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ON SPACELIKE SUBMANIFOLDS OF A PSEUDORIEMANNIAN SPACE FORM

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Abstract. *In this paper, we first prove that the mean curvature of the pseudo-umbilical submanifolds is constant, then generalize T. Ishiara's result to the submanifolds are pseudo-umbilical, last study the pseudo-umbilical submanifolds with parallel second fundamental form.*

0. INTRODUCTION

Let $N_p^{n+p}(C)$ be an $(n+p)$-dimensional pseudo-Riemannian manifold of constant curvature C, whose index is p. Let M^n be an n-dimensional specelike submanifold isometrically immersed in $N_p^{n+p}(C)$. Note that the codimension is equal to the index. T. Ishihara ([1]) proved:

Theorem A. *Let M^n be a complete maximal specelike submanifold in $N_p^{n+p}(C)$. Then either M^n is totally geodesic ($C \geq 0$) or $0 \leq S \leq -npC (C < 0)$, where S is the square of the length of the second fundamental form of M^n.*

Let h be the second fundamental form of the immersion, ξ be the mean curvature vector. $< \cdot , \cdot >$ denotes the scalar product of $N_p^{n+p}(C)$. If there exists a function λ on M^n such that

$$< h(X, Y), \xi > = \lambda < X, Y > \tag{$*$}$$

for any tangent vectors X, Y on M^n, then M^n is called a pseudo-umbilical submanifold of $N_p^{n+p}(C)$ (cf. [2]). It is clear that $\lambda \geq 0$. If the mean curvature vector $\xi = 0$ identically, then M^n is called a maximal submanifold of $N_p^{n+p}(C)$. Every maximal submanifold of $N_p^{n+p}(C)$ is itself a pseudo-umbilical submanifold of $N_p^{n+p}(C)$.

In this paper, we first prove

Theorem 1. *The mean curvature of the pseudo-umbilical submanifolds is constant.*

When the submanifold is hypersurface, Theorem 1 is correct obviously. Then using Theorem 1 we generalize Theorem A and prove

Theorem 2. *Let M^n be a complete spacelike pseudo-umbilical submanifold in $N_p^{n+p}(C)$. Then $nH^2 \leq S \leq \frac{np[H^2 - C - \sqrt{(H^2-C)^2 + 4H^2C/p}]}{2}$, where H is the mean curvature of M^n.*

It is clear that when $H \equiv 0$, by means of Theorem 2 we may obtain Theorem A. Last, we investigate the submanifolds with parallel second fundamental form and obtain

Theorem 3. *Let M^n be a spacelike pseudo-umbilical submanifold with parallel second fundamental form in $N_p^{n+p}(C)$, $p > 1$. Then*

$$3S^2 + 2n(c - H^2)S - 2n^2H^2C \geq 0$$

holds. In particular, when the equality holds, then M^n is totally geodesic or $n = p = 2$, $M^2 = H^2(\sqrt{-C})$ $(C<0)$ is a hyperbolic Veronese surface in $H_2^4\left(\sqrt{\frac{-C}{3}}\right)$. Where

$$H^2(\sqrt{-C}) = \{x \in R_1^3, <x.x> = x_1^2 + x_2^2 - x_3^2 = C, C<0\},$$

$$H_2^4\left(\sqrt{\frac{-C}{3}}\right) = \{x \in R_2^5, <x,x> = x_1^2 + x_2^2 - x_3^2 - x_4^2 - x_5^2 = \frac{C}{3}, C<0\}.$$

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1. LOCAL FORMULAS

Let $N_p^{n+p}(C)$ be an $(n+p)$-dimensional pseudo-Riemannian manifold of constant curvature C, whose index is p. Let M^n be an n-dimensional Riemannian manifold isometrically immersed in $N_p^{n+p}(C)$. As the pseudo-Riemannian metric of $N_p^{n+p}(C)$ induces the Riemannian metric of M^n, the immersion is called specelike. We choose a local field of orthonormal frames $e_1, \ldots, e_{n+p}$ in $N_p^{n+p}(C)$ such that $e_1, \ldots, e_n$ are tangent to M^n. We make use of the following convention on the ranges of indices:

$$A, B, \ldots = 1, \ldots, n+p;\ i, j, \ldots = 1, \ldots, n;\ \alpha, \beta, \ldots = n+1, \ldots, n+p.$$

Let (ω_A) be the dual frame field so that the pseudo-Riemannian metric of $N_p^{n+p}(C)$ is given by $dS^2_{N_p^{n+p}} = \sum_i \omega_i^2 - \sum_\alpha \omega_\alpha^2 = \sum_A \varepsilon_A\, \omega_A^2$, where $\varepsilon_i = 1$, $\varepsilon_\alpha = -1$. Then the structure equations of $N_p^{n+p}(C)$ are given by

$$d\omega_A = \sum_B \varepsilon_B \omega_{AB} \wedge \omega_B, \omega_{AB} + \omega_{BA} = 0,$$

$$d\omega_{AB} = \sum_C \varepsilon_C \omega_{AC} \wedge \omega_{CB} - \frac{1}{2} \sum_{CD} \varepsilon_C \varepsilon_D K_{ABCD} \omega_C \wedge \omega_D,$$

$$K_{ABCD} = c(\delta_{AC}\delta_{BD} - \delta_{AD}\delta_{BC}).$$

Restrict these forms to M^n. Then

$$\omega_\alpha = 0, \omega_{i\alpha} = \sum_j h_{ij}^\alpha \omega_j, h_{ij}^\alpha = h_{ji}^\alpha,$$

$$d\omega_i = \sum_j \omega_{ij} \wedge \omega_j,$$

$$d\omega_{ij} = \sum_k \omega_{ik} \wedge \omega_{kj} - \frac{1}{2} \sum_{kl} R_{ijkl} \omega_k \wedge \omega_l,$$

$$R_{ijkl} = C(\delta_{ik}\delta_{jl} - \delta_{il}\delta_{jk}) - \sum_\alpha (h_{ik}^\alpha h_{jl}^\alpha - h_{il}^\alpha h_{jk}^\alpha), \tag{1.1}$$

$$d\omega_\alpha = -\sum_\beta \omega_{\alpha\beta} \wedge \omega_\beta,$$

$$d\omega_{\alpha\beta} = -\sum_\gamma \omega_{\alpha\gamma} \wedge \omega_{\gamma\beta} - \frac{1}{2}\sum_{ij} R_{\alpha\beta ij}\omega_i \wedge \omega_j,$$

$$R_{\alpha\beta ij} = \sum_k (h^\alpha_{ki}h^\beta_{kj} - h^\alpha_{kj}h^\beta_{ki}). \tag{1.2}$$

We call $H = |\xi| = \frac{1}{n}\sqrt{\sum_\alpha(\sum_i h^\alpha_{ii})^2}$ the mean curvature of M^n, $S = \sum_{ij\alpha}(h^\alpha_{ij})^2$ the square of the length of h. H^α_{ijk} and h^α_{ijkl} are defined by

$$\sum_k h^\alpha_{ijk}\omega_k = dh^\alpha_{ij} + \sum_k h^\alpha_{ik}\omega_{kj} + \sum_k h^\alpha_{kj}\omega_{ki} - \sum_\beta h^\beta_{ij}\omega_{\beta\alpha} \tag{1.3}$$

and

$$\sum_l h^\alpha_{ijkl}\omega_l = dh^\alpha_{ijl} + \sum_l h^\alpha_{ijl}\omega_{lk} + \sum_l h^\alpha_{ilk}\omega_{lj} + \sum_l h^\alpha_{ljk}\omega_{li} - \sum_\beta h^\beta_{ijk}\omega_{\beta\alpha}$$

respectively.

$$h^\alpha_{ijkl} - h^\alpha_{ijlk} = \sum_m h^\alpha_{im}R_{mjkl} + \sum_m h^\alpha_{mj}R_{mikl} + \sum_\beta h^\beta_{ij}R_{\alpha\beta kl}$$

where $h^\alpha_{ijk} = h^\alpha_{ikj}$. By (1.1) we have

$$R_{ij} = C(n-1)\delta_{ij} - \sum_\alpha \left(h^\alpha_{ij}\sum_k h^\alpha_{kk}\right) + \sum_{k\alpha} h^\alpha_{ki}h^\alpha_{kj}. \tag{1.4}$$

Now, let ξ be parallel to e_{n+p}, then

$$trH_{n+p} = nH,\ trH_\alpha = 0,\ \alpha \neq n+p. \tag{1.5}$$

By a simple calculation we have ([1])

$$\frac{1}{2}\Delta S = \sum_{ijk\alpha}(h^\alpha_{ijk})^2 + \sum_{ij\alpha} h^\alpha_{ij}\Delta h^\alpha_{ij} \tag{1.6}$$

$$= \sum_{ijk\alpha}(h^\alpha_{ijk})^2 + \sum_{ijk\alpha} h^\alpha_{ij}h^\alpha_{kkij} + \sum_{ijkl\alpha} h^\alpha_{ij}h^\alpha_{lk}R_{lijk} + \sum_{ijkl\alpha} h^\alpha_{ij}h^\alpha_{li}R_{lkik}$$

$$+ \sum_{ijk\alpha\beta} h^\alpha_{ij}h^\beta_{ki}R_{\alpha\beta kj}.$$

2. PROOFS OF THEOREMS

By (*) and (1.5) we get $\sum_\alpha trH_\alpha h_{ij}^\alpha = n\lambda\delta_{ij}$, $H^2 = \lambda$ and

$$h_{ij}^{n+p} = H\delta_{ij}. \tag{2.1}$$

Using (2.1) by (1.4) we get

$$\begin{aligned} R_{ij} &= C(n-1)\delta_{ij} - nH^2\delta_{ij} + \sum_{k\alpha} h_{ik}^\alpha h_{jk}^\alpha \\ &= C(n-1)\delta_{ij} - nH^2\delta_{ij} + \sum_k h_{ik}^{n+p} h_{jk}^{n+p} + \sum_{k\alpha\neq n+p} h_{ik}^\alpha h_{jk}^\alpha \\ &= C(n-1)\delta_{ij} - nH^2\delta_{ij} + H^2\delta_{ij} + \sum_{k\alpha\neq n+p} h_{ik}^\alpha h_{jk}^\alpha \\ &= (n-1)(C-H^2)\delta_{ij} + \sum_{k\alpha\neq n+p} h_{ik}^\alpha h_{jk}^\alpha. \end{aligned} \tag{2.2}$$

For each $\alpha \neq n+p$, we may choose a frame field $e_1, \ldots, e_n$ so that H_α is diagonalized, say $h_{ij}^\alpha = \lambda_i^\alpha \delta_{ij}$, then we have $\sum_k h_{ik}^\alpha h_{kj}^\alpha = \lambda_i^\alpha \lambda_j^\alpha \delta_{ij} \geq 0$ and $\sum_{k\alpha} h_{ik}^\alpha h_{kj}^\alpha \geq 0$. Thus, combining (2.2) we may conclude following:

Lemma 1. *Let M^n be a spacelike pseudo-umbilical submanifold in $N_p^{n+p}(C)$. Then the Ricci curvature of M^n satisfies*

$$R_{ij} \geq (n-1)(C-H^2)\delta_{ij}.$$

Lemma 2.

$$\sum_{ijk} (h_{ijk}^{n+p})^2 \geq \frac{3n^2}{n+2}|\nabla H|^2.$$

Proof. Let $f_{ij} = f_{ij}^{n+p} - H\delta_{ij}$, then $\sum_i f_{ii} = 0, f_{ijk} = h_{ijk}^{n+p} - H_k\,\delta_{ij}$ and

$$\sum_{ijk} f_{ijk}^2 = \sum_{ijk} (h_{ijk}^{n+p} - H_k\delta_{ij})^2 = \sum_{ijk} (h_{ijk}^{n+p})^2 - n|\nabla H|^2. \tag{2.3}$$

From $f_{iik} = h_{iik}^{n+p} - H_k, f_{iki} = h_{iki}^{n+p} - H_i\delta_{ik}$ and noting $h_{iik}^{n+p} = h_{iki}^{n+p}$, we get

$$f_{iki} = f_{iik} + H_k - H_i\delta_{ki}.$$

Using it we get

$$\begin{aligned}\sum_{ijk} f_{ijk}^2 &\geq \sum_{i\neq k} f_{iik}^2 + \sum_{i\neq} f_{iki}^2 + \sum_{i\neq k} f_{kii}^2 + \sum_{i} f_{iii}^2 \\ &= \sum_{i\neq k} f_{iik}^2 + 2\sum_{i\neq k} f_{iki}^2 + \sum_{i} f_{iii}^2 \\ &= \sum_{i\neq k} f_{iik}^2 + 2\sum_{i\neq k} (f_{iik} + H_k - H_i\delta_{ik})^2 + \sum_{i} f_{iii}^2 \\ &= 3\sum_{i\neq k} f_{iik}^2 + 2(n-1)|\nabla H|^2 - 4\sum_{i} f_{iii}H_i + \sum_{i} f_{iii}^2 \\ &\geq \sum_{i\neq k} f_{iik}^2 + 2(n-1)|\nabla H|^2 - \frac{n+2}{(n-1)}\sum_{i} f_{iii}^2 - \frac{4(n-1)}{n+2}|\nabla H|^2 + \sum_{i} f_{iii}^2.\end{aligned} \tag{2.4}$$

On the other hand, for fixed k, noting $\sum_i f_{ii} = 0$ we have

$$\sum_{i} f_{iik}^2 = \sum_{i\neq k} f_{iik}^2 + f_{kkk}^2 = \sum_{i\neq k} f_{iik}^2 + \left(\sum_{i\neq k} f_{iik}\right)^2 \leq \sum_{i\neq k} f_{iik}^2 + (n-1)\sum_{i\neq k} f_{iik}^2 = n\sum_{n\neq k} f_{iik}^2.$$

From which we get

$$\sum_{ik} f_{iik}^2 \leq n\sum_{i\neq k} f_{iik}^2$$

and so

$$\sum_{i\neq k} f_{iik}^2 \geq \frac{1}{n-1}\sum_{i} f_{iii}^2. \tag{2.5}$$

Substituting (2.5) into (2.4) we get

$$\begin{aligned}\sum_{ijk} f_{ijk}^2 \geq \frac{3}{n-1}\sum_{i} f_{iii}^2 + 2(n-1|\nabla H|^2 - \frac{n+2}{n-1}\sum_{i} f_{iii}^2 \\ -\frac{4(n-1)}{n+2}|\nabla H|^2 + \sum_{i} f_{iii}^2 = \frac{2n(n-1)}{n+2}|\nabla H|^2.\end{aligned} \tag{2.6}$$

Combining (2.3) and (2.6) we obtain

$$\sum_{ijk} (h_{ijk}^{n+p})^2 = \sum_{ijk} f_{ijk}^2 + n|\nabla H|^2 \geq \frac{2n(n-1)}{n+2}|\nabla H|^2 + n|\nabla H|^2 = \frac{3n^2}{n+2}|\nabla H|^2.$$

This proves Lemma 2.

Now, we prove Theorem 1. Since M^n is pseudo-umbilical, by (2.1) we get

$$\sum_{ijk} (h_{ijk}^{n+p})^2 = \sum_{ijk} (H_k\delta_{ij})^2 = n|\nabla H|^2. \tag{2.7}$$

Combining (2.7) with Lemma 2 we get

$$n|\nabla H|^2 \geq \frac{3n^2}{n+2}|\nabla H|^2,$$

which implies that $|\nabla H|^2 = 0$ so that H =constant. This proves Theorem 1.

By Theorem 1 we have

$$\sum_{ijk\alpha} h_{ij}^{\alpha} h_{kkij}^{\alpha} = 0. \tag{2.8}$$

Using (1.1), (1.2) and (2.1), by a simple calculation we derive

$$\begin{aligned} &\sum_{ijkl\alpha} h_{ij}^{\alpha} h_{hl}^{\alpha} R_{lijk} + \sum_{ijkl\alpha} h_{ij}^{\alpha} h_{li}^{\alpha} r_{lkjk} + \sum_{ijk\alpha\beta} h_{ij}^{\alpha} h_{ki}^{\beta} R_{\alpha\beta jk} \\ &= nCS - n^2H^2C - nH^2S + \sum_{\alpha\beta}(trH_\alpha H_\beta)^2 + \sum_{\alpha\beta} N(H_\alpha H_\beta - H_\beta H_\alpha). \end{aligned} \tag{2.9}$$

Substituting (2.8) and (2.9) into (1.6) we get

$$\begin{aligned} \frac{1}{2}\Delta S &= \sum_{ijk\alpha}(h_{ijk}^{\alpha})^2 + n(C - H^2)S - n^2H^2C \\ &\quad + \sum_{\alpha\beta}(trH_\alpha H_\beta)^2 + \sum_{\alpha\beta} N(H_\alpha H_\beta - H_\beta H_\alpha). \end{aligned} \tag{2.10}$$

Because

$$\sum_{\alpha\beta}(trH_\alpha H_\beta)^2 \geq \sum_{\alpha}(trH_\alpha^2)^2 \geq \frac{1}{p}\left(\sum_{\alpha} trH_\alpha^2\right)^2 = \frac{1}{p}S^2$$

and

$$N(H_\alpha H_\beta - H_\beta H_\alpha) \geq 0,$$

by (2.10) we can obtain

$$\begin{aligned} \frac{1}{2}\Delta S &\geq \sum_{ijk\alpha}(h_{ijk}^{\alpha})^2 + n(C - H^2)S - n^2H^2C + \frac{1}{p}S^2 \\ &\geq n(C - H^2)S - n^2H^2C + \frac{1}{p}S^2. \end{aligned} \tag{2.11}$$

In order to prove Theorem 2, we need the following:

Lemma 3 [2,3]. *Let M be a complete Riemannian manifold with Ricci curvature bounded from below. Let f be a C^2-function which is bounded from below on M. Then for all $\varepsilon > 0$, there exists a point x in M such that, at x*

$$|\nabla f| < \varepsilon, \quad \nabla f > -\varepsilon, \quad f(x) < \inf f + \varepsilon.$$

Put $f = \frac{1}{\sqrt{S+a}}$ for any positive constant a. It is clear that f is a bounded function on M^n and when f goes to the infimum, S goes to the supremum. By the direct computation we have

$$|\nabla f|^2 = \frac{|\nabla S|^2}{4} f^6 \tag{2.12}$$

and

$$\Delta f = -\frac{1}{2} f^3 \Delta S + \frac{3}{4} |\nabla S|^2 f^5,$$

namely

$$f^4 \Delta S = \frac{3}{2} |\nabla S|^2 f^6 - 2f \Delta f. \tag{2.13}$$

Substituting (2.12) into (2.13) we get

$$f^4 \Delta S = 6|\nabla f|^2 - 2f\Delta f. \tag{2.14}$$

According to Lemma 1 we know that the Ricci curvature of M^n is bounded from below. Thus, using Lemma 3 for any positive constant $\varepsilon_m (\lim_{m\to\infty} \varepsilon_m = 0)$, there exists $x_m \in M^n$ such that

$$|\nabla f(x_m)| < \varepsilon_m, \quad \Delta f(x_m) > -\varepsilon_m, \quad f(x_m) < \inf f + \varepsilon_m.$$

Combining (2.14) we obtain

$$\Delta S(x_m) f^4(x_m) < 6\varepsilon_m^2 + 2\varepsilon_m(\inf f + \varepsilon_m). \tag{2.15}$$

From (2.11) and (2.15) we get at x_m

$$3\varepsilon_m^2 + \varepsilon_m(\inf f + \varepsilon_m) > [n(C - nH^2)S(x_m) - n^2H^2C + \frac{1}{p} S^2(x_m)] f^4(x_m). \tag{2.16}$$

Now, we put $S_0 = \sup S = \lim_{m\to\infty} S(x_m)$. Therefore when $m \to \infty$, (2.16) implies

$$0 \le n(C - H^2)S_0 - n^2H^2C + \frac{1}{p} S_0^2,$$

which yields

$$S_0 \le \frac{np[H^2 - C + \sqrt{(H^2 - C)^2 + 4H^2C/p}]}{2}.$$

Since $nH^2 \le S \le \sup S = S_0$, we get

$$nH^2 \le S \le \frac{np[H^2 - C + \sqrt{(H^2 - C)^2 + 4H^2C/p}]}{2}.$$

This completes the proof of Theorem 2.

It is obvious that when $H \equiv 0$ i.e., M^n is maximal, from Theorem 2 may obtain Theorem A, immediately. In fact, when $H \equiv 0$, from (2.17) we see

$$0 \leq S \leq \frac{-npC + np|C|}{2},$$

which follows that when $C \geq 0$, $S \equiv 0$, i.e., M^n is totally geodesic; when $C < 0$, $0 \leq S \leq -npC$. To prove Theorem 3, we need the following:

Lemma 4 [5]. *Let $H_i (i \geq 2)$ be symmetric $(n \times n)$-matrices, $S_i = trH_i^2$ and $S = \sum_i S_i$. Then*

$$\sum_{ij} N(H_iH_j - H_jH_i) + \sum_{ij}(trH_iH_j)^2 \leq \frac{3}{2}S^2$$

and the equality holds if and only if all $H_i = 0$ or there exist two H_i different from zero. Moreover, if $H_1 \neq 0$, $H_2 \neq 0$, $H_i = 0$ $(i \neq 1, 2)$, then $S_1 = S_2$ and there exists an orthogonal $(n \times n)$-matrix T such that

$$TH_1^tT = \begin{pmatrix} f & 0 & 0 \\ 0 & -f & \\ 0 & & 0 \end{pmatrix}, \quad TH_2^tT = \begin{pmatrix} 0 & f & 0 \\ f & 0 & \\ 0 & & 0 \end{pmatrix}, \quad \text{where } f = \sqrt{\frac{S_1}{2}}$$

According to the assumption of Theorem 3, the second fundamental form of M^n is parallel, i.e., $h_{ijk}^{\alpha} = 0$ for all i, j, k, α, so S is a constant. Hence from (2.10) we obtain

$$0 = \frac{1}{2}\Delta S = n(C - H^2)S - n^2H^2C + \sum_{\alpha\beta} N(H_\alpha H_\beta - H_\beta H_\alpha) + \sum_{\alpha\beta}(trH_\alpha H_\beta)^2. \quad (2.18)$$

When $p > 0$, applying Lemma 4 to (2.18) we get

$$0 = n(C - H^2)S - n^2H^2C + \sum_{\alpha\beta} N(H_\alpha H_\beta - H_\beta H_\alpha) + \sum_{\alpha\beta}(trH_\alpha H_\beta)^2 \quad (2.19)$$
$$\leq n(C - H^2)S - n^2H^2C + \frac{3}{2}S^2.$$

In particular, when the equality

$$3S^2 + 2n(C - H^2)S - 2n^2H^2C = 0 \quad , \quad (2.20)$$

holds, from (2.19) we see that the following equality

$$\sum_{\alpha\beta} N(H_\alpha H_\beta - H_\beta H_\alpha) + \sum_{\alpha\beta}(trH_\alpha H_\beta)^2 = \frac{3}{2}S^2$$

holds. Therefore by Lemma 4 we see that there exist two matrice $H_{n+1} \neq 0$, $H_{n+2} \neq 0$, $H_\alpha = 0$, $\alpha \neq n+1, n+2$. Moreover, we may assume that

$$H_{n+1} = \begin{pmatrix} a & 0 & 0 \\ 0 & -a & \\ 0 & & 0 \end{pmatrix}, \quad H_{n+2} = \begin{pmatrix} 0 & a & 0 \\ a & 0 & \\ 0 & & 0 \end{pmatrix}, \quad \text{where } a = \sqrt{-\frac{n}{6}C}.$$

Thus, we know that $trH_{n+1} = trH_{n+2} = 0$, and combining (2.1) we see that $H \equiv 0$ i.e., M^n is maximal. Therefore, (2.20) implies that when $C \geq 0$, M^n is totally goedesic; when $C < 0$ and $S \neq 0$, $S = -\frac{3}{2}C$. Now, we put

$$S_\alpha = \sum_{ij}(h_{ij}^\alpha)^2,$$
$$p\sigma_1 = \sum_\alpha S_\alpha = S,$$
$$p(p-1)\sigma_2 = \sum_{\alpha\neq\beta} S_\alpha S_\beta.$$

From those we get

$$p^2(p-1)(\sigma_1^2 - \sigma_1) = \sum_{\alpha\neq\beta}(S_\alpha - S_\beta)^2.$$

Because $S_{n+1} = S_{n+2} = 2a^2 = -\frac{1}{3}nC$, we see that

$$\begin{aligned}
&p^2(p-1)(\sigma_1^2 - \sigma_2)\\
&= p^2(p-1)\left\{\frac{1}{p^2}S^2 - \frac{1}{p(p-1)}\left[\sum_{\alpha\beta} S_\alpha S_\beta - \sum_\alpha S_\alpha^2\right]\right\}\\
&= p^2(p-1)\left[\frac{1}{p^2}S^2 - \frac{1}{p(p-1)}(S^2 - 8a^4)\right]\\
&= p^2(p-1)\left[\frac{1}{p^2}S^2 - \frac{1}{p(p-1)}(S^2 - \frac{1}{2}S^2)\right]\\
&= \frac{p-2}{2}S^2 = 0.
\end{aligned}$$

Which shows $p = 2$. On the other hand, using the same method as that of [9], under the hypothesis of Theorem 3 we can prove $n = 2$. Here we omit it. Thus we obtain $n = p = 2$ and $S = -\frac{4}{3}C$ so that $M^2 = H^2\ (\sqrt{-C})$ in the hyperbolic Veronese surface $H_2^4\left(\sqrt{-\frac{C}{3}}\right)$ (cf. [1]). The completes the proof of Theorem 3.

From Theorem 3 we can obtain

Corollary. *Let M^n be a maximal spacelike pseudo-umbilical submanifold with parallel second fundamental form in $N_p^{n+p}(C)$, $p > 1$. Then M^n is totally goedesic or $S \geq -\frac{2}{3}\,nC\ (C < 0)$ and when the equality holds, $n = p = 2$ and $M^2 = H^2\ (\sqrt{-C})$ is a hyperbolic Veronese surface in $H_2^4\left(\sqrt{-\frac{C}{3}}\right)$.*

REFERENCES

[1] T. ISHIHARA, *Maximal spacelike submanifolds of a pseudo-riemannian space of constant curvature*, Michigan Math. J. 35 (1988), 345-352.

[2] B.Y. CHEN, *Some results of Chern-docarmo-Kobayashi type and the length of second fundamental form*, Indiana Univ. Math. J. 20 (1971), 1175-1185.

[3] H. OMORI, *Isometric immersions of Riemannian manifolds*, J. Math. Soc. Japan 19 (1976), 205-214.

[4] S.T. YAU, *Harmonic function on complete Riemannian manifolds*, Comm. Pure Appl. Math. 28 (1975), 201-228.

[5] A.M. LI, J.M. LI, *An intrinsic rigidity theorem for minimal submanifolds in a sphere*, Arch. Math. 58 (1992), 52-594.

[6] H. SUM, *Pseudo-umbilical submanifolds of a space form* $N^{n+p}(C)$, Tsukuba J. Math. 20 (1996), 45-50.

[7] H. SUN, *Totally really pseudo-umbilical submanifolds of a complex projective space*, Tokyo Math. J. 20 No. 2 (1997), 163-170.

[8] H. SUM, *Totally really pseudo-umbilical submanifolds of a quaternion space form*, to appear in Glasgow Math. J. 39 (1997), 391-397.

[9] S.S. CHERN, M. DO CARMO, S. KOBAYASHI, *Minimal submanifolds of a sphere with second fundamental form of constant length*, In Functional Analysis and Related Field, Springer-Verlag, New York, 1970, 60-75.

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