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OSCILLATING SOLUTIONS OF NONLINEAR IMPULSIVE DIFFERENTIAL EQUATIONS WITH A DEVIATING ARGUMENT

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Abstract. *Sufficient conditions for oscillation of the solutions of non-linear differential equations with a deviating argument and fixed moments of impulse effect are found.*

1. INTRODUCTION

The impulsive differential equations with a deviating argument are an adequate mathematical apparatus for simulation of numerous processes which depend on their pre-history and are subject to short-time perturbations. Such processes occur in the theory of optimal control, biotechnologies, industrial robotics, economics, etc. In spite of the great possibilities for application, the theory of these equations is developing rather slowly due to obstacles of theoretical and technical character. The study of the properties of their solutions arises an ever growing interest [1, 2, 3].

We shall note that the oscillation theory of the impulsive differential equations has not been yet elaborated. The first work devoted to this theory is [4].

In the present work sufficient conditions for oscillation of the solutions of a class of nonlinear differential equations with a deviating argument and fixed moments of impulse effect are found.

2. PRELIMINARY NOTES

Let h be a positive constant, $\{\tau_k\}_{k=1}^{\infty}$ be a monotone increasing, unbounded sequence of positive numbers, $\{b_k\}_{k=1}^{\infty}$ be a sequence of real numbers.

Consider the impulsive differential equation with a deviating argument

$$x'(t) + p(t)f(x(t-h)) = 0, t \neq \tau_k$$

$$\Delta x(\tau_k) = x(\tau_k + 0) - x(\tau_k)v = b_k x(\tau_k) \tag{1}$$

and initial function

$$x(t) = \varphi(t), \quad t \in [-h, 0]. \tag{2}$$

Introduce the following conditions:

H1. $p \in C(\overline{\mathbb{R}}_+, \mathbb{R}_+), \mathbb{R}_+ = (0, \infty), \overline{\mathbb{R}}_+ = [0, \infty)$.

H2. $f \in C(\mathbb{R}, \mathbb{R}), \quad uf(u) > 0$ for $u \neq 0$ and $f(u)$ is an increasing function in $\mathbb{R}$.

H3. For any $k \in \mathbb{N}$ the inequalities $b_k > -1$ are valid.

H4. There exists a constant $T > 0$ such that $\tau_{k+1} - \tau_k \geq T > h$ for each $k \in \mathbb{N}$.

H5. There exists a constant $M > 0$ such that $|f(u)| \geq \frac{|u|}{M}$.

H6. $\varphi \in C([-h, 0], \mathbb{R}), h > 0$.

Let us construct the sequence

$$\{t_i\}_{i=1}^{\infty} = \{\tau_i\}_{i=1}^{\infty} \cup \{\tau_{ih}\}_{i=1}^{\infty}$$

where $\tau_{ih} = \tau_i + h$ and $t_i < t_{i+1}$ for $i \in \mathbb{N}$.

Definition 1. By a *solution* of equation (1) with initial function (2) we mean any function $x : [-h, \infty) \to \mathbb{R}$ for which the following conditions are valid:

1. If $-h \le t \le 0$, then $x(t) = \varphi(t)$.
2. If $0 \le t \le t_1 = \tau_1$, then x coincides with the solution of the equation

$$x'(t) + p(t)f(x(t-h)) = 0.$$

3. If $t_i < t \le t_{i+1}$, $t_i \in \{\tau_i\}_{i=1}^{\infty} \backslash \{\tau_{ih}\}_{i=1}^{\infty}$, then x coincides with the solution of the problem

$$x'(t) + p(t)f(x(t-h)) = 0$$

$$x(t_i + 0) = (1 + b_{k_i})x(t_i)$$

where the number k_i is determined from the equality $t_i = \tau_{k_i}$.

4. If $t_i < t \le t_{i+1}$, $t_i \in \{\tau_{ih}\}_{i=1}^{\infty} \backslash \{\tau_i\}_{i=1}^{\infty}$, then x coincides with the solution of the equation

$$x'(t) + p(t)f(x(t-h+0)) = 0.$$

5. If $t_i < t \le t_{i+1}$, $t_i \in \{\tau_i\}_{i=1}^{\infty} \cap \{\tau_{ih}\}_{i=1}^{\infty}$, then x coincides with the solution of the problem

$$x'(t) + p(t)f(x(t-h+0)) = 0$$

$$x(t_i + 0) = (1 + b_{k_i})x(t_i).$$

Definition 2. The nonzero solution x of equation (1) is said to be *nonoscillating* if there exists a point $t_0 \ge 0$ such that $x(t)$ has a constant sign for $t \ge t_0$. Otherwise the solution x is said to *oscillate*.

3. MAIN RESULTS

Theorem 1. *Let the following conditions hold:*

1. *Conditions* **H1 - H6** *are met.*
2. $\limsup_{k\to\infty} \frac{1}{1+b_k} \int_{\tau_k}^{\tau_k+h} p(s)ds > M$.

Then all solutions of problem (1), (2) oscillate.

Proof. Let x be a nonoscillating solution of problem (1), (2). Without loss of generality we may assume that $x(t) > 0$ for $t \ge t_0$ for some $t_0 \ge 0$. Then $x(t-h) > 0$ and $f(x(t-h)) > 0$ for $t \ge t_0 + h$.

From (1) and conditions **H1** and **H2** it follows that x is a decreasing function in the set $(t_0 + h, \tau_s) \cup [\bigcup_{i=s}^{\infty} (\tau_i, \tau_{i+1})]$ where $\tau_{s-1} < t_0 + h < \tau_s$.

Integrate (1) from τ_k to $\tau_k + h$ ($k \geq s$) and obtain that

$$x(\tau_k + h) - x(\tau_k + 0) + \int_{\tau_k}^{\tau_k+h} p(s)f(x(s-h))ds = 0. \tag{3}$$

From (3) and the fact that $x(\tau_k + 0) = (1 + b_k)x(\tau_k)$ we get to the inequality

$$x(\tau_k + h) - (1 + b_k)x(\tau_k) + \inf_{s\in[\tau_k,\tau_k+h]} f(x(s-h)) \cdot \int_{\tau_k}^{\tau_k+h} p(s)ds \leq 0. \tag{4}$$

On the other hand, we have

$$\inf_{s\in[\tau_k,\tau_k+h]} f(x(s-h)) = f(x(\tau_k)).$$

Then from (4) it follows that

$$x(\tau_k + h) - (1 + b_k)x(\tau_k) + f(x(\tau_k)) \int_{\tau_k}^{\tau_k+h} p(s)ds \leq 0.$$

From the above inequality we obtain that

$$\frac{1}{1+b_k} \int_{\tau_k}^{\tau_k+h} p(s)ds < \frac{x(\tau_k)}{f(x(\tau_k))} \leq M. \tag{5}$$

Inequality (5) contradicts condition 2 of Theorem 1. ■

Corollary 1. *Let the conditions of Theorem 1 hold. Then:*
1. The inequality

$$x'(t) + p(t)f(x(t-h)) \leq 0, t \neq \tau_k \tag{6}$$
$$\Delta x(\tau_k) = b_k x(\tau_k)$$

with initial condition (2) has no positive solution.
2. The inequality

$$x'(t) + p(t)f(x(t-h)) \geq 0, t \neq \tau_k \tag{7}$$
$$\Delta x(\tau_k) = b_k x(\tau_k)$$

with initial condition (2) has no negative solution.

Theorem 2. *Let the following conditions hold:*
1. Conditions **H1 - H6** *are met.*
2. $\limsup_{k\to\infty} b_k = N$*, N is a positive constant.*
3. $\liminf_{t\to\infty} \int_{t-h}^{t} p(s)ds > \frac{(1+N)M}{e}$*, e = exp.*
Then all solutions of problem (1), (2) oscillate.

Proof. Let x be a nonoscillating solution of problem (1), (2). Without loss of generality we may assume that $x(t) > 0$ for $t \geq t_0 > 0$. Then $x(t-h) > 0$ and $f(x(t-h)) > 0$ for $t \geq t_0 + h$.

Introduce the function $w(t) = \frac{x(t-h)}{x(t)}, t \geq t_0 + 3h$. From (1), **H1** and **H2** it follows that x is a descreasing function in the set $(t_0 + 3h, \tau_s) \cup [\bigcup_{i=s}^{\infty} (\tau_i, \tau_{i+1})]$, where $\tau_{s-1} < t_0 + 3h < \tau_s$.

Fix $t, t \geq t_0 + 3h$. The following cases are possible:

1. In the interval $(t-h, t)$ there is a point of jump τ_k. Then

$$x(t-h) > x(\tau_k) = \frac{x(\tau_k + 0)}{1 + b_k} > \frac{x(t)}{1+N}, \text{ i.e.}$$

$$w(t) > \frac{1}{1+N}. \tag{8}$$

2. In the interval $(t-h, t)$ there is no point of jump. Then $x(t-h) > x(t)$, i.e.

$$w(t) = \frac{x(t-h)}{x(t)} > 1 > \frac{1}{1+N}. \tag{9}$$

We shall prove that the function w is bounded above for $t \geq t_0 + 3h$. Let t^* be an arbitrary point such that $t^* > t_0 + 3h$. Choose a point t so that $t - h < t^* < t$ and

$$\int_{t-h}^{t^*} p(s)ds > \frac{(1+N)M}{2e} \quad \text{and} \quad \int_{t^*}^{t} p(s)ds > \frac{(1+N)M}{2e}.$$

Such a choice of the pont t is possible by virtue of condition 3 of Theorem 2.

We shall consider the following cases:

1. Let $\tau_k \in (t^*, t)$.

1.1. $\tau_{k-1} \in (t^* - h, t - h)$.

Integrate equation (1) from t^* to t and obtain that

$$x(t) - x(t^*) - b_k x(\tau_k) + \int_{t^*}^{t} p(s) f(x(s-h))ds = 0. \tag{10}$$

From (10) there follows the inequality

$$\inf_{s \in [t^*, t]} f(x(s-h)) \int_{t^*}^{t} p(s)ds \leq x(t^*) + b_k x(\tau_k) - x(t). \tag{11}$$

On the other hand, we have

$$\inf_{s \in [t^*, t]} f(x(s-h)) > f\left(\frac{x(t-h)}{1+N}\right). \tag{12}$$

From (11) and (12) we obtain that

$$f\left(\frac{(t-h)}{1+N}\right) \frac{(1+N)M}{2e} \leq x(t^*) + b_k x(\tau_k) \leq x(t^*)(1+N).$$

From the last inequality we find

$$x(t^*) \geq f\left(\frac{x(t-h)}{1+N}\right) \cdot \frac{M}{2e}. \tag{13}$$

Integrate equation (1) from $t-h$ to t^*. Since in the interval $(t-h, t^*)$ there is no point of jump, then

$$x(t^*) - x(t-h) \quad + \quad \inf_{s\in[t-h,t^*]} f(x(s-h)) \int_{t-h}^{t^*} p(s)ds \leq 0. \tag{14}$$

Since in the interval $(t-2h, t^*-h)$ neither is there a point of jump, then

$$\inf_{s\in[t-h,t^*]} f(x(s-h)) = f(x(t^*-h)). \tag{15}$$

From (14) and (15) it follows that

$$x(t-h) \geq f(x(t^*-h)) \int_{t-h}^{t^*} p(s)ds \geq$$

$$\geq f(x(t^*-h)) \frac{(1+N)M}{2e}. \tag{16}$$

From (13) and (16) we obtain that

$$\frac{x(t^*-h)}{x(t^*)} \leq \frac{x(t^*-h)}{f\left(\frac{x(t-h)}{1+N}\right)\frac{M}{2e}} =$$

$$= \frac{x(t^*-h)}{f(x(t^*-h))} \cdot \frac{\frac{x(t-h)}{1+N}}{f\left(\frac{x(t-h)}{1+N}\right)} \cdot \frac{f(x(t^*-h))}{x(t-h)} (1+N)\frac{2e}{M} \leq$$

$$\leq M^2 \left(\frac{2e}{M}\right)^2 = 4e^2 < \infty. \tag{17}$$

1.2 $\tau_{k-1} \in (t-2h, t^*-h)$. Then in the interval $(t^*-h, t-h)$ there is no point of jump. Hence

$$\inf_{s\in[t^*,t]} f(x(s-h)) = f(x(t-h)). \tag{18}$$

From (11) and (18) it follows that

$$f(x(t-h)) \int_{t^*}^{t} p(s)ds \leq x(t^*) + Nx(t^*)$$

from which we find

$$x(t^*) \geq f(x(t-h))\frac{M}{2e}. \tag{19}$$

Integrate equation (1) from $t-h$ to t^* and obtain

$$x(t^*) - x(t-h) + \int_{t-h}^{t^*} p(s) f(x(s-h))ds = 0. \tag{20}$$

In this case we have

$$x(t-2h) > x(\tau_{k-1}) = \frac{x(\tau_{k-1}+0)}{1+b_{k-1}} > \frac{x(t^*-h)}{1+N}.$$

Then

$$\inf_{s\in[t-h,t^*]} f(x(s-h)) \geq f\left(\frac{x(t^*-h)}{1+N}\right). \tag{21}$$

From (20) and (21) we get to the inequality

$$f\left(\frac{x(t^*-h)}{1+N}\right)\frac{(1+N)M}{2e} \leq x(t-h). \tag{22}$$

From (19) and (22) we obtain that

$$\begin{aligned}\frac{x(t^*-h)}{x(t^*)} &\leq \frac{x(t^*-h)}{f(x(t-h))\frac{M}{2e}} = \frac{2e}{M}\cdot\frac{\frac{x(t^*-h)}{1+N}}{f\left(\frac{x(t^*-h)}{1+N}\right)}\cdot \\ &\cdot\frac{x(t-h)}{f(x(t-h))}\cdot\frac{f\left(\frac{x(t^*-h)}{1+N}\right)}{x(t-h)}\cdot(1+N) \leq \\ &\leq \frac{2e}{M}M\cdot M\cdot\frac{2e}{(1+N)M}(1+N) = 4e^2 < \infty.\end{aligned} \tag{23}$$

1.3. Let $\tau_{k-1} < t-2h$. Then

$$\inf_{s\in[t-h,t^*]} f(x(s-h)) = f(x(t^*-h)). \tag{24}$$

Integrate equation (1) from $t-h$ to t^* and obtain (20). From (20) and (24) it follows that

$$f(x(t^*-h))\frac{(1+N)M}{2e} \leq x(t-h). \tag{25}$$

From (19) and (25) we obtain that

$$\begin{aligned}\frac{x(t^*-h)}{x(t^*)} &\leq \frac{x(t^*-h)}{f(x(t-h))}\cdot\frac{2e}{M} = \\ &= \frac{x(t^*-h)}{f(x(t^*-h))}\cdot\frac{x(t-h)}{f(x(t-h))}\cdot\frac{f(x(t^*-h))}{x(t-h)}\cdot\frac{2e}{M} \leq \\ &\leq M^2\frac{2e}{(1+N)M}\cdot\frac{2e}{M} = \frac{4e^2}{1+N} < \infty.\end{aligned} \tag{26}$$

From (17) and (26) it follows that if $\tau_k \in (t^*,t)$, then the function w is bounded above for $t \geq t_0+3h$.

2. Let $\tau_k \in (t-h,t^*)$. The considerations in this case are analogous to those in Case 1.

3. Let $\{\tau_k\}_{k=1}^{\infty} \cap (t-h,t) = \emptyset$. In this case as well, without any particular difficulties, analogously to Case 1 it is shown that the function w is bounded above.

Divide (1) by $x(t) > 0, t \geq t_0 + 3h$ and obtain

$$\frac{x'(t)}{x(t)} + p(t)\frac{f(x(t-h))}{x(t)} = 0. \tag{27}$$

Let $\tau_k \in (t-h,t]$. Integrate (27) from $t-h$ to t ($t \geq t_0 + 3h$) and obtain

$$ln\left[\frac{1}{1+b_k}\frac{x(t)}{x(t-h)}\right] + \int_{t-h}^{t} p(s)\frac{f(x(s-h))}{x(s)}ds = 0. \tag{28}$$

Introduce the notation

$$w_0 = \liminf_{t\to\infty} w(t).$$

It is clear that $0 < w_0 < \infty$. Then from (28) it follows that

$$\int_{t-h}^{t} p(s)\frac{f(x(s-h))}{x(s-h)} \cdot \frac{x(s-h)}{x(s)}ds = ln[(1+b_k)w(t)]$$

$$\int_{t-h}^{t} p(s)\frac{f(x(s-h))}{x(s-h)}w(s)ds \leq ln[(1+N)w_0]$$

$$\int_{t-h}^{t} p(s)\frac{f(x(s-h))}{x(s-h)}ds \leq \frac{1}{w_0}ln[(1+N)w_0]$$

$$\frac{1}{M}\int_{t-h}^{t} p(s)ds \leq \frac{1}{w_0}ln[(1+N)w_0]. \tag{29}$$

Using the inequality $lnx < \frac{x}{e}$ for $x > 1$, where $x = (1+N)w_0 > 1$, from (29) we get to the inequality

$$\liminf_{t\to\infty}\int_{t-h}^{t} p(s)ds \leq \frac{(1+N)M}{e}.$$

The last inequality contradicts condition 3 of Theorem 2. ■

Consider the nonhomogeneous impulsive differential equation

$$x'(t) + p(t)f(x(t-h)) = b(t), t \neq \tau_k$$

$$\Delta x(\tau_k) = b_k x(\tau_k) \tag{30}$$

with initial condition (2).

Introduce the following conditions:

H7. $b \in c(\overline{\mathbb{R}}_+, \mathbb{R})$.

H8. There exist two sequences $\{t_i'\}_{i=1}^{\infty}, \{t_i''\}_{i=1}^{\infty} \subset \overline{\mathbb{R}}_+$ and two constants $q_1, q_2 \in \mathbb{R}$ such that

(i) $\lim_{i\to\infty} t_i' = \lim_{i\to\infty} t_i'' = \infty$;

(ii) $w(t_i') = q_1 \leq w(t) \leq q_2 = w(t_i'')$

where $w'(t) = b(t), t \in \overline{\mathbb{R}}_+, i \in \mathbb{N}$.

Theorem 3. *Let the following conditions hold:*
1. Conditions **H1**, **H2**, **H5** - **H8** *are satisfied.*
2. $-1 < b_k \leq 0, k \in \mathbb{N}$.
3. $\limsup_{k\to\infty} \frac{1}{1+b_k} \int_{\tau_k}^{\tau_k+h} p(s)ds > M$.
Then all solutions of problem (30), (2) oscillate.

Proof. Let $x(t)$ be a positive solution of problem (30), (2) for $t \geq t_0 > 0$. Then $x(t-h) > 0$ and $f(x(t-h)) > 0$ for $t \geq t_0 + h$.

Set

$$z(t) = q_1 - w(t) + x(t), t \geq t_0 + h. \tag{31}$$

From (31) it follows that $x(t) \geq z(t)$ for $t \geq t_0 + h$.

From this fact and from condition **H2** it follows that

$$f(x(t)) \geq f(z(t)), t \geq t_0 + h.$$

From (30), (31) and **H8** we find that

$$0 = x'(t) - b(t) + p(t)f(x(t-h)) \geq z'(t) + p(t)f(z(t-h))$$

$$\Delta z(\tau_k) = b_k z(\tau_k) + b_k[w(\tau_k) - q_1],$$

i.e. we obtain the impulsive inequality

$$z'(t) + p(t)f(z(t-h)) \leq 0, t \neq \tau_k \tag{32}$$

$$\Delta z(\tau_k) = b_k z(\tau_k) + A_k$$

where

$$A_k = b_k[w(\tau_k) - q_1] \leq 0.$$

1. Let $z(t) > 0$ be a solution of inequality (32) for $t \geq t_1 \geq t_0$. Integrate (32) from τ_k to $\tau_k + h(\tau_k > t_1 + h)$ and obtain

$$z(\tau_k + h) - z(\tau_k + 0) + \int_{\tau_k}^{\tau_k+h} p(s)f(z(s-h))ds \leq 0$$

$$\inf_{s\in[\tau_k,\tau_k+h]} f(z(s-h)) \int_{\tau_k}^{\tau_k+h} p(s)ds \leq z(\tau_k + 0)$$

$$f(z(\tau_k)) \int_{\tau_k}^{\tau_k+h} p(s)ds \leq (1 + b_k)z(\tau_k) + A_k \leq (1 + b_k)z(\tau_k)$$

$$\frac{1}{1+b_k} \int_{\tau_k}^{\tau_k+h} p(s)ds \leq \frac{z(\tau_k)}{f(z(\tau_k))} \leq M.$$

The last inequality is valid for each k for which $\tau_k > t_1 + h$, and it contradicts condition 3 of Theorem 3.

2. Let $z(t) < 0$ be a solution of (32) for $t \geq t_1 \geq t_0$. From (31) it follows that

$$x(t_i') = z(t_i') + w(t_i') - q_1 = z(t_i') + q_1 - q_1 = z(t_i').$$

Hence for $t_i' > t_1$ we get to the contradiction $0 < x(t_i') = z(t_i') < 0$. ■

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