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DEFORMATIONS OF LEGENDRE CURVES

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1. INTRODUCTION

In contact geometry $\mathbb{R}^3$ with the standard Darboux form $\eta = \frac{1}{2}(dz - ydx)$ and Sasakian metric $g = \frac{1}{4}(dx^2 + dy^2) + \eta \otimes \eta$ is a central example. Sectional curvatures of plane sections containing the characteristic vector field $2\frac{\partial}{\partial z}$ are equal to $+1$ and sectional curvatures of planes orthogonal to the characteristic vector field are equal to -3; for this reason we denote this Sasakian manifold by $\mathbb{R}^3(-3)$. This is also the contact structure on the Heisenberg group, $\left\{ \begin{pmatrix} 1 & y & z \\ 0 & 1 & x \\ 0 & 0 & 1 \end{pmatrix} | x, y, z \in \mathbb{R} \right\} \simeq \mathbb{R}^3$, both η and g being left invariant. In this paper we first study deformations of Legendre curves in $\mathbb{R}^3(-3)$ in the direction of the principal normal, especially 2-minimal curves. In particular we show that 2-minimal Legendre curves arise from 2-minimal curves in the Euclidean plane which were characterized in [4]. For deformations of curves in Euclidean 3-space in the direction of the principal normal see [5].

Our result has an application to the theory of 2-minimal curves in the Euclidean plane. In [4] it was shown that closed 2-minimal curves have self-intersections and we show here that moreover the algebraic area of a closed 2-minimal planar curve in zero.

To prove our main result we need a lemma on Bessel functions which may be of independent interest and we devote Section 4 of the paper to this lemma.

In the last section of this paper we consider deformations of curves in a general K- contact manifold in the direction of the characteristic vector field. Critical curves for this variational problem are the so called C-loxodromes [6].

2. CONTACT MANIFOLDS

By a contact manifold we mean a C^∞ manifold M^{2n+1} together with a 1-form η such that $\eta \wedge (d\eta)^n \neq 0$. It is well known that given η there exists a unique vector field ξ such that $d\eta(\xi, X) = 0$ and $\eta(\xi) = 1$, called the *characteristic vector field.* A classical theorem of Darboux states that on a contact manifold there exist local coordinates with respect to which $\eta = dz - \sum_{i=1}^n y^i dx^i$. Roughly speaking the meaning of the contact condition, $\eta \wedge (d\eta)^n \neq 0$, is that the contact subbundle (i.e. the bundle of $2n$-planes annihilated by η) is as far from being integrable as possible. In particular the maximum dimension of an integral submanifold is only n. From the Darboux theorem it is clear that n-dimensional integral submanifolds exist, namely, those given by x^i = constant, z = constant. A 1-dimensional integral submanifold is called a *Legendre curve*, especially to avoid confusion with an integral curve of the vector field ξ.

A Riemannian metric g is an *associated metric* to a contact structure η if there exists a tensor field ϕ of type (1,1) satisfying

$$\phi^2 = -I + \eta \otimes \xi, \quad \eta(X) = g(X, \xi), \quad d\eta(X, Y) = g(X, \phi Y).$$

We refer to (η, g) or (ϕ, ξ, η, g) as a *contact metric structure*, If ξ is a Killing vector field with respect to g, the contact metric structure is called a *K-contact* structure. It is well known that on a K-contact manifold

$$\nabla_X \xi = -\phi X, \tag{2. 1}$$

where ∇ denotes the Levi-Civita connection of g. The space $\mathbb{R}^3(-3)$ discussed above is K-contact. For a general reference to the ideas of this section see [2].

In the space $\mathbb{R}^3(-3)$ discussed in the introduction, closed Legendre curves have an interesting elementary property which we now state.

Area Property of Closed Legendre Curves. The projection γ^* of a closed Legendre curve γ in $\mathbb{R}^3(-3)$ to the xy-plane must have self-intersections; moreover the algebraic area enclosed in zero.

Since $dz - ydx = 0$ along γ, this follows from the elementary formula for the area enclosed by a curve given by Green's theorem,

$$0 = -\int_\gamma dz = \int_{\gamma^*} -ydx = area,$$

the area being + for γ^* traversed counterclockwise and - for clockwise.

One of the results of [1] is the following.

Theorem. *The curvature of a Legendre curve in* $\mathbb{R}^3(-3)$ *is equal to twice the curvature of its projection to the xy-plane with respect to the Euclidean metric.*

3. k-DEFORMATIONS AND k-MINIMALITY

The theory of k-deformations, k-minimality and k-stability was developed in [4] and we briefly review this theory here. Let M be a compact Riemannian manifold and Δ the Laplacian acting on the space $C^\infty(M)$ of C^∞ functions on M. Define a metric on $C^\infty(M)$ by $(f, g) = \int_M fgdA$ where dA is the volume form on M. It is well known that Δ is a self-adjoint operator which has an infinite discrete sequence of eigenvalues $0 = \lambda_0 < \lambda_1 < \lambda_2 < \ldots \nearrow +\infty$. For each $i \in \mathbb{N}$ the eigenspace V_i of λ_i is finite dimensional; V_0 is 1-dimensional and consists of constant functions. The eigenspaces are mutually orthogonal and their sum is dense in $C^\infty(M)$. Therefore one can make a spectral decomposition $f = f_0 + \sum_{i=1}^{\infty} f_i$, for each real C^∞ function f on M, where f_0 is a constant and $\Delta f_i = \lambda_i f_i$ for $i > 0$. The set $T(f) = \{i \in \mathbb{N}_0 | f_i \neq 0\}$ is called the type of f and f is of *finite type* if $T(f)$ is a finite set.

The subject of the study in [4] was compact oriented hypersurfaces $x : M^n \longrightarrow N^{n+1}$ isometrically immersed in a Riemannian manifold N^{n+1}. For a unit vector field ζ, usually normal, defined on M, define a deformation by

$$\exp_{x(p)} tf(p)\zeta(p), \quad p \in M, \quad t \in (-\epsilon, \epsilon)$$

and in [4] the area functional for these deformations was studied. Here we are concerned with closed curves $\gamma : [0, L] \to M$ in a Riemannian manifold M parametrized by arc length and we study the length integral $L(t)$ under various deformations.

For each $q \in \mathbb{N}_0$, let $\mathcal{F}_q$ be the class of all deformations (in direction ζ) associated to functions $f \in \sum_{i \geq q} V_i$. Clearly $\mathcal{F}_0 \supset \mathcal{F}_1 \supset \mathcal{F}_2 \supset \ldots$. A deformation in $\mathcal{F}_k$ is called a *k-deformation.* A closed curve γ is said to be *k-minimal* if $L'(0) = 0$ for all deformations in $\mathcal{F}_k$. If γ is k-minimal, we say that γ is *ℓ-stable*, $\ell \geq k$, if $L''(0) \geq 0$ for all deformations in $\mathcal{F}_\ell$. One of the results of [4] is that every compact k-minimal hypersurface is q-stable for some $q \geq k$; it was also shown that a k-minimal plane curve is k-stable.

Consider a closed plane curve of length 2π. The Laplacian is just $-\frac{d^2}{ds^2}$, the eigenvalues are $\lambda_n = n^2$ and a basis of the corresponding eigenspace is given by $\{\cos ns, \sin ns\}$. By Lemma 4.1 of [4], a closed plane curve is k-minimal if and only if its curvature κ is of finite type $< k$; in particular

$$\kappa(s) = a_0 + \sum_{n=1}^{k-1} \{a_n \cos ns + b_n \sin ns\}.$$

We now state the following result from [4].

Theorem. *For each zero $j_{\ell,m}$ of the Bessel function J_ℓ of order ℓ, the curve $\gamma_{\ell,m}$ defined by*

$$\gamma_{\ell,m} = \left(\int_0^s \cos(\ell u + j_{\ell,m} \sin u) du, \int_0^s \sin(\ell u + j_{\ell,m} \sin u) du \right)$$

is a closed 2-minimal curve. Conversely up to rigid motions every 2-minimal plane curve can be obtained in this way.

4. A LEMMA ON BESSEL FUNCTIONS

In this section we prove a formula involving Bessel functions which is not found in the treatise of Watson [7] and seems to be new.

Lemma. $\sum_{m=1}^{\infty} \frac{1}{m}(J_{m-\ell}^2(x) - J_{m+\ell}^2(x)) = J_\ell(x) \sum_{k=0}^{\ell-1} \frac{2^{\ell-k}\ell!}{k!(\ell-k)} \frac{1}{x^{\ell-k}} J_k(x).$

Proof. That the series on the left converges for all x follows from Watson p. 31. We will use the following well known properties of Bessel functions

$$J_{n-1}(x) + J_{n+1}(x) = \frac{2n}{x} J_n(x), \quad [7, p. 17] \tag{4.1}$$

$$J_{n-1}(x) - J_{n+1}(x) = 2J_n'(x), \quad [7, p. 17] \tag{4.2}$$

$$J_0^2(x) + 2\sum_{r=1}^{\infty} J_r^2(x) = 1, \quad [7, p. 31] \tag{4.3}$$

$$J_{-n}(x) = (-1)^n J_n(x). \quad [7, p. 43] \tag{4.4}$$

Set $a_{\ell,k} = \frac{2^{\ell-k}\ell!}{k!(\ell-k)}$ for $0 \leq k < \ell$ and $a_{\ell,k} = 0$ for $k < 0$. The following are immediate or easy to prove

$$a_{\ell,\ell-1} = 2\ell \tag{4.5}$$

$$a_{\ell,k} = a_{\ell-1,k-1} + 2(\ell - k - 1)a_{\ell-1,k}, \tag{4.6}$$

$$a_{\ell-1,k-1} - 2ka_{\ell-1,k} - a_{\ell-2,k-2} + 2ka_{\ell-2,k-1} = 0, \quad 0 \leq k < \ell - 1. \tag{4.7}$$

for $\ell = 0$ there is nothing to prove in the formula of the Lemma. For $\ell = 1$, (4.1), (4.2) and (4.3) yield

$$\begin{aligned}
\sum_{m=1}^{\infty} \frac{1}{m}(J_{m-1}^2(x) - J_{m+1}^2(x)) &= \sum_{m=1}^{\infty} \frac{4m}{mx} J_m(x)J_m'(x) = \frac{2}{x}(\sum_{m=1}^{\infty} J_m^2(x))' \\
&= \frac{2}{x}(\frac{1}{2}(1 - J_0^2(x)))' = -\frac{2}{x}J_0(x)J_0'(x) \\
&= \frac{2}{x}J_0(x)J_1(x).
\end{aligned}$$

For $l = 2$, proceeding in the same manner we have

$$\begin{aligned}
\sum_{m=1}^{\infty}\frac{1}{m}(J_{m-2}^2(x) - J_{m+2}^2(x)) &= \sum_{m=1}^{\infty} \frac{1}{m}(J_{m-2}^2(x) - J_m^2(x) + J_m^2(x) - J_{m+2}^2(x)) \\
&= \sum_{m=1}^{\infty}(\frac{4(m-1)}{mx}J_{m-1}(x)J_{m-1}'(x) + \frac{4(m+1)}{mx}J_{m+1}(x)J_{m+1}'(x)) \\
&= \frac{2}{x}(\sum_{m=1}^{\infty}(J_{m-1}^2(x) + J_{m+1}^2(x)))' - \frac{2}{x}(\sum_{m=1}^{\infty} \frac{1}{m}(J_{m-1}^2(x) - J_{m+1}^2(x)))' \\
&= \frac{2}{x}(J_0^2(x) + \frac{1}{2}(1 - J_0^2(x)) + \frac{1}{2}(1 - J_0^2(x)) - J_1^2(x))' - \frac{2}{x}(\frac{2}{x}J_0(x)J_1(x))' \\
&= -\frac{4}{x}J_1(x)J_1'(x) + \frac{4}{x^3}J_0(x)J_1(x) + \frac{4}{x^2}J_1^2(x) - \frac{4}{x^2}J_0(x)J_1'(x) \\
&= (\frac{4}{x^2}J_0(x) + \frac{4}{x}J_1(x))(-J_1'(x) + \frac{1}{x}J_1(x)) \\
&= (\frac{4}{x^2}J_0(x) + \frac{4}{x}J_1(x))J_2(x).
\end{aligned}$$

Now we come to the induction step of the proof for $\ell \geq 3$.

$$\begin{aligned}
\sum_{m=1}^{\infty}\frac{1}{m}(J_{m-\ell}^2(x) - J_{m+\ell}^2(x)) &= \sum_{m=1}^{\infty} \frac{1}{m}(J_{m-\ell}^2(x) - J_{m-\ell+2}^2(x) + J_{m-\ell+2}^2(x) \\
&\quad - J_{m+\ell-2}^2(x) + J_{m+\ell-2}^2(x) - J_{m+\ell}^2(x)) \\
&= \sum_{m=1}^{\infty}(\frac{4(m-\ell+1)}{mx}J_{m-\ell+1}(x)J_{m-\ell+1}'(x) + \frac{4(m+\ell-1)}{mx}J_{m+\ell-1}(x)J_{m+\ell-1}')(x)
\end{aligned}$$

$$
\begin{aligned}
&+(\sum_{k=0}^{\ell-3}\frac{a_{\ell-2,k}}{x^{\ell-2-k}}J_k(x))J_{\ell-2}(x)\\
=&\frac{2}{x}(\sum_{m=1}^{\infty}J_{m-\ell+1}^2(x)+J_{m+\ell-1}^2(x))'-\frac{2(\ell-1)}{x}(\sum_{m=1}^{\infty}\frac{1}{m}(J_{m-\ell+1}^2(x)-J_{m+\ell-1}^2(x)))'\\
&+(\sum_{k=0}^{\ell-3}\frac{a_{\ell-2,k}}{x^{\ell-2-k}}J_k(x))J_{\ell-2}(x)\\
=&\frac{2}{x}(1-J_{\ell-1}^2(x))'-\frac{2(\ell-1)}{x}((\sum_{k=0}^{\ell-2}\frac{a_{\ell-1,k}}{x^{\ell-1-k}}J_k(x))J_{\ell-1}(x))'\\
&+(\sum_{k=0}^{\ell-3}\frac{a_{\ell-2,k}}{x^{\ell-2-k}}(\frac{2(k+1)}{x}J_{k+1}(x)-J_{k+2}(x)))J_{\ell-2}(x)\\
=&-\frac{2}{x}J_{\ell-1}(x)(J_{\ell-2}(x)-J_\ell(x))+\frac{2(\ell-1)}{x}(\sum_{k=0}^{\ell-2}\frac{(\ell-1-k)a_{\ell-1,k}}{x^{\ell-k}}J_k(x))J_{\ell-1}(x)\\
&-\frac{2(\ell-1)}{x}(\sum_{k=0}^{\ell-2}\frac{a_{\ell-1,k}}{x^{\ell-1-k}}J_k'(x))J_{\ell-1}(x)-\frac{2(\ell-1)}{x}(\sum_{k=0}^{\ell-2}\frac{a_{\ell-1,k}}{x^{\ell-1-k}}J_k(x))J_{\ell-1}'(x)\\
&+\sum_{k=0}^{\ell-2}\frac{2ka_{\ell-2,k-1}}{x^{\ell-k}}J_k(x)J_{\ell-2}(x)-\sum_{k=0}^{\ell-1}\frac{a_{\ell-2,k-2}}{x^{\ell-k}}J_k(x)J_{\ell-2}(x).
\end{aligned}
$$

Reindexing and using (4.5) this becomes

$$
\begin{aligned}
&-\frac{2}{x}J_{\ell-1}(x)(J_{\ell-2}(x)-J_\ell(x))-\frac{2(\ell-2)}{x}J_{\ell-1}(x)J_{\ell-2}(x)\\
&+\sum_{k=0}^{\ell-2}\frac{(2ka_{\ell-2,k-1}-a_{\ell-2,k-2})}{x^{\ell-k}}J_k(x)J_{\ell-2}(x)\\
&+(\sum_{k=0}^{\ell-2}\frac{(\ell-1-k)a_{\ell-1,k}}{x^{\ell-k}}J_k(x))(J_{\ell-2}(x)+J_\ell(x))\\
&-(\sum_{k=0}^{\ell-2}\frac{a_{\ell-1,k}}{x^{\ell-1-k}}\frac{1}{2}(J_{k-1}(x)-J_{k+1}(x)))(J_{\ell-2}(x)+J_\ell(x))\\
&-(\ell-1)(\sum_{k=0}^{\ell-2}\frac{a_{\ell-1,k}}{x^{\ell-k}}J_k(x))(J_{\ell-2}(x)-J_\ell(x)).
\end{aligned}
$$

Using (4.1) on the next to last line and reindexing again, this line is

$$
\sum_{k=0}^{\ell-1}\frac{a_{\ell-1,k-1}}{x^{\ell-k}}J_k(x)(J_{\ell-2}(x)+J_\ell(x))-\sum_{k=0}^{\ell-2}\frac{ka_{\ell-1,k}}{x^{\ell-k}}J_k(x)(J_{\ell-2}(x)+J_\ell(x)).
$$

Now looking at the coefficients of $J_\ell(x)$ and $J_{\ell-2}(x)$, applying (4.6) and (4.7), and using (4.5) one more time gives the result.

5. DEFORMATION OF LEGENDRE CURVES IN DIRECTION OF PRINCIPAL NORMAL

Let $\gamma : [0, L] \to \mathbb{R}^3(-3)$ be a closed Legendre curve parametrized by arc length in the space $\mathbb{R}^3(-3)$. Differentiating $\eta(\gamma') = 0$ along γ we see from (2.1) that $\nabla_{\gamma'}\gamma'$ is orthogonal to ξ and hence that $\nabla_{\gamma'}\gamma'$ is in the direction $\phi\gamma'$. Thus

$$\nabla_{\gamma'}\gamma' = \pm\kappa\phi\gamma'$$

where $\kappa \geq 0$ is the curvature and $\pm\phi\gamma'$ the principal normal.

Now consider a deformation of γ in the direction of the principal normal,

$$\gamma_t(s) = \exp_{\gamma(s)} tf(s)\phi\gamma'(s)$$

and the length

$$L(t) = \int_0^L g(\gamma_t', \gamma_t')^{1/2} ds.$$

Computing $L'(0)$ in the usual manner we have

$$L'(0) = -\int_0^L fg(\phi\gamma', \nabla_{\gamma'}\gamma')ds.$$

Using the orthonormal basis $e = 2\frac{\partial}{\partial y}$, $\phi e = 2(\frac{\partial}{\partial x} + y\frac{\partial}{\partial z})$, $\xi = 2\frac{\partial}{\partial z}$, we have $\phi\gamma' = -\frac{1}{2}x'e + \frac{1}{2}y'\phi e$ and hence

$$\begin{aligned} g(\phi\gamma', \nabla_{\gamma'}\gamma') &= \frac{1}{4}(y'x'' - x'y'' - (x'^2 + y'^2)(z' - yx')) \\ &= \frac{1}{4}(y'x'' - x'y'') \end{aligned}$$

since $\eta(\gamma') = \frac{1}{2}(z' - yx') = 0$ for a Legendre curve. Thus in view of the theorem in Section 2, Legendre k-minimal curves in $\mathbb{R}^3(-3)$ arise from k-minimal curves in the xy-plane $[0, 2\pi] \to E^2$ by $z = \int_0^s ydx$, s being arc length on the plane curve. The condition for Legendre k-minimal curves becomes

$$\frac{1}{4}(y'x'' - x'y'') = a_0 + \sum_{n=1}^{k-1} a_n \cos ns + b_n \sin ns.$$

If the plane curve is closed, the Legendre curve is closed if $z(2\pi) = 0$.

Thus 0-minimal curves correspond to lines in the plane. Integration of ydx gives a parabola as the Legendre curve; this parabola is a geodesic in $\mathbb{R}^3(-3)$ but clearly there are no closed 0-minimal Legendre curves.

Since $x'^2 + y'^2 = 4$ for a Legendre curve, if $y'x'' - x'y'' =$ const., the curve in the xy-plane is a circle. Thus from the area property of closed Legendre curves given in Section 2, there are no closed 1-minimal Legendre curves. Since some closed 3-minimal curves in the plane (see e.g. fig. 7 of [4]) fail to satisfy this property we cannot expect a general result. For 2-minimal curves we have the following theorem.

Theorem 1. *Every closed 2-minimal curve in the plane gives rise to a closed 2-minimal Legendre curve γ in $\mathbb{R}^3(-3)$ by integration of $z' = yx'$ and conversely.*

Proof. Closed 2-minimal curves in the plane were described explicity by the theorem stated in Section 3 and we have just noted the correspondence between k-minimal curves in the plane and k-minimal Legendre curves in $\mathbb{R}^3(-3)$. Thus it remains to prove the closure of the Legendre curve γ, i.e. to show that

$$z(2\pi) = \int_0^{2\pi} \cos(\ell v + j_{\ell,m} \sin v) \int_0^v \sin(\ell u + j_{\ell,m} \sin u) du dv = 0.$$

Set

$$f_1(v) = \cos(\ell v + j_{\ell,m} \sin v), \quad f_2(v) = \int_0^v \sin(\ell u + j_{\ell,m} \sin u) du$$

each of which is a periodic function of period 2π. Now consider the Fourier expansions of f_1 and f_2,

$$f_1(v) = A_0 + \sum_{m=1}^{\infty} A_m \cos mv + B_m \sin mv$$
$$f_2(v) = A_0^* + \sum_{m=1}^{\infty} A_m^* \cos mv + B_m^* \sin mv$$

and we must show that $A_0A_0^* + \sum_{m=1}^{\infty} A_mA_m^* + \sum_{m=1}^{\infty} B_mB_m^* = 0$. Now since $j_{\ell,m}$ is a zero of J_ℓ, it follows from ([7], p. 19) that $A_0 = \frac{1}{2\pi}\int_0^{2\pi} \cos(\ell v + j_{\ell,m} \sin v) dv = 0$. Moreover $B_m = \frac{1}{\pi}\int_0^{2\pi} f_1(v) \sin mv dv = 0$ as is easily seen by shifting the interval to $[-\pi, \pi]$ and noting that the integrand is an odd function. Thus we must show that $\sum_{m=1}^{\infty} A_mA_m^* = 0$.

First of all by [7], p. 19,

$$\begin{aligned} A_m &= \frac{1}{\pi}\int_0^{2\pi} \cos(\ell v + j_{\ell,m} \sin v) \cos mv dv \\ &= \frac{1}{\pi}\{\int_0^{2\pi} \cos((m-\ell)v - j_{\ell,m} \sin v) dv + \int_0^{2\pi} \cos((m+\ell)v + j_{\ell,m} \sin v) dv\} \\ &= 2\{J_{m-\ell}(j_{\ell,m}) + J_{m+\ell}(-j_{\ell,m})\}. \end{aligned}$$

Secondly

$$
\begin{aligned}
A_m^* &= \frac{1}{\pi}\int_0^{2\pi}\left(\int_0^v \sin(\ell u + j_{\ell,m}\sin u)du\right)\cos mv dv \\
&= -\frac{1}{m\pi}\int_0^{2\pi} \sin(\ell v + j_{\ell,m}\sin v)\sin mv dv \\
&= -\frac{1}{m\pi}\left\{\int_0^{2\pi}\cos((m-\ell)v - j_{\ell,m}\sin v)dv - \int_0^{2\pi}\cos((m+\ell)v + j_{\ell,m}\sin v)dv\right\} \\
&= -\frac{2}{m}\{J_{m-\ell}(j_{\ell,m}) - J_{m+\ell}(-j_{\ell,m})\}.
\end{aligned}
$$

Thus the proof reduces to showing that

$$\sum_{m=1}^{\infty}\frac{4}{m}(J_{m-\ell}^2(j_{\ell,m}) - J_{m+\ell}^2(j_{\ell,m})) = 0$$

but since $j_{\ell,m}$ is zero of J_ℓ, this is consequence of the lemma of Section 4.

We have just seen that the theory of 2-minimal curves in the plane contributes to the theory of Legendre curves and we now show that our result on 2-minimal Legendre curves contributes to the theory of 2-minimal curve in the plane. In [4] it was shown that every closed 2-minimal curve in the plane has a line of symmetry and a point of self-intersection, the proof of the latter assertion being somewhat extensive. Now we see from the area property of closed Legendre curves that by projecting a 2-minimal Legendre curve in $\mathbb{R}^3(-3)$ to the xy-plane, the self-intersection is immediate and that the algebraic area vanishes. Thus we have the following result.

Theorem 2. *Every closed 2-minimal curve in E^2 has a point of self-intersection and algebraic area zero.*

In particular we reproduce the following six figures from [4] which illustrate the area property in an interesting and attractive manner.

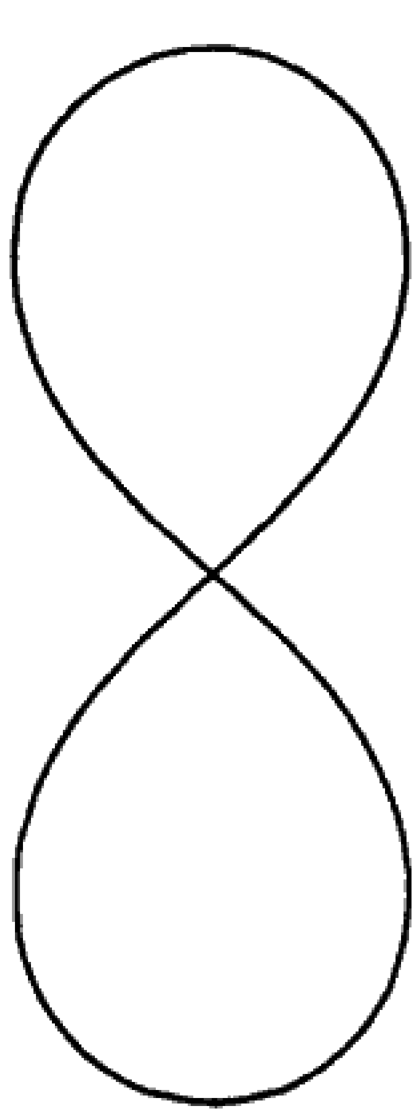

FIGURE 1.
$k = j_{0,1}\cos(s)$

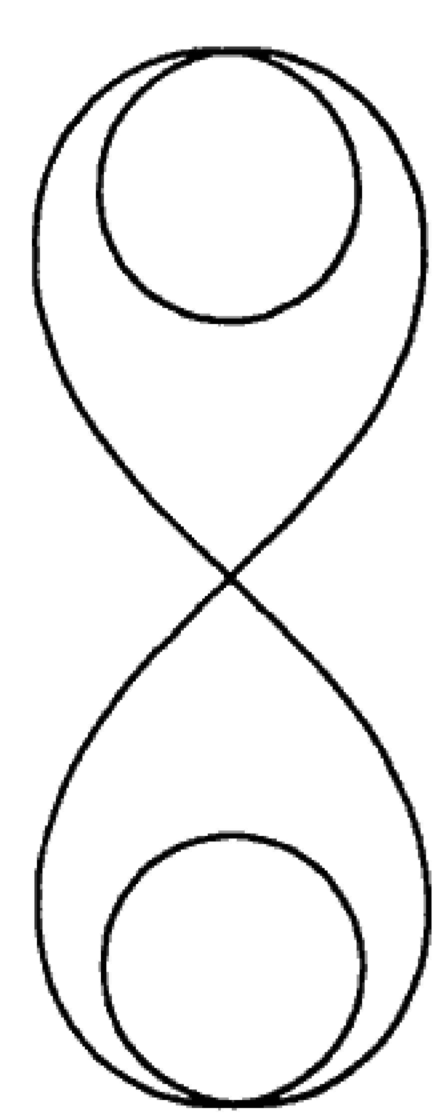

FIGURE 2.
$k = j_{0,2}\cos(s)$

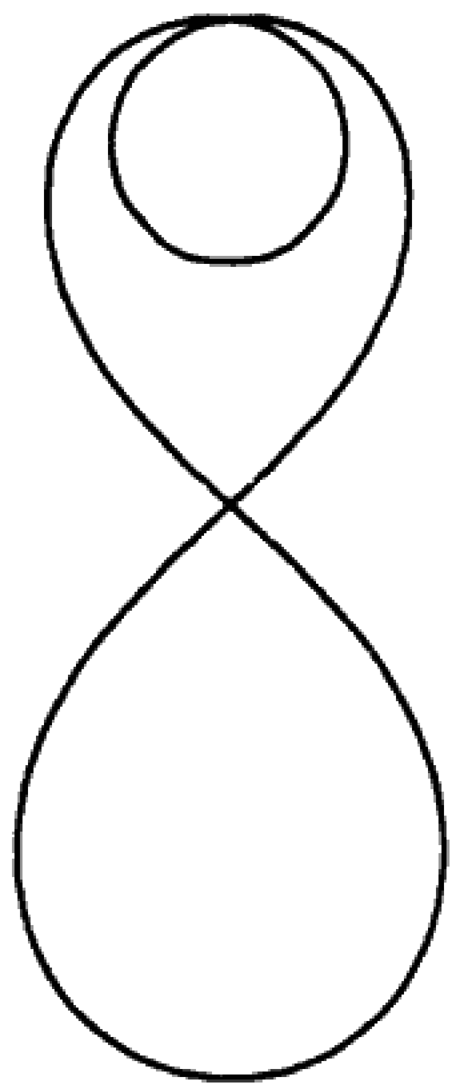

FIGURE 3.
$k = 1 + j_{1,1} \cos(s)$

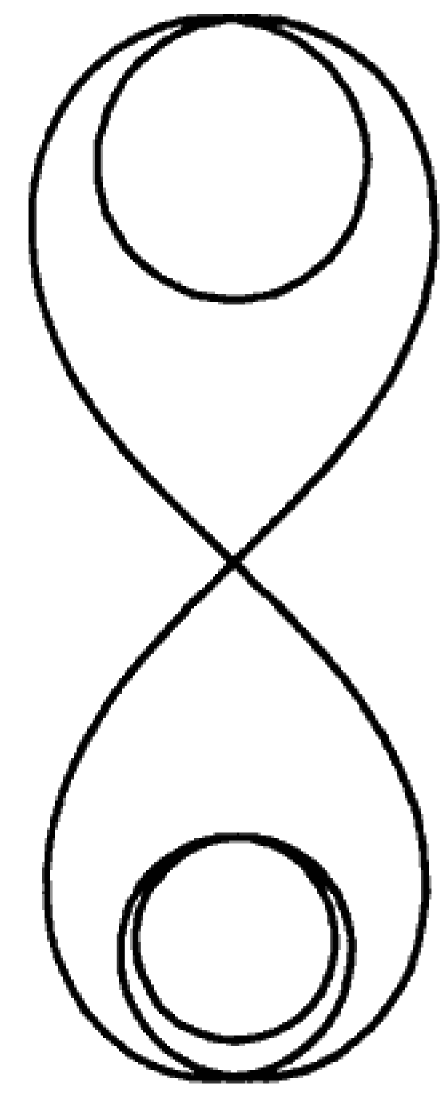

FIGURE 4.
$k = 1 + j_{1,2} \cos(s)$

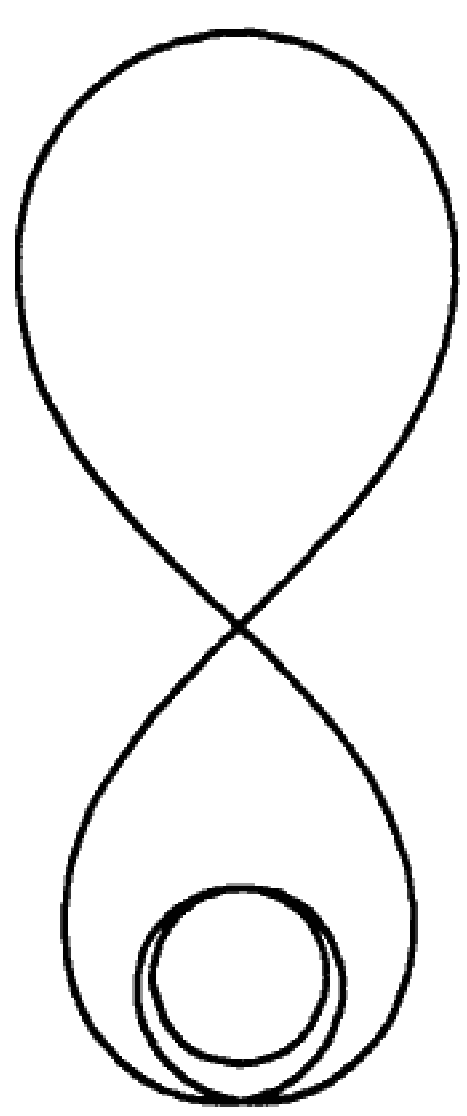

FIGURE 5.
$k = 2 + j_{2,1} \cos(s)$

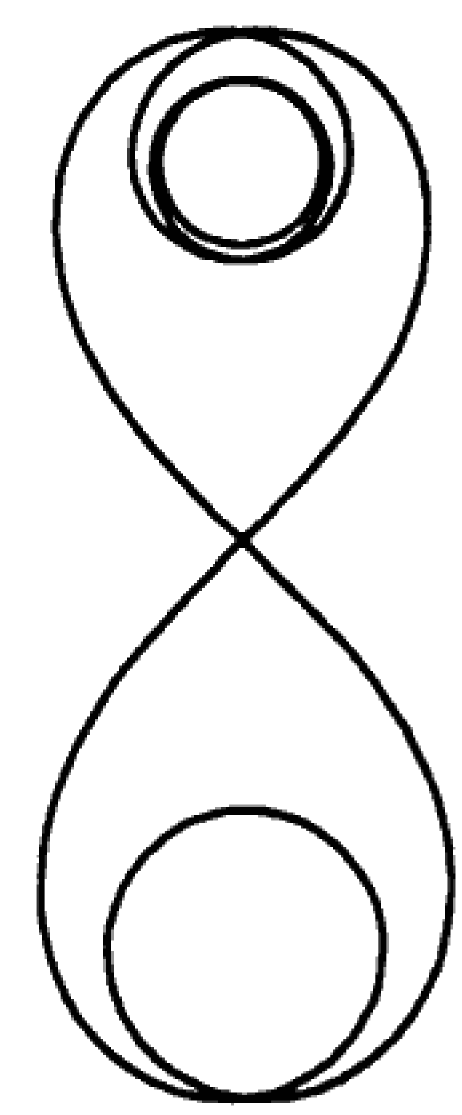

FIGURE 6.
$k = 2 + j_{2,2} \cos(s)$

6. ξ-DEFORMATIONS

In this section we consider deformations of curves in the direction of the characteristic vector field ξ in a K-contact manifold M^{2n+1}, we call such a deformation a *ξ-deformation*. For a curve $\gamma : [0, L] \to M^{2n+1}$ set

$$\gamma_t(s) = \exp_{\gamma(s)} tf(s)\xi(\gamma(s))$$

and consider the length integral $L(t) = \int_0^L g(\gamma_t', \gamma_t')^{1/2} ds$. Calculating $L'(0)$ now yields

$$L'(0) = -\int_0^L fg(\xi, \nabla_{\gamma'}\gamma')ds.$$

Therefore by virtue of (2.1), $L'(0) = 0$ for every C^∞ function f if and only if

$$g(\xi, \nabla_{\gamma'}\gamma') = \gamma' g(\xi, \gamma') = 0,$$

i.e. the angle θ between ξ and γ' is constant along the curve; such curves were called *C-loxodromes* by Tachibana and Tashiro [6]. Note that even in the space $\mathbb{R}^3(-3)$ a closed C-loxodrome which is not a Legendre curve ($\theta = \pi/2$) is possible. For example

$$\gamma = (\sin\frac{2s}{\sqrt{5}}, \cos\frac{2s}{\sqrt{5}}, \frac{1}{4}\sin\frac{4s}{\sqrt{5}})$$

is a closed C-loxodrome in $\mathbb{R}^3(-3)$ for which $\theta \neq 0, \frac{\pi}{2}$.

Computing the second variation in the usual manner and noting that the spectrum of the Laplacian for closed curves of length L is $\{(2\pi k/L)^2 | k \in \mathbb{N}\}$ we have

$$\begin{aligned} L''(0) &= \int_0^L f'2(1-\eta(\gamma')^2) - f^2\eta(\gamma')^2 ds \\ &\geq \int_0^L ((\frac{2\pi k}{L})^2(1-\eta(\gamma')^2) - \eta(\gamma')^2)f^2 ds \\ &= \frac{(2\pi k)^2(1-\eta(\gamma')^2) - L^2\eta(\gamma')^2}{L^2}\int_0^L f^2 ds. \end{aligned}$$

Thus we have the following result.

Proposition. *A Legendre curve in a K-contact manifold is 0-stable and a C-loxodrome is ℓ-stable for some ℓ.*

For the question of 1-minimal curves under ξ-deformations, $\gamma' g(\xi, \gamma') = \text{const} \neq 0$ implies $\cos\theta = As + B$ and hence there are no closed 1-minimal curves for ξ-deformations.
For 2-minimal curves, one expects closed 2-minimal curves to exist. The condition of 2-minimality is

$$\gamma' g(\xi, \gamma') = a_0 + a_1 \cos\left(\frac{2\pi s}{L}\right) + b_1 \sin\left(\frac{2\pi s}{L}\right).$$

In $\mathbb{R}^3(-3)$ this becomes

$$\frac{1}{2}(z' - yx')' = a_0 + a_1 \cos\left(\frac{2\pi s}{L}\right) + b_1 \sin\left(\frac{2\pi s}{L}\right).$$

So the problem is to choose $x(s), y(s)$ periodic such that

$$z = \int yx' ds - a_1 \frac{L^2}{2\pi^2}\cos\left(\frac{2\pi s}{L}\right) - b_1 \frac{L^2}{2\pi^2}\sin\left(\frac{2\pi s}{L}\right) + cs + d$$

is periodic and the arc length condition

$$\frac{1}{4}(x'^2 + y'^2 + (z' - yx')^2) = 1$$

is satisfied. For example the vertical circle

$$x = 2\sin s, \quad y = 0, \quad z = 2\cos s$$

easily satisfies these conditions.

The above situation is analogous to studying variations of curves in a fixed direction a in Euclidean space. If T is the unit tangent field the k-minimality condition is

$$(T \cdot a)' = a_0 + \sum_{n=1}^{k-1} a_n \cos\left(\frac{2\pi n}{L}s\right) + b_n \sin\left(\frac{2\pi n}{L}s\right).$$

Thus 0-minimal curves would be the generalized helices but they are not closed; 1-minimality means $T \cdot a$ is linear in s and hence the curve is not closed. For 2-minimal curves for deformations in the direction of the z-axis with $a_0 = 0, z'' = a_1 \cos \frac{2\pi}{L}s + b_1 \sin \frac{2\pi}{L}s$. Integrating and taking the first constant of integration to be zero,

$$z = -\frac{a_1 L^2}{4\pi^2} \cos\left(\frac{2\pi}{L}s\right) - \frac{b_1 L^2}{4\pi^2} \sin\left(\frac{2\pi}{L}s\right) + c$$

with $x(s), y(s)$ subject only to being periodic and $x'^2 + y'^2 + z'^2 = 1$.

Deformations of this type for all directions give a variational characterization of curves of finite type [3].

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