\pi^2(0.82527)^3\left[1 - \frac{3\lambda^2}{\sqrt{6}(0.82527)^2}\right]}\right\}^{1/2} \tag{42}$$

where $\bar{I}(Q, \lambda)$ must be calculated from (35) for $b = \lambda/\omega$ and for selected λ. For example, for $\lambda = 0.7$ the final result is

$$\|\delta q\|_{L_\infty} \leq 0.18764 \cdot q_0. \tag{43}$$

In the time interval $t \in [0, b]$, where (41) is satisfied, conditions of the Theorem B are also satisfied. Hence, the variational principle (35) has a global maximum on the exact solution of the equations (33) and (34) in the subclass of monotonically decreasing functions satisfying (34) and in the time interval $t \in [0, b]$.

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