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ON A LINEAR POSITIVE OPERATOR ASSOCIATED WITH THE MULTIDIMENSIONAL PEARSON'S - χ^2 DISTRIBUTION

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Summary. *In this paper we defined and studied a new linear positive operator which was associated with the multidimensional Pearson's - χ^2 distribution.*

Subject Classifications: *AMS, 41A36, 41A63.*

1. INTRODUCTION

In our papers [3], [5] we defined and investigeted a linear positive operator which was associated with the Pearson's -χ^2 distribution:

$$(C_nf)(x) = E\left[f\left(\frac{1}{n}\sum_{k=1}^{n} X_k^2\right)\right] = \frac{1}{(2x)^{n/2}\Gamma(n/2)}\int_0^\infty t^{\frac{n}{2}-1}e^{\frac{-t}{2x}}f\left(\frac{t}{n}\right)dt \qquad (1.1)$$

where the sequence of independent random variables $(X_k)_{k\in N^*}$ having the same normal distribution $N(0,\sqrt{x}), x>0$, $E(X_k) = 0$, $D^2(X_k) = x$, $(\forall)k \in N^*$ and f is a real function bounded on $(0,+\infty)$ such that the mean value of the random variable $f\left(\frac{1}{n}\sum_{k=1}^{n} X_k^2\right)$ exists, for any $n \in N^*$. This operator was called by F. Altomare, M. Campiti [2] "the n^{-th} Cismasiu operator". The version of this operator, was studied by W. Feller [7], R.A. Khan [8]. In another way, J.A. Adell, J. De la Cal [1] considered the operator C_t where t is a continuous parameter. Using the Gamma distribution A. Lupaş, M. Muller [9] and A. Lupaş, D.H. Mache, M.W. Muller [10] defined and studied a version of the operator (1.1). The linear positive operator (1.1) was extension in the case of two variables [6], when f is a given function defined and bounded over

$$\Omega_2 = \{(x,y) \in R^2 | x>0, y>0\}:$$

$$(C_nf)(x,y) = \frac{1}{(4xy)^{n/2}(\Gamma(n/2))^2}\int_0^\infty\int_0^\infty (uv)^{\frac{n}{2}-1}e^{-\frac{1}{2}\left(\frac{u}{x}+\frac{v}{y}\right)}f\left(\frac{u}{n},\frac{v}{n}\right)dudv \qquad (1.2)$$

Now, in according with [11], we consider the following extension of the operators (1.1) and (1.2) to the case of several variables:

$$(C_nf)(x_1,x_2,\ldots,x_s) = \frac{1}{(2^s x_1,x_2,\ldots,x_s)^{n/2}\left(\Gamma\left(\frac{n}{2}\right)\right)^s}\int_0^\infty \cdots$$

$$\cdots\int_0^\infty (t_1,t_2\ldots t_s)^{\frac{n}{2}-1}e^{\frac{-1}{2}\left(\frac{t_1}{x_1}+\ldots+\frac{t_s}{x_s}\right)}f\left(\frac{t_1}{x_1,\ldots,\frac{t_s}{x_s}}\right)dt_1\ldots dt_s \qquad (1.3)$$

where f is a given function defined and bounded over $\Omega_s = \{(x_1,x_2,\ldots,x_s) \in R^s | x_1>0, x_2>0,\ldots,x_s>0\}$. Indeed, let be a sequence of s-dimensional random vectors $\{X_k = (X_{k1},$

$X_{k2}, \ldots, X_{ks})\}_{k\in N^*}$, where $X_{kj}, j = \overline{1,s}$ are independent random variables, having the same normal distribution $N(0, \sqrt{xj})$, $x_j > 0$, $E(X_{kj}) = 0$, $D^2(X_{kj} = x_j, k \in N^*, j = \overline{1,s}$.

We assume that the components Y_{nv} of the random vector Y_n reprezent the arithmetic means of the first n components X^2_{kv}, $k = \overline{1,n}$, $v = \overline{1,s}$: $Y_{nv} = \frac{1}{n}\sum_{k=1}^{n} X^2_{kv}$, $Y_n = (Y_{n1}, Y_{n2}, \ldots, Y_{ns})$.

These components Y_{nv} have a Pearson's-χ^2 distribution with n degrees of freedom and parameters $x_v > 0$, $v = 1, s$. If f is a real function bounded on $\Omega_s = (0, +\infty)\, x \ldots x\, (0, +\infty)$ such that the mean value of the random variables $f(Y_{n1}, Y_{n2}, \ldots, Y_{ns})$ exists for any $n \in N^*$, then (1.1) become (1.3).

2. APPROXIMATION PROPERTY OF OPERATORS

In this section we investigate the approximation properties of the operators (1.3).

Theorem 2.1 *If f is a bounded unform continuous function on* $(0,a)x\ldots x(0,a)$, $a > 0$, *then the sequence* $\{(C_nf)\,(x_1, x_2, \ldots, x_s)\}_{n\in N^*}$ *convergence uniformly to* $f(x_1, x_2, \ldots, x_s)$ *on* $(0,a)$ $x \ldots x\, (0,a)$, $a > 0$.

Proof. In accordance with the limit theorem of [6] is sufficient that $\lim_{n\to\infty} \sigma^2_{n,v} = 0$, $(\forall)v = \overline{1,s}$ where $\sigma^2_{m,v} = D^2\left(\frac{1}{n}\sum_{k=1}^{n} X^2_{k,v}\right) = \frac{1}{(2^s x_1 x_2 \ldots x_s)^{n/s}(\Gamma(\frac{n}{2}))^2} \int_0^\infty \ldots \int_0^\infty \left(\frac{t_v}{n} - x_v\right)^2$ $(t_1 t_2 \ldots t_s)^{\frac{n}{2}-1} e^{\frac{-1}{2}\left(\frac{t_1}{x_1}+\frac{t_2}{x_2}+\ldots+\frac{t_s}{x_s}\right)}\, dt_1 \ldots dt_s$ is the variances of the Pearson's - χ^2 distribution. But $\sigma^2_{n,v} = (2x_v^2 / n)$, $v = \overline{1,s}$ and $\lim_{n\to\infty} \sigma^2_{n,v} = 0$, $(\forall)v = \overline{1,s}$. We conclude that $\lim_{n\to\infty}$ $(C_nf)\,(x_1, \ldots, x_s) = f\,(x_1, \ldots, x_s)$ uniformly on $(0,a)\, x \ldots x\, (0,a)$, $a > 0$ for any function f uniform continuous.

3. ESTIMATE OF THE ORDER OF APPROXIMATION

We shall now proceed to estimate the order of approximation of the function f by the operator (1.3). It is convenient to make use of the modulus of continuity, defined as follows: $\omega(f, \delta_1, \delta_2, \ldots, \delta_s) = \sup\,\{|f\,(x''_1, x''_2, \ldots, x''_s) - f\,(x'_1, x'_2, \ldots, x'_s)\mid;\ |x''_1 - x'_1| < \delta_1, \ldots, |x''_s - x'_s\,| < \delta_s\}$ where $(x''_1, x''_2, \ldots, x''_s)$ and $(x'_1, x'_2, \ldots, x'_s)$ are point of $(0,a)\, x \ldots x\, (0,a)$, $a > 0$.

Theorem 2.1 *If f is a bounded and uniform continuous function on* $(0,a)\, x \ldots x\, (0,a)$, $a > 0$, *then*

$$|f(x_1, x_2, \ldots, x_s) - (C_nf)(x_1, x_2, \ldots, x_s)| < (1 + sa\sqrt{2})\omega\left(f, \frac{1}{\sqrt{n}}, \ldots, \frac{1}{\sqrt{n}}\right).$$

Proof. Using the following properties to the modulus of continuity: $|f(x''_1, x''_2, \ldots, x''_s)$ $- f(x'_1, x'_2, \ldots, x'_s)\,| < \omega\,\left(f, |x''_1 - x'_1|, \ldots, |x''_s - x'_s|\right)$ and $\omega(f, \lambda_1\,\delta_1, \lambda_2\,\delta_2, \ldots, \lambda_s\,\delta_s)$ $< (1 + \lambda_1 + \ldots + \lambda_s)\,\omega\,(f, \delta_1, \delta_2, \ldots, \delta_s)$ where $\lambda_1 > 0, \ldots, \lambda_s > 0$, we have $|f\,(x''_1, x''_2, \ldots,$ $x''_s) - f(x'_1, x'_2, \ldots, x'_s)\,| < \omega\,(f; \frac{1}{\delta_1}|x''_1 - x'_1\,|\delta_1, \ldots, \frac{1}{\delta_s}\,|x''_s - x'_s\,|\delta_s)$.

Now, $|f(x_1, x_2, \ldots, x_s)\ - (C_nf)\,(x_1, x_2, \ldots, x_s)\mid\ \leq \int_0^\infty \ldots \int_0^\infty |f\,(x_1, x_2, \ldots, x_s) - f$ $\left(\frac{t_1}{x_1}, \ldots, \frac{t_s}{x_s}\right) \mid \rho_n\,(t_1 \ldots t_s; x_1 \ldots x_s)\, dt_1 \ldots d$ where

$$\rho_n(t_1, t_2, \ldots t_s, x_1, x_2, \ldots, x_s) =$$

$$= \begin{cases} \dfrac{1}{(2^s x_1 x_2 \ldots x_s)^{\frac{n}{2}}} (t_1 t_2 \ldots t_s)^{\frac{n}{2}-1} e^{-\frac{1}{2}\left(\frac{t_1}{x_1}+\ldots+\frac{t_s}{x_s}\right)}, & t_i > 0, x_i > 0, i = \overline{1,s} \\ 0, & t_i \le 0, \end{cases}$$

We may therefore write: $|f(x_1,x_2,\ldots,x_s) - (C_n f)(x_1,x_2,\ldots,x_s)| < \left[1 + \sum_{v=1}^{s} \left(\frac{1}{\delta_v} C_n\left(\left|x_v - \frac{t_v}{n}\right|;\ x_1,\ldots,x_s\right)\right)\right] \omega(f;\ \delta_1,\ \ldots,\ \delta_s)$. In accordance with the Cauchy-Schwarz inequality, we have:

$$C_n\left(\left|x_v - \frac{t_v}{n}\right|, x_1, \ldots, x_s\right) \le$$

$$\le \left(\int_0^\infty \cdots \int_0^\infty \left(x_v - \frac{t_v}{n}\right)^2 \rho_n(t_1,\ldots,t_s,x_1,\ldots,x_s) dt_1 \ldots dt_s\right)^{1/2} = \sigma_{nv} = \left(\frac{2x_v^2}{n}\right)^{1/2}$$

So: $|f(x_1,x_2,\ldots,x-s) - (C_n f)(x_1,x_2,\ldots,\ldots,x_s)| < \left(1 + \sum_{v=1}^{s} \frac{x_v\sqrt{2}}{\delta_v\sqrt{n}}\right) \omega(f,\delta_1,\ldots,\delta_s)$.

For $\delta_v = 1/\sqrt{n}$, $v = \overline{1,s}$ and $\sup\{x_v\sqrt{2}\ |x_v \in (0,a)\} = a\sqrt{2}$, we obtain:

$$|f(x_1,x_2,\ldots,x_s) - (C_n f)(x_1,x_2,\ldots,x_s)| < (1 + sa\sqrt{2})\omega(f, 1/\sqrt{n}, \ldots, 1/\sqrt{n}).$$

4. ASYMPTOTIC ESTIMATE OF THE REMAINDER

We next turn to the task of establishing an asymptotic estimate of the remainder $R_n(f;\ x_1,\ x_2,\ \ldots,\ x_s) = f(x_1,\ x_2,\ \ldots,\ x_s) - (C_n f)\ (x_1,\ x_2,\ \ldots,\ x_s)$ which corresponds to a result of Voronovskaya about Bernstein polynomials.

Theorem 4.1 *If f is a function defined and bounded on $(0,+\infty)\ x \ldots x\ (0,+\infty)$ and at an interior point $(x_1,\ x_2,\ \ldots,\ x_s)$ of Ω_s the second differential $d^2 f(x_1,\ x_2,\ \ldots,\ x_s)$ exists, then we have the asymptotic formula*:

$$\lim_{n\to\infty} \left[|f(x_1,x_2,\ldots,x_s) - (C_n f)(x_1,x_2,\ldots,x_s)|\right] = -\sum_{v=1}^{s} x_v^2 f''_{x_v^2}(x_1,\ldots,x_s).$$

Proof. Let $(t_1,t_2,\ldots,t_s) \in (0,+\infty)x\ldots x(0,+\infty)$ be. Under the hypothesis of the theorem, exists a function $g(t_1,t_2,\ldots,t_s)$ defined on $(0,+\infty)\ x \ldots x\ (0,+\infty)$ such that when $(t_1,\ t_2,\ \ldots,t_s) \to (x_1,x_2,\ldots,x_s)$ we have $g(t_1,\ t_2,\ \ldots,\ t_s) \to 0$ and $f\left(\frac{t_1}{n},\ldots,\frac{t_s}{n}\right) = f(x_1,\ x_2,\ \ldots,x_s) + \sum_{v=1}^{s}\left(\frac{t_v}{n} - x_v\right) f'_{x_v}\ (x_1,\ \ldots,x_s) + \frac{1}{2}\sum_{v,j=1}^{s}\left(\frac{t_v}{n} - x_v\right)\left(\frac{t_j}{n} - x_j\right) f''_{x_v x_j}\ (x_1,\ \ldots,x_s) + \left(\sum_{v=1}^{s}\left(\frac{t_v}{n} - x_v\right)^2\right) g\left(\frac{t_1}{n},\ldots,\frac{t_s}{n}\right)$.

Multiply by $\rho_n(t_1,\ldots,t_s,x_1,\ldots,x_s)$ and then integrate into $t_1,t_2,\ldots,t_s$ with $t_1 > 0,\ \ldots,\ t_s > 0$, we have: $R_n(f,\ x_1,\ x_2,\ \ldots,\ x_s) = -\frac{1}{2}\sum_{v=1}^{s}\frac{2x_v^2}{n} f''_{x_v^2}\ (x_1,\ x_2,\ \ldots,\ x_s) + \alpha_n\ (x_1,\ x_2,\ \ldots,\ x_s)$ where $\alpha_n\ (x_1,x_2,\ldots,x_s) = \int_0^\infty \ldots \int_0^\infty \left(\sum_{v=1}^{s}\left(\frac{t_v}{n} - x_v\right)^2\right) g\left(\frac{t_1}{n},\ \ldots,\ \frac{t_s}{n}\right) \rho_n\ (t_1,\ \ldots,t_s;\ x_1,\ldots,x_s)\ dt_1\ \ldots\ dt_s$.

Since $g\left(\frac{t_1}{n},\ldots,\frac{t_s}{n}\right) \to 0$ as $\frac{t_v}{n} \to x_v$, $v = \overline{1,s}$, it follows that for any positive $\varepsilon > 0$ there are the positive numbers $\delta_1,\ \ldots,\ \delta_s$ such that $\left|g\left(\frac{t_1}{n},\ldots,\frac{t_s}{n}\right)\right| > \varepsilon$ whenever $\frac{t_v}{n} \to x_v$, $v = \overline{1,s}$.

In view of the fact that

$$\alpha_n(x_1,x_2,\ldots,x_s) \leq \int_0^\infty \cdots \int_0^\infty \left(\sum_{\nu=1}^{s}\left(\frac{t_\nu}{n}-x_\nu\right)^2\right)\left|g\left(\frac{t_1}{n},\ldots,\frac{t_s}{n}\right)\right|$$

$$\rho_n(t_1,\ldots,t_s,x_1,\ldots,x_s)dt_1\ldots dt_s$$

we may proceed further in the same way as in the case of one variable [4] and reach the conclusion that $\alpha_n(x_1,x_2,\ldots,x_s) = \frac{\varepsilon_n(x_1,x_2,\ldots,x_s)}{n}$, where $\varepsilon_n\,(x_1,x_2,\ldots,x_s) \to 0,\, n \to \infty$.

REFERENCES

[1] J.A. ADELL, K. DE LA CAL, *Using stochastica processes for studiyng Bernstein - type operators*, Rend. Circ. Mat. Pallermo (2) Suppl. 33 (1993), 125-141.

[2] F. ALTOMARE, M. CAMPITI, *Korovkin-type Approximation Theory and its Applications*, Walter De Gruyter, Berlin New York 1994.

[3] C.S. CISMAŞIU, *About an Infinitely Divisible Distribution*, Proceedings of the Colloquim on Approximation and Optimization, Cluj-Napoca 1984, 53-58.

[4] C.S. CISMAŞIU, *Probabilistic interpretation of Voronovskaya's Theorem*, Bul. Univ. Brasov, Seria C, vol. XXVII, (1985), 7-12.

[5] C.S. CISMAŞIU, *A Linear positive operator associated with the Pearson's - χ^2 distribution*, Studia Univ. Babes-Bolyai, Matematica, XXXII, 4, 1987, 21-23.

[6] C.S. CISMAŞIU, *A new linear positive operator in two variables associated with the Pearson's - χ^2 distribution*, (to appear).

[7] W. FELLER, *An Introduction to Probability Theory and its Application*, Vol. II 3 -rd edition, J. Willey, New-York, 1966, 219.

[8] R.A. KHAN, *Some probabilistic methods in the theory of approximation operators*, Acta Math. Acad. Sci. Hungar, 39 (1980), 193-203.

[9] A. LUPAŞ, M.W. MULLER, *Punktweise und gleichmaßige Approximation durch Gamma-operatorn*, Math. Zeitschrift, 103 (1968), 227-238.

[10] A. LUPAŞ, D.H. MACHE, M.W. MULLER, *Weighted L_p - approximation of derivatives by the method of gamma operators*, Results in Mathematics, 28 (1995), 277-286.

[11] D.D. STANCU, *Probabilistic Methods in the theory of Approximation of functions of several variables by liniar positive operators*, Approximation Theory, Intern. Symp. Univ. Lancaster, July 1969, London 1970, 329-342.

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