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A CHARACTERIZATION OF CURVES OF MINIMAL ORDER AS REGARDS SINGULAR POINTS AND THEIR MULTIPLICITIES

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1. INTRODUCTION

In [6] the following theorem was proved: "Let $\mathcal{K}$ be a set of order-characteristics in the plane with fundamental number k. Then a normal arc or curve of $\mathcal{K}$-order $k+1$ contains at most $k+1$ $\mathcal{K}$-singular points".

Moreover, $\mathcal{K}$-multiplicities were assigned to $\mathcal{K}$-singular points and a stronger result was obtained:
"The sum of the $\mathcal{K}$-multiplicities of the $\mathcal{K}$-singular points is at most $k+1$".

Here it is shown:

Theorem. *Let $\mathcal{K}$ be a set of order-characteristics in the plane with fundamental number k. Then the sum of the $\mathcal{K}$-multiplicities of the $\mathcal{K}$-singular points of a curve C_{k+1} of $\mathcal{K}$-order $k+1$ is at least $k+1$.*

2. ORDER CHARACTERISTICS

2.1. Let $\mathcal{K}$ be a family of order characteristic arcs or curves in $G = \overline{G}$, a closed disk in the euclidean plane with fundamental number k satisfying the following axions ([1]; 1.1 and 2.4).

(I) If $K \varepsilon \mathcal{K}$ is an arc then K has exactly its two end-points e', e'' in common with the boundary G_b of G.

If $K \varepsilon \mathcal{K}$ is a curve, then K has at most one point in common with G_b. Hence for each $K \varepsilon \mathcal{K}$, $\underline{G} - \underline{G} \cap K$ ($\underline{G}$ is the interior of G) is the union of two disjoint open connected sets $K(\alpha, G)$ in which $\alpha = +$ or $\alpha = -$; these two global sides of K in G are denoted by $K(\pm, G)$.

(II) There exists a natural number $k \geq 1$, the so-called fundamental number with the following properties:

1. Let $x_\lambda, \lambda = 1, 2, \ldots, k$ be k distinct points of G. Then there is a unique $K \varepsilon \mathcal{K}$ with $x_\lambda \varepsilon K$.

2. Let x'_λ be close to x_λ, $\lambda = 1, 2, \ldots, k$. Then there exists $K' \varepsilon \mathcal{K}$, $K' = K\,(x'_1, x'_2, \ldots, x'_k)$ and $K(x'_1, x'_2, \ldots, x'_k)$ varies continuously with the x'_λ. Note: The class of all compact sets in G is a metric space where $d(A_1, A_2) = \inf(\varepsilon > 0; A_1 \subseteq U_2, A_2 \subseteq U_1)$ where $U_i = U(A_i, \varepsilon)$ is the $\varepsilon - G$-neighbourhood of A_i (i.e., the union of $\varepsilon - G$-neighbourhoods of all points of A_i). We use this metric for $\mathcal{K}$.

(III) Let $K_n \varepsilon \mathcal{K}$, $n = 1, 2, \ldots$, with $x_{nu} \varepsilon K_n$ where $x_u = \lim x_{nu}$, $u = 1, 2, \ldots, k$, x_u distinct. Then there exists $K \varepsilon \mathcal{K}$ with $x_u \varepsilon K$, $u = 1, 2, \ldots, k$ (by Axiom II 2), $K = \lim K_n$).

(IV) For any $K \varepsilon \mathcal{K}$ and any point $a \varepsilon K$, let $y_1, y_2, \ldots, y_i$ be i arbitrary, distinct points of G, $1 \leq i \leq k-1$, $y_i \neq a$. Further, let $x_1, x_2, \ldots, x_{k-i}$ be distinct points of K converging to a on

K. Then $\lim K(y_1, y_2, \ldots, y_i, x_1, x_2, \ldots, x_{k-i}) = K\ (y_1, y_2, \ldots, y_i;\ a^{k-i})$ exists uniquely in $\mathcal{K}$. This condition saying that the order characteristics *themselves* are strongly differentiable arcs or cuves was previously denoted EP_k by Haupt and Künneth ([1], 4.1.4).

Remark. *These Axioms I - IV are more restrictive than those in 1.1 and 2.4 of [1]. Also one can see 1.1 of [5] for the conformal definition of strong differentiability.*

2.2. Let A be an arc in G (we use the same letter for a point on the parameter interval and its image).

Definition. An order characteristic K has j-point contact (is j-osculating) with an arc $\mathcal{A}$ at $a \varepsilon \mathcal{A}$ if for any two-sided neighbourhood (subarc) N of a there exists a $\tilde{K}$ close to K that intersects N at j distinct points.
In particular one can consider the subsystem $\mathcal{K}(a^j)$ of $\mathcal{K}$ having j-point concact with each other studied in ([6]; 2.3). It was shown that:
1) j-point contact is a "transitive" relation on $\mathcal{K}$.
2) The subsystem $\mathcal{K}(a^j)$ also satisfies Axioms (I) - (IV) with fundamental number $k - j$.

3. MULTIPLICITIES OF $\mathcal{K}$-SINGULAR POINTS

3.1. Let a be an end-point of a normal arc $\mathcal{A}_{k+1}$ of $\mathcal{K}$-order $k + 1$. Assume that $a < r$ for all $r \varepsilon \mathcal{A}_{k+1}$. It is also assumed that $\mathcal{K}$ is a family of order characteristics with fundamental number k satisfying Ax. I - IV.

3.2. Next let $i = k,\ k - 1,\ \ldots,\ 2, 1$. There are subfamilies $\mathcal{K}\ (a^{k-i})$ of $\mathcal{K}$ for each i with fundamental number i having $k - i$ point contact at a. Moreover, at each interior point $t \in \mathcal{A}_{k+1}$ there are unique one-sided osculating characteristics $K_i^- = K(a^{k-i}, t^i)$ and $K_i^+ = K\ (a^{k-i}, t^i)$ of $\mathcal{A}_{k+1}$ ([1]; 4.2.6.3 and 4.2.6.4).

3.3. An interior point $t \varepsilon \mathcal{A}_{k+1}$ is said to be $\mathcal{K}\ (a^{k-i})$-singular if for any two-sided neighbourhood $N = L \cup \{t\} \cup M$ (L below t on the parameter interval, M above) of t on $\mathcal{A}_{k+1}$ there is a member of $\mathcal{K}\ (a^{k-i})$ that intersects N at $i + 1$ distinct points. Note that this member of $\mathcal{K}\ (a^{k-i})$ cannot then meet $\mathcal{A}_{k+1}$ again; otherwise the order of $\mathcal{A}_{k+1}$ is $> k + 1$.

One can now classify all the $\mathcal{K}(a^{k-i})$-singular points t of $\mathcal{A}_{k+1}$ as follows by specifying the kinds of pairs of one-sided osculating characteristic curves of $\mathcal{A}_{k+1}$ at t:
(a) $(i, 1)(1, i)$ point
(b) $(i, 1)(0, i)$ point
(c) $(i, 0)(1, i)$ point
(d) $(i, 0)(0, i)$ point
where t is assigned the symbol $(i, r)\ (s, i)$ if for any neighbourhood $N = L \cup \{t\} \cup M$ of t on $\mathcal{A}_{k+1}$ there are members K_{Ni}^-, K_{Ni}^+ of $\mathcal{K}\ (a^{k-i})$ one, K_{Ni}^-, that intersects L at i points, M at r points and one, K_{Ni}^+, that intersects M at i points and L at s points; $r, s = 0, 1$.

Remarks. *1)* $\lim K_{Ni}^- = K_i^-$, $\lim K_{Ni}^+ = K_i^+$ *([1]; 4.2.6.3 and 4.2.6.4).*
2) In words
(a) *t is a $\mathcal{K}\ (a^{k-i})$-singular point with respect to both $K_i^-(a^{k-i}, t^i)$ and $K_i^+\ (a^{k-i}, t^i)$,*

(b) *t is $\mathcal{K}$ (a^{k-i})-singular with respect to $K_i^-(a^{k-i}, t^i)$ only,*
(c) *t is $\mathcal{K}$ (a^{k-i})-singular with respect to $K_i^+(a^{k-i}, t^i)$ only,*
(d) *t is $\mathcal{K}$ (a^{k-i})-singular with respect to neither K_i^- (a^{k-i}, t^i) nor K_i^+ (a^{k-i}, t^i).*

3.4. Again let z be an interior $\mathcal{K}$-singular point and let $i = k, k-1, \ldots, 2$. At each stage each of the types (a), (b), (c), (d) may be subclassified as

α^i: z is of type α^i if z is $\mathcal{K}$, $\mathcal{K}$ $(a), \ldots, \mathcal{K}$ (a^{k-i})-singular but not $\mathcal{K}$ (a^{k-i+1})-singular,
β_1^i: z is β_1^i if z is $\mathcal{K}$, $\mathcal{K}(a), \ldots, \mathcal{K}(a^{k-i})$, $\mathcal{K}$ (a^{k-i+1})-singular and both K_i^-, K_i^+ do not meet the arc (a, z) again.
β_2^i: z is β_2^i if z is $\mathcal{K}$, $\mathcal{K}(a), \ldots, \mathcal{K}(a^{k-i})$, $\mathcal{K}$ (a^{k-i+1})-singular and both K_i^-, K_i^+ meet (a, z) again.
γ_1^i: z is γ_1^i if z is $\mathcal{K}$, $\mathcal{K}$ $(a), \ldots, \mathcal{K}$ (a^{k-i}), $\mathcal{K}$ (a^{k-i+1})-singular, K_i^-, meets (a, z) again but K_i^+ does not.
γ_2^i: z is γ_2^i if z is $\mathcal{K}$, $\mathcal{K}(a), \ldots, \mathcal{K}$ (a^{k-i}), $\mathcal{K}$ (a^{k-i+1})-singular, K_i^-, does not meet (a, z) again but K_i^+ does.

Definition. Each $\mathcal{K}$-singular point z is given an initial multiplicity equal to 1. At each stage (i) changes to the multiplicities may be assigned as indicated by the following chart. The total is then called the $\mathcal{K}$ – multiplicity of z.

	(a) ((i,1),(1,i))	(b) ((i,1),(0,i))	(c) ((i,0),(1,i))	(d) ((i,0),(0,i))
α^i	0	0	0	0
β_1^i	+1	0	+1	0
β_2^i	-	-	-	0
γ_1^i	-	-	0	-1
γ_2^i	-	+1	-	-

The entry - indicates that this situation cannot occur. The motivation for this definition comes from [5] for the conformal case with $k = 3$. Note that the $\mathcal{K}$-multiplicity is always non-negative even though there is a possibility to decrease it in the case where z is γ_1^i.

Remarks.
(A) *In the case $k = 2$ ([1]; 3.2.1) the points of inflection and the cusps of the second kind each have multiplicity* 1 *(using the definition of multiplicity in 3.4) while a cusp of the first kind has multiplicity* 2.
(B) *In the case $k = 3$ in [2] the differentiable singular points with the characteristic* $(1, 1, 2)$, $(1, 1, 2)_0$, $(1, 2, 1)_0$, $(2, 1, 1)_0$ *have multiplicities* $1, 1, 2, 3$, *respectively.*

4. $\mathcal{K}(a)$-SINGULAR POINTS

Since induction will be the method of proof for the main result, it is necessary to see how $\mathcal{K}(a)$-singular points with certain $\mathcal{K}(a)$-multiplicities give rise to $\mathcal{K}$-singular points with their $\mathcal{K}$-multiplicities using the monotony, contraction and expansion theorems of Haupt and Künneth ([1]; 2.4).

4.1. Let z be a $\mathcal{K}(a)$-singular point of $\mathcal{A}_{k+1}$ with $\mathcal{K}(a)$-multiplicity s. If z is $\mathcal{K}$-singular, then

Possible type	$\mathcal{K}$-multiplicity
$\beta_1^k\ (k,1)\ (1,k)$	$s+1$
$\beta_1^k\ (k,0)\ (1,k)$	$s+1$
$\gamma_2^k\ (k,1)\ (0,k)$	$s+1$
$\gamma_1^k\ (k,0)\ (0,k)$	$s-1$
$\beta_1^k\ (k,1)\ (0,k)$	s
$\gamma_1^k\ (k,0)\ (1,k)$	s
$\beta_1^k\ (k,0)\ (0,k)$	s
$\beta_2^k\ (k,0)\ (0,k)$	s

Notice that an α^k $\mathcal{K}$-singular point is not possible since z is already $\mathcal{K}\ (a)$-singular.

4.2. Let z be a $\mathcal{K}\ (a)$-singular point of $\mathcal{A}_{k+1}$ which is not $\mathcal{K}$-singular. Then z has $\mathcal{K}\ (a)$-multiplicity 1.

Proof. Since z is of $\mathcal{K}$-order k, $\mathcal{A}_{k+1}$ satisfies EP_{k-1} at z (i.e. the $(k-2)$ strong differentiability condition). Hence there is only one $\mathcal{K}\ (a)$-osculating characteristic at a and z is not $\mathcal{K}\ (a^2)$-singular. Thus z has $\mathcal{K}\ (a)$-multiplicity 1.

4.3. Let $z_1 < z_2$ be two $\mathcal{K}\ (a)$-singular points on $\mathcal{A}_{k+1}$.

If z_1 is one of $\begin{cases} \beta_1^k & (k,1)(0,k) \\ \beta_1^k & (k,0)(0,k) \\ \gamma_1^k & (k,0)(0,k) \end{cases}$ and if z_2 is one of $\begin{cases} \gamma_1^k & (k,0)(1,k) \\ \gamma_1^k & (k,0)(0,k) \\ \beta_2^k & (k,0)(0,k) \end{cases}$, then there is at least one $\mathcal{K}$-singular point in (z_1, z_2).

Proof. Consider the case where z_1 is $\beta_1^k(k,1)(0,k)$ and z_2 is $\gamma_1^k\ (k,0)\ (1,k)$. The proof for the other cases is similar.

Since z_1 is $\beta_1^k(k,1)\ \underline{(0,k)}$ there is an order-characteristic close to $\mathcal{K}_k^+\ (z_1)$ that meets $\mathcal{A}_{k+1}$ at k points $z_1 < y_1 < y_2 < \ldots < y_k$ near z_1 and does not meet (a, z_1). Since z_2 is $\gamma_1^k\ \underline{(k,0)}\ (1,k)$ there is a member of $\mathcal{K}$ close to $\mathcal{K}_k^-\ (z_2)$ that meets $\mathcal{A}_{k+1}$ at k points $x_2 < x_3 < \ldots < x_{k+1} < z_2$ near z_2 and meets (a, z_1) at one point x_1.

Now let a point t_2 move from x_2 toward y_1, keeping $x_3, \ldots, x_{k+1}$ fixed. Then there is a point u which moves from x_1 toward z_1. If u reaches z_1 first, then there is a $k+1$-tuple of points in $[z_1, z_2)$ and an order-characteristic containing these points. By contraction, one obtains a $\mathcal{K}$-singular point in (z_1, z_2). If t_2 reaches y_1 first, then let t_3 move from x_3 toward y_2, t_4 move from x_4 toward $y_3, \ldots, t_{k+1}$ move from x_{k+1} toward y_k, if necessary. The point u must reach z_1 first. Otherwise there is a member $\mathcal{K}(t_2, t_3, \ldots, t_{k+1}) = \mathcal{K}(y_1, y_2, \ldots, y_k)$ meeting (a, z_1); contradiction. Then one obtains a $\mathcal{K}$-singular point in (z_1, z_2) as above.

Similarly one obtains

4.4. Let z_1 be $\mathcal{K}$(a)-singular but not $\mathcal{K}$-singular.

If z_2 is $\begin{cases} \mathcal{K}(\text{a})\text{-singular} & \text{but not } \mathcal{K}\text{-singular} \\ \gamma_1^k & (k,0)(1,k) \\ \gamma_1^k & (k,0)(0,k) \\ \beta_2^k & (k,0)(0,k) \end{cases}$, where $z_1 < z_2$, then there is a $\mathcal{K}$-singular point in (z_1, z_2).

4.5. Let z_2 be $\mathcal{K}$(a)-singular but not $\mathcal{K}$-singular.

If z_1 is $\begin{cases} \mathcal{K}(\text{a})\text{-singular} & \text{but not } \mathcal{K}\text{-singular} \\ \beta_1^k & (k,1)(0,k) \\ \beta_1^k & (k,0)(0,k) \\ \gamma_1^k & (k,0)(0,k) \end{cases}$, where $z_1 < z_2$, then there is a $\mathcal{K}$-singular point in (z_1, z_2).

4.6. Let $a < z_1 < z_2 < \ldots < z_r < a$ where z_j is a $\mathcal{K}(a)$-singular point of $\mathcal{K}(a)$-multiplicity m_j, $j = 1, 2, \ldots, r$, on a curve $\mathcal{C}_{k+1}$ of $\mathcal{K}$-order $k+1$.

Then the sum of the $\mathcal{K}$-multiplicities of the $\mathcal{K}$-singular points is at least $\left(\sum_{j=1}^{r} m_j\right) + 1$.

Proof. (A) Each z_j is $\mathcal{K}$-singular and not γ_1^k $(k,0)(0,k)$.

By 4.1, the sum of the $\mathcal{K}$-multiplicities is at least $\sum_{j=1}^{r} m_j$ and will be at least $\left(\sum_{j=1}^{r} m_j\right)$ $+1$ if any of the z_j are $\begin{cases} \beta_1^k & (k,1)(1,k) \\ \beta_1^k & (k,0)(1,k) \\ \gamma_2^k & (k,1)(0,k) \end{cases}$.

If z_1 is $\begin{cases} \gamma_1^k & (k,0)(1,k) \\ \beta_2^k & (k,0)(0,k) \end{cases}$ then $\mathcal{K}_k^-(z_1)$ meets (a, z_1) and a characteristic close to $\mathcal{K}_k^-(z_1)$ meets (a, z_1) and meets $\mathcal{C}_{k+1}$ at k points $x_1 < x_2 < \ldots < x_k < z_1$. By contraction there is a $\mathcal{K}$-singular point in (a, z_1). Similarly, if z_r is $\begin{cases} \beta_1^k & (k,1)(0,k) \\ \beta_1^k & (k,0)(0,k) \end{cases}$ there is a $\mathcal{K}$-singular point in (z_r, a).

Hence assume that z_1 is $\begin{cases} \beta_1^k & (k,1)(0,k) \\ \beta_1^k & (k,0)(0,k) \end{cases}$, z_r is $\begin{cases} \gamma_1^k & (k,0)(1,k) \\ \beta_2^k & (k,0)(0,k) \end{cases}$, and $z_2, \ldots, z_{r-1}$ are $\begin{cases} \beta_1^k & (k,1)(0,k) \\ \gamma_1^k & (k,0)(1,k) \\ \beta_1^k & (k,0)(0,k) \\ \beta_2^k & (k,0)(0,k) \end{cases}$

If z_2 is $\begin{cases} \gamma_1^k & (k,0)(1,k) \\ \beta_2^k & (k,0)(0,k) \end{cases}$ then there is a $\mathcal{K}$-singular point in (z_1, z_2), by 4.3 and the desired result is obtained. If z_2 is not $\begin{cases} \gamma_1^k & (k,0)(1,k) \\ \beta_2^k & (k,0)(0,k) \end{cases}$ then z_2 is $\begin{cases} \beta_1^k & (k,1)(0,k) \\ \beta_1^k & (k,0)(0,k) \end{cases}$ the same as z_1.

Continue this process with $z_3, z_4, \ldots$. Either one obtains a $\mathcal{K}$-singular point in one of the (z_j, z_{j+1}) or all of $z_1, z_2, \ldots, z_{r-1}$ are $\left\{\begin{array}{ll} \beta_1^k & (k,1)(0,k) \\ \beta_1^k & (k,0)(0,k) \end{array}\right.$. But then z_r is $\left\{\begin{array}{ll} \gamma_1^k & (k,0)(1,k) \\ \beta_2^k & (k,0)(0,k) \end{array}\right.$ and there is a $\mathcal{K}$-singular point in (z_{r-1}, z_r) by 4.3.

Hence the sum of the $\mathcal{K}$-multiplicities of the $\mathcal{K}$-singular points is at least $\left(\sum_{j=1}^r m_j\right) + 1$.

(B) Each z_j is $\mathcal{K}$-singular and at least one of the z_j is γ_1^k $(k,0)$ $(0,k)$.

If z_1 is $\gamma_1^k(k,0)(0,k)$ then z_1 is of $\mathcal{K}$-multiplicity $m_1 - 1$. Also $P_k^-(z_1)$ meets (a, z_1) and one obtains a $\mathcal{K}$-singular point in (a, z_1).

If z_r is $\gamma_1^k(k,0)(0,k)$ then again z_r is of $\mathcal{K}$-multiplicity $m_r - 1$ and there is a $\mathcal{K}$-singular point in (z_r, a).

Let J be the first index other than $J = 1$ or $J = r$ for which z_j is γ_1^k $(k,0)$ $(0,k)$. Hence the $\mathcal{K}$-multiplicity of z_J is $m_J - 1$. If z_{J-1} is $\left\{\begin{array}{ll} \beta_1^k & (k,1)(0,k) \\ \beta_1^k & (k,0)(0,k) \end{array}\right.$ then there is a $\mathcal{K}$-singular point in (z_{J-1}, z_J) by 4.3.

If z_{J-1} is $\left\{\begin{array}{ll} \beta_1^k & (k,1)(1,k) \\ \beta_1^k & (k,0)(1,k) \\ \gamma_2^k & (k,1)(0,k) \end{array}\right.$ then z_{J-1} has $\mathcal{K}$-multiplicity m_{J-1} $+1$ by 4.1. Hence one is left with z_{J-1} being $\left\{\begin{array}{ll} \gamma_1^k & (k,0)(1,k) \\ \beta_2^k & (k,0)(0,k) \end{array}\right.$. Considering z_{J-2}, one either obtains one more multiplicity in (z_{J-2}, z_{J-1}) or one more multiplicity for z_{J-2}, or z_{J-2} is $\left\{\begin{array}{ll} \gamma_1^k & (k,0)(1,k) \\ \beta_2^k & (k,0)(0,k) \end{array}\right.$. Proceeding one obtains one more multiplicity or all of $z_{J-1}, z_{J-2}, \ldots z_1$ are $\left\{\begin{array}{ll} \gamma_1^k & (k,0)(1,k) \\ \beta_2^k & (k,0)(0,k) \end{array}\right.$ in which case there is a $\mathcal{K}$ (a)-singular point in (a, z_1) since $\mathcal{K}_k^-$ (z_1) meets (a, z_1). Hence the total sum of the $\mathcal{K}$-multiplicities of the $\mathcal{K}$-singular points on $(a, z_J]$ is at least $\sum_{j=1}^J m_J$. But now if z_{J+1} is $\left\{\begin{array}{ll} \gamma_1^k & (k,0)(1,k) \\ \beta_2^k & (k,0)(0,k) \end{array}\right.$, there is a $\mathcal{K}$-singular point in (z_J, z_{J+1}), by 4.3 and the total sum of the $\mathcal{K}$-multiplicities of the $\mathcal{K}$-singular points on $(a, z_{J+1}]$ is at least $\left(\sum_{j=1}^{J+1} m_j\right) + 1$. Also if z_{J+1} is $\left\{\begin{array}{ll} \beta_1^k & (k,1)(1,k) \\ \beta_1^k & (k,0)(1,k) \\ \gamma_2^k & (k,1)(0,k) \end{array}\right.$ then z_{J+1} is of $\mathcal{K}$-multiplicity m_{J+1} $+1$ and again the total sum of the $\mathcal{K}$-multiplicities of the $\mathcal{K}$-singular points on $(a, z_{J+1}]$ is at least $\left(\sum_{j=1}^{J+1} m_j\right) + 1$. If z_{J+1} is $\left\{\begin{array}{ll} \beta_1^k & (k,1)(0,k) \\ \beta_1^k & (k,0)(0,k) \end{array}\right.$ we move to the next index where a γ_1^k $(k,0)$ $(0,k)$ occurs and treat as we did for z_{J-1} being $\left\{\begin{array}{ll} \beta_1^k & (k,1)(0,k) \\ \beta_1^k & (k,0)(0,k) \end{array}\right.$ as above. If z_{J+1} is γ_1^k $(k,0)$ $(0,k)$ then treat z_{J+1} as we did z_J above.

The only possibility for which we do not get the total sum of the $\mathcal{K}$-multiplicities of the $\mathcal{K}$-singular points as being at least $\left(\sum_{j=1}^r m_j\right)$ $+1$ occurs if $z_J, x_{J+1}, \ldots, z_r$ are all γ_1^k $(k,0)$ $(0,k)$. But then there are $\mathcal{K}$-singular points in all of $(z_J, z_{J+1}), (z_{J+1}, z_{J+2}), \ldots, (z_{r-1}, z_r)$ and one in (z_r, a); altogether $\left(\sum_{j=1}^r m_j\right) + 1$.

(C) At least one of the z_j is not $\mathcal{K}$-singular.

If z_1 is not $\mathcal{K}$-singular then there is a $\mathcal{K}$-singular point in (a, z_1) since z_1 is $\mathcal{K}$ (a)-singular. If z_r is not $\mathcal{K}$-singular then there is a $\mathcal{K}$-singular point in (z_r, a).

Let J be the first index other than $J = 1$ or $J = r$ for which z_J is not $\mathcal{K}$-singular. If z_{J-1} is $\left\{\begin{array}{ll} \beta_1^k & (k,1)(0,k) \\ \beta_1^k & (k,0)(0,k) \\ \gamma_1^k & (k,0)(0,k) \end{array}\right.$, then there is a $\mathcal{K}$-singular point in (z_{J-1}, z_J), by 4.5, and if z_{J-1} is $\left\{\begin{array}{ll} \beta_1^k & (k,1)(1,k) \\ \beta_1^k & (k,0)(1,k) \\ \gamma_2^k & (k,1)(0,k) \end{array}\right.$ then the $\mathcal{K}$-multiplicity of z_{J-1} is $m_{J-1} + 1$. Hence one is left with z_{J-1} being $\left\{\begin{array}{ll} \gamma_1^k & (k,0)(1,k) \\ \beta_2^k & (k,0)(0,k) \end{array}\right.$. As in (B) the total sum of the $\mathcal{K}$-multiplicities of the $\mathcal{K}$-singular points on $(a, z_J]$ is at least $\sum_{j=1}^{J} m_j$. But if z_{J+1} is $\left\{\begin{array}{ll} \gamma_1^k & (k,0)(1,k) \\ \beta_2^k & (k,0)(0,k) \\ \gamma_1^k & (k,0)(0,k) \end{array}\right.$, there is a $\mathcal{K}$-singular point in $(z_J, z_{J+1}$, by 4.4. If z_{J+1} is $\left\{\begin{array}{ll} \beta_1^k & (k,1)(1,k) \\ \beta_1^k & (k,0)(1,k) \\ \gamma_2^k & (k,1)(0,k) \end{array}\right.$ then z_{J+1} has $\mathcal{K}$-multiplicity m_{J+1} $+1$. Hence the total sum of the $\mathcal{K}$-multiplicities of the $\mathcal{K}$-singular points on $(a, z_{J+1}]$ is at least $\left(\sum_{j=1}^{J+1} m_j\right) + 1$. If z_{J+1} is $\left\{\begin{array}{ll} \beta_1^k & (k,1)(0,k) \\ \beta_1^k & (k,0)(0,k) \end{array}\right.$ we move to the next index where a non $\mathcal{K}$-singular, $\mathcal{K}$ (a)-singular point occurs and treat as we did above for z_{J-1} being $\left\{\begin{array}{ll} \beta_1^k & (k,1)(0,k) \\ \beta_1^k & (k,0)(0,k) \end{array}\right.$. Finally, if z_{J+1} is not $\mathcal{K}$-singular, then treat z_{J+1} as we did z_J above.

The only possibility for which we do not get the total sum of the $\mathcal{K}$-multiplicities of the $\mathcal{K}$-singular points as being at least $\left(\sum_{j=1}^{r} m_j\right) + 1$ occurs if $z_J, z_{J+1}, \ldots, z_r$ are all not $\mathcal{K}$-singular. But then there are $\mathcal{K}$-singular points in all of $(z_J, z_{J+1}), (z_{J+1}, z_{J+2}), \ldots, (z_{r-1}, z_r)$ and one in (z_r, a); altogethere $\left(\sum_{j=1}^{r} m_j\right) + 1$.

5. THE MAIN RESULT

Theorem 1. *Let $\mathcal{C} = \mathcal{C}_{k+1}$ be a curve of $\mathcal{K}$-order $k + 1$ with respect to a system $\mathcal{K}$ of order-characteristics with fundamental number k. Then the sum of the $\mathcal{K}$-multiplicities of the $\mathcal{K}$-singular points of $\mathcal{C}$ is at least $k + 1$.*

Proof. The proof is by induction. The result is valid for $k = 2$ ([1]; 3.2.6).

Now assume that the result is true for $n = k - 1$ and show that it is true for $n = k$. Take any strongly differentiable ($\mathcal{C}$ satisfies EP_k at a) non-singular point a on $\mathcal{C}$. Now $\mathcal{C}$ is of $\mathcal{K}$ (a)-order k with respect to the system $\mathcal{K}$ (a) whose fundamental number is $k - 1$.

By induction the sum of the $\mathcal{K}$ (a)-multiplicities of the $\mathcal{K}$ (a)-singular points is at least k. Denote the $\mathcal{K}$ (a)-singular points z_j with $\mathcal{K}$ (a)-multiplicity m_j and $a < z_1 < z_2 < \ldots < z_r < a$. Then $\sum_{j=1}^{r} m_j \geq k$. By 4.6, the sum of the $\mathcal{K}$-multiplicities of the $\mathcal{K}$-singular points is at least $\left(\sum_{j=1}^{r} m_j\right) + 1$; i.e. $\geq k + 1$.

Hence the theorem is true by induction.

Theorem 2. *Let $\mathcal{C} = \mathcal{C}_{k+1}$ be a curve of $\mathcal{K}$-order $k + 1$ with respect to a system $\mathcal{K}$ of* *order-characteristics with fundamental number k. Then the sum of the $\mathcal{K}$-multiplicities of the $\mathcal{K}$-singular points is exactly $k + 1$.*

Proof. Combine Theorem 1 and section 5 of [6].

Corollary. *A curve C_{k+1} of $\mathcal{K}$-order $k + 1$ satisfying EP_k at each point contains exactly $k + 1$ singular points.*

Proof. Use Theorem 2 and the fact that $\mathcal{K}$-singular points satisfying EP_k are of $\mathcal{K}$-multiplicity 1.

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