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# ON RUND'S CONNECTION

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**Abstract.** *We show that the holomorphic curvature $K_F$ (associated with a complex Finsler metric $F$) in the sense of M. Suzuki, [13], and B. Wong, [15], is (in the smooth case) precisely the holomorphic curvature of a connection essentially due to H. Rund, [12] (and reproposed in the bundle-theoretic setting by S. Kobayashi, [8]). We prove a complex analogue of Deike's theorem in real Finsler geometry. The indicatrix in each fibre of a convex complex Finsler bundle is shown to be an extrinsic sphere.*

## 1. INTRODUCTION

Let $M$ be a complex manifold of complex dimension $n$ and $E \rightarrow M$ a holomorphic vector bundle (with standard fibre $\mathbf{C}^r$) over $M$. Let $\sigma : M \rightarrow E$ be the zero section, i.e. $\sigma(x) = 0_z \in E_x$, $x \in M$, and set $E^0 = E - \sigma(M)$. Let $\Omega \subseteq E$ be an open subset to that $\sigma(M) \subset \Omega$. A $C^1$ function $F : \Omega \rightarrow \mathbf{R}$ is a *complex Finsler metric* on $E$ if i) $F \in C^\infty(\Omega - \sigma(M))$, ii) $F(v) \geq 0$ and $F(v) = 0 \Longleftrightarrow v = 0$, and iii) $F(\lambda v) = |\lambda|^2 F(v)$ for any $\lambda \in \mathbf{C}$, $v \in \Omega$. A pair $(E, F)$ consisting of a holomorphic vector bundle and a complex Finsler metric is a *complex Finsler bundle*. When $E = T^{1,0}(M)$, with any complex Finsler metric $F : T^{1,0}(M) \rightarrow [0, +\infty)$ one may associate a concept of holomorphic curvature $K_F$ (cf. M. Suzuki, [13], B. Wong, [15]). This coincides with the usual holomorphic sectional curvature if $F$ is a (smooth) Hermitian metric on $M$, and several results in Hermitian geometry carry over to the case of complex Finsler bundles. For instance, if $M$ is a complex manifold with a complex Finslar metric $F$ whose holomorphic curvature is bounded from above by a negative constant ($K_f(v) \leq c < 0$ for any $v \in T^{0,1}(M)$, $v \neq 0$) then $M$ is hyperbolic (cf. Corollary 1.5 by M. Abate & G. Patrizio, [1], p. 7). On the other hand, let $\pi : E^0 \rightarrow M$ be the natural projection and $\pi^{-1}E \rightarrow E^0$ the pullback of $E$ by $\pi$. Any convex (in the sense of S. Kobayashi, [8], p. 155) complex Finsler metric on $E$ induces a natural Hermitian structure $h$ on $\pi^{-1}E$. Let then $D$ be the Hermitian connection of $(\pi^{-1}E, h)$. We show (cf. Theorem 2) that $K_F$ is the holomorphic curvature of $D$. Consequently, in the smooth convex case Corollary 1.5 in [1], p. 7, is a consequence of Theorem 6.2 in [8], p. 164.

In section 6 we obtain a complex analogue (cf. Theorem 3) of the well known Deicke's theorem in real Finsler geometry (cf. A. Deicke, [4]).

Let $(E, F)$ be a complex Finsler bundle. Then each fibre $E_x$ is a complex Minkowski space, yet fibres over distinct points of $M$ are not congruent, in general. A complex manifold $M$ carrying the complex Finsler metric $F : T^{1,0}(M) \rightarrow [0, +\infty)$ is *locally Minkowski* if there is an atlas $\mathcal{A} = \{(U, z^1, \ldots, z^n)\}$ on $M$ so that $F$ depends on the directional arguments alone (and consequently $T^{1,0}(M)_x \approx T^{1,0}(M)_y$ as complex Minkowski spaces, for any $x, y \in M$). If $(M, F)$ is locally Minkowski we show that any transition function of $\mathcal{A}$ is a complex affine transformation (cf. Theorem 4). On the other hand, for any convex complex Finsler bundle $(E, F)$ each fibre $E_x^0$ carries a natural Kählerian metric $g_x$. We show that the *indicatrix* $I(E)_x = \{v \in E_x : F(v) = 1\}$ of $(E, F)$ at $x \in M$ is an extrinsic sphere in $(E_x, g_x)$ (cf. Theorem 5).

## 2. COMPLEX FINSLER METRICS

We proceed by looking at several examples of complex Finsler metrics.

1) Let $M = \mathbf{C}^n, n \geq 2$, and $E = T^{1,0}(M) = \mathbf{C}^n \times \mathbf{C}^n$. Let $\Omega = \{(z, \zeta) : z \neq \zeta\} \subset E$ and define $F : \Omega \to [0, +\infty)$ by $F(z, \zeta) = |PQ|^2 / |OH|^2$, where $P = (z^1, \dots, z^n), Q = (z^1 + \zeta^1, \dots, z^n + \zeta^n)$ and $OH \perp PQ, H \in PQ$. Also $|PQ|$ denotes the Euclidean distance in $\mathbf{C}^n \approx \mathbf{R}^{2n}$. This is the *complex Wrona metric* (a complex analogue of the well known Wrona metric on $\mathbf{R}^2$, cf. W. Wrona, [16], and also M. Matsumoto, [9], p. 107). To see that $F$ is a complex Finsler metric let $Z = (Z^1, \dots, Z^n)$ be the cartesian coordinates on $\mathbf{C}^n$ and $\langle z, w \rangle = z^1 \bar{w}^1 + \dots + z^n \bar{w}^n$. Note that $H$ is the intersection point of the complex lines $Z = z + \lambda\zeta, Z = \mu w$, where $\langle w, \zeta \rangle = 0$ and $\lambda, \mu \in \mathbf{C}$. Thus $H = z - \langle z, \zeta \rangle |\zeta|^{-2} \zeta$ and consequently:

$$F(z, \zeta) = \frac{|\zeta|^4}{|z|^2 |\zeta|^2 - |\langle z, \zeta|^2} \tag{1}$$

for any $z \neq \zeta$.

The real Wrona metric (on $\mathbf{R}^2$) is known (cf. also [17]) to possess a number of interesting features. For instance, its geodesics are either arcs of circles or logarithmic spirals. It is an open problem to study the real (or complex) geodesics of the complex Wrona metric (1) (on $\mathbf{C}^n$). With the above notations set $[PQ] = F(z, \zeta)^{1/2}$. If $\gamma : [a, b] \to \mathbf{C}^n$ is a parametrized curve, we define the *lenght* of $\gamma$ with respect to (1) by $L(\gamma) = \lim_{n \to \infty} \sum_{i=0}^{n-1} [\gamma(t_i)\gamma(t_{i+1})]$ where $t_1 = a + i(b - a)/n$. We obtain:

**Proposition.** *Let* $0 < \alpha < \frac{\pi}{2}$ *and* $\gamma(t)$, $0 \leq t \leq \alpha$, *any geodesic of* $S^{2n+1} \subset \mathbf{C}^{n+1}$ *parametrized by arc lenght. Then* $L(\gamma) = \alpha$.

**Proof.** Let $z \in S^{2n+1}$ and write $\gamma(t) = z \cos t + w \sin t$, $0 \leq t \leq \alpha$, $|z| = 1$, $|w| = 1$, $\langle z, w \rangle = 0$. Set $t_i = \frac{\alpha}{n} i$, $0 \leq i \leq n$. Then $\langle \gamma(t_{i+1}), \gamma(t_i) \rangle = \cos \frac{\alpha}{n}$, $|\gamma(t_{i+1}) - \gamma(t_i)|^2 = 2(1 - \cos \frac{\alpha}{n})$ and consequently $G(\gamma(t_i), \gamma(t_{i+1}) - \gamma(t_i)) = 2 \tan \frac{\alpha}{2n}$ where $G = F^{1/2}$. Finally $\lim_{n \to \infty} \sum_{i=0}^{n-1} G(\gamma(t_i), \gamma(t_{i+1}) - \gamma(t_i)) = \lim_{n \to \infty} 2n \tan \frac{\alpha}{2n} = \alpha$, Q.E.D..

2) Let $S^1 \to S^{2n+1} \xrightarrow{p} \mathbf{C}P^n$, be the Hopf fibration and $g$ any Riemannian metric on $S^{2n+1}$ so that $S^1 \subset Isom(S^{2n+1}, g)$. Let $x \in \mathbf{C}P^n$ and $v \in T_x(\mathbf{C}P^n)$. Choose $z \in S^{2n+1}$ with $p(z) = x$ and let $H_z$ be the orthogonal complement of $Ker(d_z p)$ in $T_z(S^{2n+1})$, with respect to $g_z$. As $p$ is a submersion $d_z p : H_z \to T_x(\mathbf{C}P^n)$ is a $\mathbf{R}$-linear isomorphism. Let $u \in H_z$ so that $(d_z p)u = v$ and set:

$$F_g(v) = F(u) \tag{2}$$

where $F$ is the complex Wrona metric (1) on $\Omega = \{(z, \zeta) : z \neq \zeta\} \subset T(\mathbf{C}^{n+1})$. Note that $(z, \zeta)$ is tangent to $S^{sn+1}$ iff $Re\langle z, \zeta \rangle = 0$ and consequently $T(S^{2n+1}) \subset \Omega$. Next, the definition of $F_g(v)$ does not depend upon the choice of homogeneous coordinates $z$ of $x \in \mathbf{C}P^n$. Indeed, let $z' \in S^{2n+1}$ so that $z' = \lambda z$ for some $\lambda \in S^1 \subset \mathbf{C}$. Let $R_\lambda : S^{2n+1} \to S^{2n+1}$ be the right translation with $\lambda$. As $p \circ R_\lambda = p$ it follows that $(d_z R_\lambda) Ker(d_z p) = Ker(d_{z'} p)$. Next, as $R_\lambda \in Isom(S^{2n+1}, g)$, we also have $(d_z R_\lambda)H_z = H_{z'}$. Finally, let $u' = (d_z R_\lambda)u$ and then $F(u') = F(u)$ because $S^1 \subset Isom(F)$, as a consequence of the complex homogeneity of $F$. This construction furnishes a complex Finsler metric $F_g : T(\mathbf{C}P^n) \to [0, +\infty)$. Indeed, the natural map $\rho : \mathbf{C}^{n+1} - \{0\} \to \mathbf{C}P^n$ is holomorphic so that $p = \rho \circ \iota$ (where $\iota : S^{2n+1} \subset \mathbf{C}^{n+1}$) is a CR map. Then the $\mathbf{C}$-linearity of $d_z p$ yields the complex homogeneity of $F_g$.

3) Let $V$ be a linear space over $\mathbf{C}$. A *complex Minkowski norm* on $V$ is a map $v \mapsto \|v\|$ so that i) $\|v\| \geq 0$ and $\|v\| = 0 \iff v = 0$, ii) $\|\lambda v\| = |\lambda|\|v\|$, and iii) for any linear basis $\{e_1, \dots, e_n\}$ of $V$ the map $f(z^1, \dots, z^n) = \|z^i e_i\|$ is at least of class $C^4$ at $z \neq 0$. A pair $(V, \|\cdot\|)$ is a *complex Minkowski space*. Two complex Minkowski spaces $(V, \|\cdot\|)$ and $(W, \|\cdot\|)$ are *congruent* if there is a $\mathbf{C}$-linear isomorphism $\varphi : V \to W$ so that $\|\varphi v\| = \|v\|$ for any $v \in V$.

Let $F : T^{1,0}(M) \to [0, +\infty)$ be a complex Finsler metric on $M$. Each holomorphic tangent space $T^{1,0}(M)_x$ is a complex Minkowski space in a natural way. Then $(M, F)$ is *modelled* on $(V, \|\cdot\|)$ if $T^{1,0}(M)_x \approx V$ (congruent complex Minkowski spaces) for any $x \in M$.

Let $(V, \|\cdot\|)$ and $\{e_1, \dots, e_n\}$ a fixed basis of $V$. Let $G = \{(g^i_j) \in GL(n, \mathbf{C}) : f(g^1_j z^j, \dots, g^n_j z^j) = f(z^1, \dots, z^n)\}$. Then $G$ is a closed subgroup of $GL(n, \mathbf{C})$. Let $H \subset G$ be a Lie subgroup and $B \to M$ a $H$-substructure of the complex structure of $M$. A pair $(M, B)$ is referred to as a *complex* $\{V, H\}$*-manifold* (in analogy with Y. Ichijyo, [6]). Let $v \in T^{1,0}(M)_x$ and $U \subseteq M$ an open neighborhood of $x$. Let $\{X_i, JX_i\}$ be a (local) frame on $U$ adapted to the $H$-structure $B$. Here $J$ denotes the complex structure of $M$. Set $Z_j = X_j - \sqrt{-1}JX_j$. Then $v = v^j Z_j(x)$ for some $v^j \in \mathbf{C}$. We set:

$$F(v) = \|v^j e_j\|^2 \tag{3}$$

Let $(U, z^1, \dots, z^n)$ be local complex coordinates at $x$. Then $Z_j = \lambda^k_j\, \partial / \partial z^k$, for some $\lambda^k_j \in C^\infty(U)$, and (3) may be written:

$$F(z, \zeta) = f(\mu^1_j(z)\zeta^j, \dots, \mu^n_j(z)\zeta^j)^2 \tag{4}$$

where $\mu = \lambda^{-1}, \lambda = (\lambda^j_k)$. Therefore, any complex $\{V, H\}$-manifold carries the natural complex Finsler metric (4) and $(M, F)$ is modelled on $(V, \|\cdot\|)$.

Let $(E, F)$ be a complex Finsler bundle. Let $(U, z^\alpha)$ be a local system of complex coordinates on $M$ and $f = (\sigma_1, \dots, \sigma_r)$, $\sigma_j \in \mathcal{O}(E)$, $1 \leq j \leq r$, a local holomorphic frame in $E$ on $U$. Let $(\pi^{-1}(U), z^\alpha, \zeta^j)$ be the naturally induced complex coordinates on $E^0$ (i.e. if $v \in \pi^{-1}(U)$ then $\zeta^j(v) = v^j$, where $v = v^j\sigma_j(x), x = \pi(v)$). As usual, we set:

$$F_i = \partial F / \partial \zeta^i, F_{\bar{j}} = \partial F / \partial \bar{\zeta}^j, F_{i\bar{j}} = \partial^2 F / \partial \zeta^i \partial \bar{\zeta}^j$$

etc. Then $F$ is *convex* if $(F_{i\bar{j}}) > 0$, i.e. $(F_{i\bar{j}})$ is positive definite. If $F$ is convex then the indicatrix $I(E)_x$ is a strictly pseudoconvex CR hypersurface in $E^0_x$, for any $x \in M$.

## 3. RUND'S CONNECTION

As $E \to M$ is a holomorphic vector bundle, the pullback bundle $\pi^{-1}E \to E^0$ is holomorphic, as well. Also $s_j \in \mathcal{O}(\pi^{-1}E)$ where $s_j$ is the *natural lift* of $\sigma_j$, i.e. $s_j(v) = (v, \sigma_j(\pi(v)))$, for any $v \in \pi^{-i}(U)$, $1 \leq j \leq r$. Thus $(s_1, \dots, s_r)$ is a holomorphic frame of $\pi^{-1}E$ on $\pi^{-1}(U)$. The Cauchy-Riemann operator of $E^0$ induces a differential operator:

$$\bar{\partial} : \Gamma^\infty(\pi^{-1}E) \to \Gamma^\infty((T^{0,1}E^0)^* \otimes \pi^{-1}E)$$

in a natural manner. Any complex Finsler metric $F$ on $E$ induces a Hermitian metric $h$ on $\pi^{-1}E$, as follows. Let $v \in E^0$ and $Z, W \in (\pi^{-1}E)_v$. Let $x = \pi(v)$ and choose complex coordinates $(U, z^\alpha)$ at $x$ and a holomorphic frame $f$ of $E$ on $U$. Set $h_v(Z, W) = F_{i\bar{j}}(v)Z^i\overline{W}^j$ where $Z = Z^i s_i(v)$ and $W = W^j s_j(v)$. The definition of $h_v(Z, W)$ does not depend upon the choice of $(z^\alpha)$ and $f$. Let then:

$$D : \Gamma^\infty(\pi^{-1}E) \to \Gamma^\infty(\mathbf{C}T^*E^0 \otimes \pi^{-1}E)$$

be the canonical Hermitian connection of the Hermitian bundle $(\pi^{-1}E, h)$. Cf. [14], p. 79, $D$ is given by:

$$D'Z = (\partial Z^i + \omega^i_j Z^j) \otimes s_i \tag{5}$$

$$D'' = \bar{\partial} \tag{6}$$

$$\omega^i_j = F^{i\bar{k}}\,\partial F_{j\bar{k}} \tag{7}$$

for any $Z = Z^i s_i \in \Gamma^\infty(\pi^{-1}E)$. Here $(F^{i\bar{j}} = (F_{i\bar{j}})^{-1}$ and $D = D' + D''$ corresponding to the decomposition $\mathbf{C}TE^0 = T^{1,0}E^0 \otimes T^{0,1}E^0$. Set:

$$\omega^i_j = \Gamma^i_{j\alpha} dz^\alpha + C^i_{jk} d\zeta^k$$

Then (7) yields:

$$\Gamma^i_{j\alpha} = F^{i\bar{k}} \frac{\partial F_{j\bar{k}}}{\partial x^\alpha} \tag{8}$$

$$C^i_{j\ell} = F^{i\bar{k}} \frac{\partial F_{j\bar{k}}}{\partial \zeta^\ell} \tag{9}$$

Then $D$ is referred to as the *Rund connection* (of the convex complex Finsler bundle $(E, F)$). H. Rund (cf. [12]) was the first to find (8)-(9) (when $E = T^{1,0}(M)$) though complex Finsler metrics were first studied by G.B. Rizza (cf. [11]). S. Kobayashi (cf. [8]) observed (in conenction with the geometry of ample vector bundles) that $F$ induces a Hermitian structure $\tilde{F}$ in the pullback bundle $\tilde{E} \to P(E)$ of $E \to M$ by the map $P(E) \to M$ (cf. the notations in [8], and that (8)-(9) are the connection coefficients of the Hermitian connection of $(\tilde{E}, \tilde{F})$. Our treatment of the Rund connection of $(E, F)$ follows the ideas in [8]. However we replace $P(E) \to M$ by $E^0 \to M$ (and therefore $\tilde{E} \to P(E)$ by $\pi^{-1}E \to E^0$) with the advantage that the 'correct' complex analogues of real Finsler geometry notions are more apparent. For instance, we may produce complex analogues of nonlinear connections (in the sense of A. Kawaguchi, [7]) and of the Liouville vector field (which turns out to be a global holomorphic section in $\pi^{-1}E$).

## 4. NONLINEAR CONNECTIONS

A *nonlinear connection* $N$ on $E^0$ is a complex subbundle $N$ of $T^{1,0}E^0$ so that:

$$T^{1,0}E^0 = N \oplus Ker(\partial\pi) \tag{10}$$

(direct sum of complex vector bundles) where $\partial\pi$ is the map $T^{1,0}E^0 \to \mathbf{C}TE^0 \overset{d\pi}{\to} \mathbf{C}TM \to T^{1,0}M$. The *Liouville vector field* is the cross section $\mathcal{L}$ in $\pi^{-1}E$ defined by:

$$\mathcal{L}(v) = (v, v)$$

for any $v \in E^0$. Note that $\bar{\partial}\mathcal{L} = 0$, i.e. $\mathcal{L} \in \mathcal{O}(\pi^{-1}E)$. We have:

**Theorem 1.** *Let $(E, F)$ be a convex complex Finsler bundle and $D$ its Rund connection. The solutions $X \in \Gamma^\infty(T^{1,0}, E^0)$ of:*

$$D_X\mathcal{L} = 0 \tag{11}$$

*determine a nonlinear connection $N$ on $E^0$ given by the complex Pfaffian system:*

$$d\zeta^j + N^j_\alpha(z, \zeta)dz^\alpha = 0$$

*where:*

$$N^j_\alpha = \Gamma^j_{k\alpha}\zeta^k \tag{12}$$

**Proof.** As a consequence of the complex homogeneity property of $F$ we have:

$$F = F_i\zeta^i = F_{\bar{j}}\bar{\zeta}^j \tag{13}$$

$$F = F_{i\bar{j}}\zeta^i\bar{\zeta}^j \tag{14}$$

$$F_{i\bar{j}}(\lambda v) = F_{i\bar{j}}(v) \tag{15}$$

for any $\lambda \in \mathbf{C}, \lambda \neq 0$, and $v \in \pi^{-1}(U)$. Differentiate (15) with respect to $\lambda$ and $\bar{\lambda}$ and set $\lambda = 1$ in the resulting identity so that to get:

$$F_{i\bar{j}k}\zeta^k = F_{i\bar{j}\bar{\ell}}\bar{\zeta}^\ell = 0 \tag{16}$$

Next (9), (16) yield:

$$C^i_{jk}\zeta^k = 0 \tag{17}$$

Using (17) and $\bar{\partial}\mathcal{L} = 0$ we obtain:

$$D\mathcal{L} = (d\zeta^j + \Gamma^j_{k\alpha}\zeta^k dz^\alpha) \otimes s_j$$

Let $X = A^\alpha\, \partial / \partial z^\alpha + B^j\, \partial / \partial \zeta^j$. The natural contraction of $X \otimes D\mathcal{L}$ brings (11) to the form:

$$B^j = -\Gamma^j_{k\alpha}\zeta^k A^\alpha$$

so that $N = \{X \in T^{1,0}E^0 : D_X\mathcal{L} = 0\}$ is locally spanned by:

$$\frac{\delta}{\delta z^\alpha} = \frac{\partial}{\partial z^\alpha} - \Gamma^j_{k\alpha}\zeta^k \frac{\partial}{\partial \zeta^j}$$

Finally, it is easy to see that (12) must hold.

The functions $N^i_\alpha$ are referred to as the *coefficients of the nonlinear connection N*. In analogy with the treatment of real Finsler metrics we may set:

$$\Gamma^{*j}_{k\alpha} = \omega^j_k \left( \frac{\delta}{\delta z^\alpha} \right)$$

Then:

$$\Gamma^{*i}_{j\alpha} = \Gamma^i_{j\alpha} - C^i_{jk} N^k \alpha$$

with $N^i_\alpha$ given by (12). By (17) we also get:

$$N^i_\alpha = \Gamma^{*i}_{j\alpha} \zeta^j$$

We shall need the bundle morphism $L : T^{1,0}E^0 \to \pi^{-1}$ given by $L_v Z = (v, (\partial\pi)_v Z)$ for any $Z \in (T^{1,0}E^0)_v$ and any $v \in E^0$. The *vertical lift* is the bundle morphism $\gamma : \pi^{-1}E \to T^{1,0}E^0$ given by $\gamma s_i = \partial / \partial \zeta^i, 1 \le i \le r$. The definition of $\gamma$ does not depend upon the choice of holomorphic frame $f = (\sigma_1, \dots, \sigma_r)$ in $E$. The following short sequence (of bundles and bundle morphisms):

$$0 \to \pi^{-1}E \overset{\gamma}{\to} T^{1,0}(E^0) \overset{L}{\to} \pi^{-1}E \to 0 \tag{18}$$

is exact. Consequently, for any nonlinear connection $N$ on $E^0$, the restriction $L \circ \iota$ (where $\iota_v : N_v \subset (T^{1,0}E^0)_v, v \in E^0$) is a bundle isomoprhism. The *horizontal lift* (associated with the nonlinear connection $N$) is the bundle isomoprhism $\beta : \pi^{-1}E \to N$ given by $\beta = (L \circ \iota)^{-1}$. We shall also need the bundle morphism $K : T^{1,0}E^0 \to \pi^{-1}E$ defined by $K = \gamma^{-1} \circ Q$ where $Q : T^{1,0}E^0 \to Ker(\partial\pi)$ is the natural projection associated with the direct sum decomposition (10). Then $K$ is referred to as the *Dombrowski map* (in analogy with its real countempart in P. Dombrowski, [5]). There is a short exact sequence:

$$0 \to \pi^{-1}E \overset{\beta}{\to} T^{1,0}(E^0) \overset{K}{\to} \pi^{-1}E \to 0 \tag{19}$$

Let $E = T^{1,0}M$ and $F : T^{1,0}M \to [0, +\infty)$ a convex complex Finsler structure. Let $D$ be the Rund connection of $(T^{1,0}M, F)$. Two concepts of torsion may be associated with $D$, namely $T_L(Z, W) = D_Z LW - D_W LZ - L[Z, W]$ and $T_K(Z, W) = D_Z KW - D_W KZ - K[Z, W]$, for any $Z, W \in \Gamma^\infty(T^{1,0}E^0)$, where $E^0 = T^{1,0}M - \sigma(M)$. Here $K$ is the Dombrowski map associated with the nonlinear connection $N$ given by (11), cf. Theorem 1. Several fragments of $T_L$ and $T_K$ may be defined in terms of the vertical lift and horizontal lift (associated with $N$), that is we set $T(X, Y) = T_L(\beta X, \beta Y), C(X, Y) = T_L(\gamma X, \beta Y)$ and $R^1(X, Y) = T_K(\beta X, \beta Y), P^1(X, Y) = T_K(\gamma X, \beta Y)$, for any $X, Y \in \Gamma^\infty(\pi^{-1}T^{1,0}M)$. Then $T$ (respectively $C$) is referred to as the *horizontal component* (respectively as the *mixed component*) of $T_L$, etc. There is no 'vertical' component of $T_L$ (respectively of $T_K$) because $T_L(\gamma X, \gamma Y) = 0$ by the exactness of (18) (respectively because:

$$T_K(\gamma X, \gamma Y) = 0$$

as a consequence of (9)). Note that the horizontal and mixed components of $T_L$ determine $T_L$, i.e. $T_L(Z, W) = T(LZ, LW) + C(KZ, LW) - C(KW, LZ)$ (and similarly $R^1$, $P^1$ determine $T_K$). As to local coordinates computations, one may set:

$$T(s_j, s_k) = T^i_{jk} s_i, C(s_j, s_k) = C^i_{jk} s_i$$

$$R^1(s_j, s_k) = R^i_{jk} s_i, P^1(s_j, s_k) = P^i_{jk} s_i$$

The commutation formulae:

$$\left[\frac{\delta}{\delta z^\alpha}, \frac{\delta}{\delta \zeta^\beta}\right] = \left(\frac{\delta N^j_\alpha}{\delta z^\beta} - \frac{\delta N^j_\beta}{\delta z^\alpha}\right) \frac{\partial}{\partial \zeta^j}$$

$$\left[\frac{\partial}{\partial \zeta^j}, \frac{\delta}{\delta z^\beta}\right] = -\frac{\partial N^k_\beta}{\partial \zeta^j} \frac{\partial}{\partial \zeta^k}, \left[\frac{\partial}{\partial \zeta^j}, \frac{\partial}{\partial \zeta^k}\right] = 0$$

lead to:

$$T^j_{\alpha\beta} = \Gamma^{*j}_{\beta\alpha} - \Gamma^{*j}_{\alpha\beta}$$

$$R^j_{\alpha\beta} = \frac{\delta N^j_\beta}{\delta z^\alpha} - \frac{\delta N^j_\alpha}{\delta z^\beta}, P^i_{j\beta} = \frac{\partial N^i_\beta}{\partial \zeta^j} - \Gamma^{*i}_{j\beta}$$

while $C^i_{jk}$ are given by (9). Note that $R^j_{\alpha\beta}$ is the obstruction for the complete integrability of the complex Pfaffian system $d\zeta^j + N^j_\alpha dz^\alpha = 0$ (i.e. $R^1 = 0$ if and only if $N$ is involutive). M. Abate & G. Patrizio consider the functions $T_{j\alpha\bar{k}} \in C^\infty(\pi^{-1}(U))$ defined by:

$$T_{j\alpha\bar{k}} = F_{i\bar{k}}(\Gamma^i_{\alpha;j} - \Gamma^i_{j;\alpha}) \tag{20}$$

(cf. (3.5) in [1], p. 14) where $\Gamma^i_{j;\alpha}$ are given by:

$$\Gamma^i_{j;\alpha} = F^{i\bar{\ell}} \frac{\partial F_{j\bar{\ell}}}{\partial z^\alpha} - F^{k\bar{\ell}} F^{i\bar{s}} F_{k\bar{s}j} \frac{\partial F_{\bar{\ell}}}{\partial z^\alpha} \tag{21}$$

We show that $T_{j\alpha\bar{k}} F^{i\bar{k}} = T^i_{j\alpha}$, that is (20) is nothing byt the horizontal component of $T_L$. Indeed, by (8)-(9) and (21) we have:

$$\Gamma^{*i}_{j\alpha} = \Gamma^i_{j\alpha} - C^i_{j\ell} N^\ell_\alpha = F^{i\bar{k}} \frac{\partial F_{j\bar{k}}}{\partial z^\alpha} - F^{i\bar{k}} \frac{\partial F_{j\bar{k}}}{\partial \zeta^\ell} \Gamma^\ell_{s\alpha} \zeta^s = \Gamma^i_{j;\alpha}$$

## 5. THE CURVATURE THEORY

Let $(E, F)$ be a convex complex Finsler bundle and $D$ its Rund connection. Let:

$$\Omega : \Gamma^\infty(\pi^{-1}E) \to \Gamma^\infty(\Lambda^{1,1}(E^0) \otimes \pi^{-1}E)$$

be its curvature form. Cf. [We], p. 79, $\Omega$ is locally given by:

$$\Omega s_j = \Omega^i_j \otimes s_i$$

$$\Omega^i_j = \bar{\partial} \omega^i_j$$

Therefore (taking into account (8)-(9)) we get:

$$\Omega^i_j = K^i_{j\alpha\bar{\beta}} dz^\alpha \wedge d\bar{z}^\beta +$$

$$+ H^i_{j\alpha\bar{\ell}} dz^\alpha \wedge d\bar{\zeta}^\ell + H^i_{jk\bar{\beta}} d\zeta^k \wedge d\bar{z}^\beta + Q^i_{jk\bar{\ell}} d\zeta^k \wedge d\bar{\zeta}^\ell$$

where we have set:

$$K^i_{j\alpha\bar{\beta}} = -\frac{\partial \Gamma^i_{j\alpha}}{\partial \bar{z}^\beta} \tag{22}$$

$$H^i_{j\alpha\bar{\ell}} = -\frac{\partial \Gamma^i_{j\alpha}}{\partial \bar{\zeta}^\ell}, H^i_{kk\bar{\beta}} = -\frac{\partial C^i_{jk}}{\partial \bar{z}^\beta} \tag{23}$$

$$Q^i_{jk\bar{\ell}} = -\frac{\partial C^i_{jk}}{\partial \bar{\zeta}^\ell} \tag{24}$$

Let $\mathbf{B}^1 = \{z : |z| < 1\}$ be the unit disk in $\mathbf{C}$. Let $x \in M$ and $v \in T^{1,0}(M)_x$, $v \neq 0$. Set:

$$K_F(v) = \sup K(\varphi^* F)(0) \tag{25}$$

where the supremum is taken over all holomorphic maps $\varphi : \mathbf{B}^1 \to M$ so that $\varphi(0) = x$ and $\varphi'(0) = \lambda v$ for some $\lambda \in \mathbf{C}$, $\lambda \neq 0$. Also $\varphi^* F = f dz \otimes d\bar{z}, f(z) = F(\varphi(z), \varphi'(z))$, and:

$$K(\varphi^* F) = -\frac{1}{2f} \Delta \log f$$

while:

$$\Delta = 4 \frac{\partial^2}{\partial z \partial \bar{z}}$$

**Theorem 2.** *Let $E = T^{1,0}M$ and $F$ a convex complex Finsler metric on $E$. Then:*

$$K_F(\hat{\pi}\mathcal{L}) = 2F^{-2} K_{i\bar{j}\alpha\bar{\beta}} \zeta^i \bar{\zeta}^j \zeta^\alpha \bar{\zeta}^\beta \tag{26}$$

*where:*

$$K_{i\bar{j}\alpha\bar{\beta}} = F_{k\bar{j}} K^k_{i\alpha\bar{\beta}}$$

*while $K^i_{j\alpha\bar{\beta}}$ is given by (22) and $\hat{\pi} : \pi^{-1}E \to E$ is the natural projection.*

**Proof.** We start from (2.18) in [1], p. 13, that is:

$$K_F(\hat{\pi}\mathcal{L}) = -2R^{-2} F_\alpha \Gamma^\alpha_{;i\bar{j}} \zeta^i \bar{\zeta}^j \tag{27}$$

where:

$$\Gamma^\alpha_{;i\bar{j}} = F^{\alpha\bar{\mu}} \frac{\partial F_{\bar{\mu}}}{\partial z^i \partial \bar{z}^j} - F^{\alpha\bar{\nu}} F^{\beta\bar{\mu}} \frac{\partial F_{\beta\bar{\nu}}}{\partial \bar{z}^j} \frac{\partial F_{\bar{\mu}}}{\partial z^i}$$

Using (22) we obtain:

$$K_{i\bar{j}\alpha\bar{\beta}} \zeta^i \bar{\zeta}^j \zeta^\alpha \bar{\zeta}^\beta = -F_k \frac{\partial N^k_\alpha}{\partial \bar{z}^\beta} \zeta^\alpha \bar{\zeta}^\beta \tag{28}$$

On the other hand, by (9):

$$N^j_\alpha = F^{j\bar{k}} \frac{\partial F_{\bar{k}}}{\partial z^\alpha} \tag{29}$$

Finally (28)-(29) and the identity:

$$\frac{\partial F^{i\bar{\ell}}}{\partial \bar{z}^\beta} = -F^{i\bar{j}} F^{k\bar{\ell}} \frac{\partial F_{k\bar{j}}}{\partial \bar{z}^\beta}$$

lead to:

$$K_{i\bar{j}\alpha\bar{\beta}} \zeta^i \bar{\zeta}^j \zeta^\alpha \bar{\zeta}^\beta =$$

$$= F_i \left( F^{i\bar{j}} F^{k\bar{\ell}} \frac{\partial F_{k\bar{j}}}{\partial \bar{z}^\beta} \frac{\partial F_{\bar{\ell}}}{\partial z^\alpha} - F^{k\bar{\ell}} \frac{\partial^2 F_{\bar{\ell}}}{\partial z^\alpha \partial \bar{z}^\beta} \right) \zeta^\alpha \bar{\zeta}^\beta \tag{30}$$

At this point (26) follows from (27) and (30).

Let $(E, F)$ be a convex complex Finsler bundle and $B$ the curvature tensor field of its Rund connection, i.e. $B(Z, W)s_j = ([D_Z, D_W] - D_{[Z,W]})s_j$. As in the case of the torsions of $D$, we may define *horizontal components* of $B$ as follows. Set $B(\delta / \delta z^\alpha, \delta / \delta z^\beta)s_j = R^i_{j\alpha\beta} s_i$, $B(\delta / \delta z^\alpha, \delta / \delta \bar{z}^\beta)s_j = R^i_{j\alpha\bar{\beta}} s_j$, etc. Here:

$$\frac{\delta}{\delta \bar{z}^\alpha} = \frac{\partial}{\partial \bar{z}^\alpha} - N^{\bar{j}}_{\bar{\alpha}} \frac{\partial}{\partial \bar{\zeta}^j}$$

where $N^{\bar{j}}_{\bar{\alpha}} = \overline{N^j_\alpha}$. Then:

$$R^i_{j\alpha\beta} = \frac{\delta \Gamma^{*i}_{j\beta}}{\delta z^\alpha} - \frac{\delta \Gamma^{*i}_{j\alpha}}{\delta z^\beta} + \Gamma^{*k}_{j\beta} \Gamma^{*i}_{k\alpha} - \Gamma^{*i}_{j\alpha} \Gamma^{*i}_{k\beta} + R^k_{\alpha\beta} C^i_{jk} \tag{31}$$

Next, we may use:

$$\left[ \frac{\delta}{\delta z^\alpha}, \frac{\delta}{\delta \bar{z}^\beta} \right] = \frac{\delta N^i_\alpha}{\delta \bar{z}^\beta} \frac{\partial}{\partial \zeta^i} - \frac{\delta N^{\bar{j}}_{\bar{\beta}}}{\delta z^\alpha} \frac{\partial}{\partial \bar{\zeta}^j}$$

so that to yield:

$$R^i_{j\alpha\bar{\beta}} = -\frac{\delta \Gamma^{*i}_{j\alpha}}{\delta \bar{z}^\beta} - \frac{\delta N^k_\alpha}{\delta \bar{z}^\beta} C^i_{jk} \tag{32}$$

$$R^i_{j\bar{\alpha}\bar{\beta}} = 0 \tag{33}$$

Note that, by (31), we have:

$$R^i_{j\alpha\beta} \zeta^j = R^i_{\alpha\beta}$$

Similarly, we consider *mixed components* of $B$, i.e. $B(\partial / \partial \zeta^k, \delta / \delta z^\beta)s_j = P^i_{jk\beta}\ s_i$ and $B(\partial / \partial \zeta^k, \delta / \delta \bar{z}^\beta)s_i = P^i_{jk\bar{\beta}} s_i$, etc. Then:

$$P^i_{jk\beta} = \frac{\partial \Gamma^{*i}_{j\beta}}{\partial \zeta^k} - \frac{\delta C^i_{jk}}{\delta z^\beta} + \Gamma^{*s}_{j\beta} C^i_{sk} - \Gamma^{*i}_{s\beta} C^s_{jk} + \frac{\partial N^s_\beta}{\partial \zeta^k} C^i_{js} \tag{34}$$

Next, by the commutation formula:

$$\left[\frac{\partial}{\partial \zeta^k}, \frac{\delta}{\delta \bar{z}^\beta}\right] = -\frac{\partial N^{\bar{j}}_{\bar{\beta}}}{\partial \zeta^k}\frac{\partial}{\partial \bar{\zeta}^j}$$

we obtain:

$$P^i_{jk\bar{\beta}} = -\frac{\delta C^i_{jk}}{\delta \bar{z}^\beta} \tag{35}$$

$$P^i_{j\bar{k}\beta} = \frac{\partial \Gamma^{*i}_{j\beta}}{\partial \bar{\zeta}^k} + \frac{\partial N^\ell_\beta}{\partial \bar{\zeta}^k} C^i_{j\ell} \tag{36}$$

$$P^i_{j\bar{k}\bar{\beta}} = 0 \tag{37}$$

Finally, the identity:

$$D_{\partial / \partial \zeta^j}\mathcal{L} = s_j$$

and (34) yield:

$$P^i_{jk\beta}\zeta^j = P^i_{k\beta}$$

As to the *vertical components* of $B$, we set $B(\partial / \partial \zeta^k, \partial / \partial \zeta^\ell)\, s_j = S^i_{jk\ell} s_i$ and $B(\partial / \partial \zeta^k, \partial / \partial \bar{\zeta}^\ell)\, s_j = S^i_{jk\bar{\ell}} s_i$, etc., and derive the indentities:

$$S^i_{jk\ell} = \frac{\partial C^i_{j\ell}}{\partial \zeta^k} - \frac{\partial C^i_{jk}}{\partial \zeta^\ell} + C^s_{j\ell}C^i_{sk} - C^s_{jk}C^i_{s\ell} \tag{38}$$

$$S^i_{jk\bar{\ell}} = -\frac{\partial C^i_{jk}}{\partial \bar{\zeta}^\ell} \tag{39}$$

$$S^i_{j\bar{k}\bar{\ell}} = 0 \tag{40}$$

Note that:

$$S^i_{jk\ell}\zeta^j = 0$$

Also, by comparing (24) and (39):

$$Q^i_{jk\bar{\ell}} = S^i_{jk\bar{\ell}}$$

Let $M$ be a real $n$-dimensional, $n > 2$, $C^\infty$ manifold. By a result of F. Brickell, [2], any homogeneous real Finsler metric on $M$ with vanishing vertical curvature must be a Riemannian metric. Let $(E, F)$ be a convex complex Finsler bundle and set $S(X, Y)Z = B(\gamma X, \gamma Y)Z$, for any $X, Y, Z \in \Gamma^\infty(\pi^{-1}E)$. Clearly, if $F$ is a Hermitian metric in $E$ then $S = 0$. The converse is an open problem, as yet.

## 6. DEIKE'S THEOREM

The purpose of this section is to establish the following:

**Theorem 1.** *Let* $F : \mathbf{C}^{n+1} \to [0, +\infty)$ *be a convex complex Finsler metric. If* $\det(F_{j\bar{k}}) =$ *const. on* $\mathbf{C}^{n+1} - \{0\}$ *then* $|F_{j\bar{k}}|$ = *const. on* $\mathbf{C}^{n+1} - \{0\}$.

**Remarks.**

1. This is a obvious complex analogue of Deicke's theorem in real Finsler geometry (cf. [4]). Let:

$$F(z, \zeta) = f(z)(\zeta^1 \dots \zeta^n \bar{\zeta}^1 \dots \bar{\zeta}^n)^{1/n} \tag{41}$$

Then:

$$F_{j\bar{k}} = \frac{1}{n^2} \frac{F}{\zeta^j \bar{\zeta}^k}$$

and consequently $\det(F_{j\bar{k}}) = 0$ (yet $|F_{j\bar{k}}|$ depends of $\zeta^j$). Therefore the assumption $F(v) = 0 \Longleftrightarrow v = 0$ is necessary (for Theorem 3 to be true).

2. As $F$ is convex the hypothesis in Theorem 3 is equivalent to:

$$C_j = 0 \tag{42}$$

where $C_j = C^i_{ji}$ and $C^i_{ik}$ is given by (9). Indeed, if $H = \det(F_{j\bar{k}})$ then $C_j = \partial\,(\log H) / \partial\,\zeta^j$. The proof of Theorem 3 is organized in several steps, as follows. First, we show that $F^{j\bar{k}}\,(z)\,F_{j\bar{k}}\,(w) \geq n+1$ for any $z, w \in \mathbf{C}^{n+1} - \{0\}$. Indeed, as $F$ is convex $(F^{\bar{j}k}(z)F_{k\bar{h}}(w))$ has positive eigenvalues. Thus:

$$F^{j\bar{k}}(z)F_{j\bar{k}}(w) \geq (n+1)\det(F^{\bar{j}k}(z)F_{k\bar{h}}(w))^{1/(n+1)} = n+1$$

(by the inequality between the arithmetic and geometric means). We shall need the differential operator $\mathcal{L}$ given by:

$$\mathcal{L} = F^{\bar{j}k} \frac{\partial^2}{\partial\,\bar{z}^j\,\partial\,z^k}$$

Next, we show that $(\mathcal{L}F_{j\bar{k}})$ is positive semidefinite. Indeed, let $f_z : \mathbf{C}^{n+1} - \{0\} \to (0, +\infty)$ be given by $f_z(w) = F^{j\bar{k}}(z)F_{j\bar{k}}(w)$. Then $f_z(z) = n+1$ hence $f_z$ has a minumum at $w = z$. Thus $(\partial^2 f_z / \partial\,w^j\,\partial\,\bar{w}^k)$ is positive semidefinite at $w = z$. Finally, one may check that:

$$(\mathcal{L}F_{j\bar{k}})(z) = \frac{\partial^2 f_z}{\partial\,w^j\,\partial\,\bar{w}^k}(z)$$

Note that $\mathcal{L}F_{j\bar{j}} \geq 0$. Next $F_{j\bar{j}}$ is complex homogeneous of degree zero and $F_{j\bar{j}} > 0$. In particular $F_{j\bar{j}}$ is positive homogeneous of degree zero, so that $F_{j\bar{j}}$ attains a maximum on $\mathbf{C}^{n+1} - \{0\}$. But $\mathcal{L}$ is elliptic, so that $F_{j\bar{j}} = const$. Define $\varphi(z, w) = (\mathcal{L}F_{j\bar{k}})\ z^j \bar{w}^k$. But $\varphi$ satisfies the Cauchy-Schwarz inequality, so that $\mathcal{L}F_{j\bar{j}} = 0$ yields $\mathcal{L}F_{j\bar{k}} = 0$. But $\mathcal{L}$ is a real operator, so that:

$$\mathcal{L}|F_{j\bar{k}}| = 0$$

as well. Finally $|F_{j\bar{k}}|$ is positive homogeneous of degree zero and thus (again by the Hopf maximum principle) $|F_{j\bar{k}}| = const.$, Q.E.D..

## 7. LOCALLY MINKOWSKI MANIFOLDS

We establish the following:

**Theorem 4.** *Let $(M, F)$ be a locally Minkowski manifold and $\mathcal{A}$ the corresponding atlas of adapted local coordinate systems (i.e. $(U, z^\alpha) \in \mathcal{A} \Longrightarrow \partial F / \partial z^\alpha = 0$). Then any transition function of $\mathcal{A}$ is of the form:*

$$z'^\alpha = A^\alpha_\beta z^\beta + C^\alpha \tag{43}$$

*with $A^\alpha_\beta$, $C^\alpha \in \mathbf{C}$, $\det(A^\alpha_\beta) \neq 0$. Conversely, any local coordinate system obtained from an adapted one by a complex affine transformation (43) is again adapted.*

**Proof.** Let $(U, z^\alpha)$, $(U', z'^\alpha)$ be complex local coordinate neighborhoods on $M$, $U \cap U' \neq \emptyset$. Under a transformation:

$$\begin{cases} z'^\alpha = z'^\alpha(z^1, \dots, z^n) \\ \det\left(\dfrac{\partial z'^\alpha}{\partial z^\beta}\right) \neq 0 \qquad on \quad U \cap U' \end{cases} \tag{44}$$

the (induced) local coordinates on $T^{1,0}M$ change as:

$$\begin{cases} z'^\alpha = z'^\alpha(z^1, \dots, z^n) \\ \zeta'^\alpha = \dfrac{\partial z'^\alpha}{\partial z^\beta} \zeta^\beta \end{cases}$$

Consequently:

$$\frac{\partial F}{\partial z^\alpha} = \frac{\partial z'^\beta}{\partial z^\alpha} \frac{\partial F}{\partial z'^\beta} + \zeta^\beta \frac{\partial^2 z^{ij}}{\partial z^\alpha \partial z^\beta} \frac{\partial F}{\partial \zeta'^j}$$

If $(U, z^\alpha)$ is adapted and $z'^\alpha$ are given by (43) then $\partial F / \partial z'^\beta = 0$, Q.E.D.. Viceversa, assume that both $(z^\alpha)$, $(z'^\alpha)$ are adapted. The transformation law of the coefficients of the nonlinear connection of the Rund connection (under a transformation (44)) reads:

$$N^i_\alpha = \frac{\partial z^i}{\partial z'^j} \frac{\partial z'^\beta}{\partial z^\alpha} N'^j_\beta + \zeta^\beta \frac{\partial^2 z'^j}{\partial z^\alpha \partial z^\beta} \frac{\partial z^i}{\partial z'^j} \tag{45}$$

By (8), if $\partial F / \partial z^\alpha = 0$ then $N^i_\alpha = 0$, $N'^j_\beta = 0$ and (45) yields:

$$\frac{\partial^2 z'^j}{\partial z^\alpha \partial z^\beta} = 0 \tag{46}$$

It is an elementary consequence of (46) that $z'^j$ must be given by (43) (indeed, if $f^j_\beta = \partial z'^j / \partial z^\beta$ then $f^j_\beta$ are holomorphic; thus $f^j_\beta = A^j_\beta = const.$, by (46). This may be written as $\partial(z'^j - A^j_\alpha z^\alpha) / \partial z^\beta = 0$ and $z'^j - A^j_\alpha z^\alpha$ are holomorphic, etc.).

## 8. INDICATRICES

We may state the following:

**Theorem 5.** *Let $(E, F)$ be a convex Finsler bundle. Let $x \in M$. Then $E^0_x$ is a Kähler manifold and $I(E)_x = \{v \in E_x : F(v) = 1\}$ is an extrinsic sphere of $E^0_x$.*

We recall that given a submanifold $M$ of a Riemannian manifold $X, M$ is an *extrinsic sphere* in $X$ if $M$ is totally umbilical with a nonzero parallel (in the normal bundle of $M \subset X$) mean curvature vector. Let $(E, F)$ be a convex complex Finsler bundle. The map $F_x$ given by $E_x^0 \hookrightarrow E \xrightarrow{F} [0, +\infty)$ is a Hermitian metric on $E_x^0$ and the coefficients of the Hermitian connection of $(E_x^0, F_x)$ are $C^i_{jk}$. By (9), $C^i_{jk} - C^i_{kj} = 0$ so that $(E_x^0, F_x)$ is Kählerian. Set $G = F^{1/2}$. Let $(u^1, \dots, u^{2r-1})$ be local coordinates on $I(E)_x$ and let:

$$\zeta^j = \zeta^j(u^1, \dots, u^{2r-1})$$

be the equations of $I(E)_x \hookrightarrow E_x^0$. Set $\ell_j = F_{j\bar{k}}\ell^{\bar{k}}$ where $\ell^j = G^{-1}\zeta^j, \ell^{\bar{j}} = \overline{\ell^j}$. Then $\ell_j = G^{-1}F_j$ and $F(z, \zeta(u)) = 1$ yields:

$$\ell_j \frac{\partial \zeta^j}{\partial u^\alpha} + \ell_{\bar{j}} \frac{\partial \bar{\zeta}^j}{\partial u^\alpha} = 0 \tag{47}$$

where $\ell_{\bar{j}} = \overline{\ell_j}$. Let $h$ be the Hermitian from on $T^{1,0}(E_x^0)$ associated with $F_x$, i.e. $h_v(Z, W) = F_{j\bar{k}}(z, \zeta)Z^j\overline{W^k}$, for any $Z, W \in (T^{1,0}E_x^0)_v$, $v \in E_x^0$, $Z = Z^j(\partial / \partial \zeta^j)_v$, $W = W^j(\partial / \partial \zeta^j)_v$. Here, the arguments $z, \zeta$ of $F_{j\bar{k}}$ are respectively the coordinates of the (fixed) point $x$, and the components of $v$ (with respect to $\{\sigma_1(x), \dots, \sigma_r(x)\}$). As customary in Hermitian geometry, we extend $h$ to a complex bilinear form $H$ on $\mathbf{C}TE_x^0$ by $H(Z, \overline{W}) = h(Z, W)$, $H(Z, W) = H(\overline{Z}, \overline{W}) = 0$, $H(\overline{Z}, W) = \overline{H(Z, \overline{W})}$ for any $Z, W \in T^{1,0}(E_x^0)$. Set:

$$N = \ell^j \frac{\partial}{\partial \zeta^j} + \ell^{\bar{j}} \frac{\partial}{\partial \bar{\zeta}^j}$$

Then $H(X_\alpha, N) = 0$ by (47), where $X_\alpha = \iota_* \partial / \partial u^\alpha$ (and $\iota : I(E)_x \subset E_x^0$). Consequently $N \in T(I(E)_x)^\perp$. Also $H(N, N) = 2$. At this point we may prove Theorem 5. Let $A_N$ be the shape operator of $I(E)_x$ in $E_x^0$. Taking into account the identity:

$$D_{\partial / \partial \zeta^j} N = G^{-1}(\delta^i_j - \ell_j \zeta^i) \frac{\partial}{\partial \zeta^i}$$

the Weingarten formula (cf. e.g. [3], p. 40):

$$D_{X_\alpha} N = -A_N X_\alpha$$

may be written:

$$A_N X_\alpha = -G^{-1} X_\alpha$$

Yet $G = 1$ on $I(E)_x$, so that $I(E)_x$ is a totally umbilical real hypersurface of constant mean curvature (in $E_x^0$).

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