ON A LINEAR POSITIVE OPERATOR ASSOCIATED WITH THE MULTIDIMENSIONAL PEARSON’S - $\chi^2$ DISTRIBUTION

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Summary. In this paper we defined and studied a new linear positive operator which was associated with the multidimensional Pearson’s - $\chi^2$ distribution.

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1. INTRODUCTION

In our papers [3], [5] we defined and investigated a linear positive operator which was associated with the Pearson’s - $\chi^2$ distribution:

$$(C_nf)(x) = E \left[ f \left( \frac{1}{n} \sum_{k=1}^{n} X_k \right) \right] = \frac{1}{(2\pi)^n/2\Gamma(n/2)} \int_{0}^{\infty} t^{n/2-1} e^{-\frac{t}{n}} \int_{0}^{\infty} f \left( \frac{t}{n} \right) dt$$ (1.1)

where the sequence of independent random variables $(X_k)_{k \in N^*}$ having the same normal distribution $N(0, \sqrt{x}), x > 0$, $E(X_k) = 0$, $D^2(X_k) = x, (\forall) k \in N^*$ and $f$ is a real function bounded on $(0, +\infty)$ such that the mean value of the random variable $f \left( \frac{1}{n} \sum_{k=1}^{n} X_k \right)$ exists, for any $n \in N^*$. This operator was called by F. Altomare, M. Campiti [2] "the $n$-th Cismasiu operator". The version of this operator, was studied by W. Feller [7], R.A. Khane [8]. In another way, J.A. Adell, J. De la Cal [1] considered the operator $C_i$ where $i$ is a continuous parameter. Using the Gamma distribution A. Lupaș, M. Muller [9] and A. Lupaș, D.H. Mache, M.W. Muller [10] defined and studied a version of the operator (1.1). The linear positive operator (1.1) was extension in the case of two variables [6], when $f$ is a given function defined and bounded over

$$\Omega_2 = \{ (x, y) \in \mathbb{R}^2 | x > 0, y > 0 \} :$$

$$(C_nf)(x, y) = \frac{1}{(4\pi)^n/2(\Gamma(n/2))^2} \int_{0}^{\infty} \int_{0}^{\infty} (uv)^{n/2-1} e^{-\frac{1}{4}(\frac{x+y}{n})^2} f \left( \frac{u}{n}, \frac{v}{n} \right) du dv$$ (1.2)

Now, in according with [11], we consider the following extension of the operators (1.1) and (1.2) to the case of several variables:

$$(C_nf)(x_1, x_2, \ldots, x_s) = \frac{1}{(2\pi x_1, x_2, \ldots, x_s)^n/2 \Gamma(n/2)^s} \int_{0}^{\infty} \cdots$$

$$\cdots \int_{0}^{\infty} \left( \frac{t_1}{x_1}, \ldots, \frac{t_s}{x_s} \right)^{n/2-1} e^{-\frac{1}{2} \left( \frac{t_1}{x_1} + \cdots + \frac{t_s}{x_s} \right)^2} f \left( \frac{t_1}{x_1}, \ldots, \frac{t_s}{x_s} \right) dt_1 \cdots dt_s$$ (1.3)

where $f$ is a given function defined and bounded over $\Omega_s = \{ (x_1, x_2, \ldots, x_s) \in \mathbb{R}^s | x_1 > 0, x_2 > 0, \ldots, x_s > 0 \}$. Indeed, let be a sequence of s-dimensional random vectors $\{ X_k = (X_{k1},$
$X_{k1}, \ldots, X_{ks})_{k \in N^*}$, where $X_{kj}, j = 1, s$ are independent random variables, having the same normal distribution $N(0, \sqrt{x_j})$, $x_j > 0$, $E(X_{kj}) = 0$, $D^2(X_{kj}) = x_j$, $k \in N^*$, $j = 1, s$.

We assume that the components $Y_{nv}$ of the random vector $Y_n$ represent the arithmetic means of the first $n$ components $X_{kv}^2, k = 1, n, v = 1, s$: $Y_{nv} = \frac{1}{n} \sum_{k=1}^n X_{kv}^2$, $Y_n = (Y_{n1}, Y_{n2}, \ldots, Y_{ns})$.

These components $Y_{nv}$ have a Pearson's-$\chi^2$ distribution with $n$ degrees of freedom and parameters $x_v > 0$, $v = 1, s$. If $f$ is a real function bounded on $\Omega_s = (0, +\infty) x \ldots x (0, +\infty)$ such that the mean value of the random variables $f(Y_{n1}, Y_{n2}, \ldots, Y_{ns})$ exists for any $n \in N^*$, then (1.1) become (1.3).

2. APPROXIMATION PROPERTY OF OPERATORS

In this section we investigate the approximation properties of the operators (1.3).

**Theorem 2.1** If $f$ is a bounded uniform continuous function on $(0, a)x \ldots x (0, a)$, $a > 0$, then the sequence $\{C_{nf}(x_1, x_2, \ldots, x_s)\}_{n \in N^*}$ convergence uniformly to $f(x_1, x_2, \ldots, x_s)$ on $(0, a)x \ldots x (0, a)$, $a > 0$.

**Proof.** In accordance with the limit theorem of [6] is sufficient that $\lim_{n \to \infty} \sigma_{n,v}^2 = 0$, $(\forall) \nu = 1, s$ where $\sigma_{n,v}^2 = D^2 \left( \frac{1}{n} \sum_{k=1}^n X_{kv}^2 \right) = \frac{1}{(2\pi)^{n/2}} \int_0^\infty \cdots \int_0^\infty \left( \frac{1}{n} \sum_{k=1}^n X_{kv}^2 \right)^{n/2} dt_1 \ldots dt_s$ is the variances of the Pearson's-$\chi^2$ distribution. But $\sigma_{n,v}^2 = (2x_v^2 / n), \nu = 1, s$ and $\lim_{n \to \infty} \sigma_{n,v}^2 = 0$, $(\forall) \nu = 1, s$. We conclude that $\lim_{n \to \infty} (C_{nf})(x_1, \ldots, x_s) = f(x_1, \ldots, x_s)$ uniformly on $(0, a)x \ldots x (0, a)$, $a > 0$ for any function $f$ uniform continuous.

3. ESTIMATE OF THE ORDER OF APPROXIMATION

We shall now proceed to estimate the order of approximation of the function $f$ by the operator (1.3). It is convenient to make use of the modulus of continuity, defined as follows:

$\omega(f, \delta_1, \delta_2, \ldots, \delta_s) = \sup \{|f(x''_1, x''_2, \ldots, x''_s) - f(x'_1, x'_2, \ldots, x'_s)| : |x''_1 - x'_1| < \delta_1, \ldots, |x''_s - x'_s| < \delta_s\}$

where $(x''_1, x''_2, \ldots, x''_s)$ and $(x'_1, x'_2, \ldots, x'_s)$ are point of $(0, a)x \ldots x (0, a)$, $a > 0$.

**Theorem 2.1** If $f$ is a bounded and uniform continuous function on $(0, a)x \ldots x (0, a)$, $a > 0$, then

$$|f(x_1, x_2, \ldots, x_s) - (C_{nf})(x_1, x_2, \ldots, x_s)| < (1 + sa\sqrt{2})\omega \left( f, \frac{1}{\sqrt{n}}, \ldots, \frac{1}{\sqrt{n}} \right).$$

**Proof.** Using the following properties to the modulus of continuity: $|f(x''_1, x''_2, \ldots, x''_s) - f(x'_1, x'_2, \ldots, x'_s)| < \omega (f, |x''_1 - x'_1|, \ldots, |x''_s - x'_s|)$ and $\omega(f, \lambda_1 \delta_1, \lambda_2 \delta_2, \ldots, \lambda_s \delta_s) < (1 + \lambda_1 + \ldots + \lambda_s) \omega (f, \delta_1, \delta_2, \ldots, \delta_s)$ where $\lambda_i > 0, \ldots, \lambda_s > 0$, we have $|f(x''_1, x''_2, \ldots, x''_s) - f(x'_1, x'_2, \ldots, x'_s)| < \omega (f; \frac{1}{\sqrt{n}} |x''_1 - x'_1|, \delta_1, \ldots, \frac{1}{\sqrt{n}} |x''_s - x'_s|, \delta_s)$. Now, $\left|f(x_1, x_2, \ldots, x_s) - (C_{nf})(x_1, x_2, \ldots, x_s)\right| \leq \int_0^\infty \cdots \int_0^\infty |f(x_1, x_2, \ldots, x_s) - f(t_1, \ldots, t_s)| dt_1 \ldots d$ where

$\rho_n(t_1, \ldots, t_s; x_1, x_2, \ldots, x_s)$ =

$$\rho_n(t_1, t_2, \ldots, t_s, x_1, x_2, \ldots, x_s) =$$
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\[
\frac{1}{(2\pi x_1 x_2 \ldots x_s)^{\frac{s}{2}}} e^{-\frac{1}{2} \sum_{i=1}^{s} \left( \frac{x_i}{x_i^2} + \frac{x_i^2}{\alpha_i} \right)}, \quad t_i > 0, x_i > 0, i = 1, s
\]

0, \quad t_i \leq 0,

We may therefore write: \(|f(x_1, x_2, \ldots, x_s) - (C_n f)(x_1, x_2, \ldots, x_s)| < \left[ 1 + \sum_{v=1}^{s} \left( \frac{1}{\delta_v} \right) C_n \left( \left| x_v - \frac{t_v}{n} \right| ; \ x_1, \ldots, x_s \right) \right] \omega(f; \delta_1, \ldots, \delta_s)\). In accordance with the Cauchy-Schwarz inequality, we have:

\[
C_n \left( \left| x_v - \frac{t_v}{n} \right| ; x_1, \ldots, x_s \right) \leq \left( \int_0^\infty \cdots \int_0^\infty (x_v - \frac{t_v}{n})^2 \rho_n(t_1, \ldots, t_s, x_1, \ldots, x_s) dt_1 \cdots dt_s \right)^{1/2} = \sigma_{nv} = \left( \frac{2t_v^2}{n} \right)^{1/2}
\]

So: \(|f(x_1, x_2, \ldots, x-s) - (C_n f)(x_1, x_2, \ldots, x_s)| < \left( 1 + \sum_{v=1}^{s} \frac{v k_v^2}{\delta_v \sqrt{n}} \right) \omega(f, \delta_1, \ldots, \delta_s)\). For \(\delta_v = 1 / \sqrt{n}, v = 1, s\) and \(\sup \{x_v \sqrt{2} \mid x_v \in (0, a) \} = a \sqrt{2}\), we obtain:

\[
|f(x_1, x_2, \ldots, x_s) - (C_n f)(x_1, x_2, \ldots, x_s)| < (1 + sa \sqrt{2}) \omega(f, 1 / \sqrt{n}, \ldots, 1 / \sqrt{n})
\]

4. ASYMPTOTIC ESTIMATE OF THE REMAINDER

We next turn to the task of establishing an asymptotic estimate of the remainder \(R_n(f; x_1, x_2, \ldots, x_s) = f(x_1, x_2, \ldots, x_s) - (C_n f)(x_1, x_2, \ldots, x_s)\) which corresponds to a result of Voronovskaya about Bernstein polynomials.

**Theorem 4.1** If \(f\) is a function defined and bounded on \((0, +\infty) x \ldots x (0, +\infty)\) and at an interior point \((x_1, x_2, \ldots, x_s)\) of \(\Omega_s\) the second differential \(d^2 f(x_1, x_2, \ldots, x_s)\) exists, then we have the asymptotic formula:

\[
\lim_{n \to \infty} \left[ \left| f(x_1, x_2, \ldots, x_s) - (C_n f)(x_1, x_2, \ldots, x_s) \right| \right] = -\sum_{v=1}^{s} \frac{2f''_{x_v}}{x_v^2}(x_1, \ldots, x_s).
\]

**Proof.** Let \((t_1, t_2, \ldots, t_s) \in (0, +\infty) x \ldots x (0, +\infty)\) be. Under the hypothesis of the theorem, exists a function \(g(t_1, t_2, \ldots, t_s)\) defined on \((0, +\infty) x \ldots x (0, +\infty)\) such that when \((t_1, t_2, \ldots, t_s) \to (x_1, x_2, \ldots, x_s)\) we have \(g(t_1, t_2, \ldots, t_s) \to 0\) and \(f \left( \frac{t_1}{n}, \ldots, \frac{t_s}{n} \right) = f(x_1, x_2, \ldots, x_s) + \sum_{v=1}^{s} \left( \frac{t_v}{n} - x_v \right) f'_{x_v}(x_1, \ldots, x_s) + \frac{1}{2} \sum_{v,j=1}^{s} \left( \frac{t_v}{n} - x_v \right) \left( \frac{t_j}{n} - x_j \right) f''_{x_vx_j}(x_1, \ldots, x_s) + \left( \sum_{v=1}^{s} \frac{t_v^2}{n} - x_v \right)^2 \right) g \left( \frac{t_1}{n}, \ldots, \frac{t_s}{n}, \ldots, \frac{t_s}{n} \right) \right) \).

Multiply by \(\rho_n(t_1, t_2, \ldots, t_s)\) and then integrate into \(t_1, t_2, \ldots, t_s\) with \(t_1 > 0, \ldots, t_s > 0\), we have: \(R_n(f; x_1, x_2, \ldots, x_s) = -\frac{1}{2} \sum_{v=1}^{s} \frac{2f''_{x_v}}{x_v^2}(x_1, x_2, \ldots, x_s) + \alpha_n(x_1, x_2, \ldots, x_s)\) where \(\alpha_n(x_1, x_2, \ldots, x_s) = \int_0^\infty \cdots \int_0^\infty \left( \sum_{v=1}^{s} \left( \frac{t_v}{n} - x_v \right)^2 \right) g \left( \frac{t_1}{n}, \ldots, \frac{t_s}{n} \right) \rho_n(t_1, \ldots, t_s; x_1, \ldots, x_s) dt_1 \cdots dt_s\).

Since \(g \left( \frac{t_1}{n}, \ldots, \frac{t_s}{n} \right) \to 0\) as \(\frac{t_v}{n} \to x_v, v = 1, s,\) it follows that for any positive \(\varepsilon > 0\) there are the positive numbers \(\delta_1, \ldots, \delta_s\) such that \(|g \left( \frac{t_1}{n}, \ldots, \frac{t_s}{n} \right)| > \varepsilon\) whenever \(\frac{t_v}{n} \to x_v, v = 1, s.\)
In view of the fact that

$$\alpha_n(x_1, x_2, \ldots, x_s) \leq \int_0^\infty \ldots \int_0^\infty \left( \sum_{v=1}^s \left( \frac{t_v}{n} - x_v \right)^2 \right) \left| g \left( \frac{t_1}{n}, \ldots, \frac{t_s}{n} \right) \right|$$

$$\rho_n(t_1, \ldots, t_s, x_1, \ldots, x_s) dt_1 \ldots dt_s$$

we may proceed further in the same way as in the case of one variable [4] and reach the conclusion that $\alpha_n(x_1, x_2, \ldots, x_s) = \frac{\epsilon_n(x_1, x_2, \ldots, x_s)}{n}$, where $\epsilon_n(x_1, x_2, \ldots, x_s) \to 0, n \to \infty$. 
REFERENCES


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