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RIEMANNIAN MANIFOLDS SATISFYING [RIC $\wedge g, W] = 0$

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Abstract. *We study the condition [Ric $\wedge g, W] = 0$ on 2n-dimensional Riemannian manifolds which also have non-negative curvature on totally isotropic two-planes. We show that if, in addition, certain holomorphic bisectional curvatures are positive, then the manifold is biholomorphically isometric to* $\mathbb{CP}^n$.

1. INTRODUCTION

Let M be a $2n$-dimensional Riemannian manifold, $n \geq 2$, with curvature tensor R. We denote the corresponding curvature operator by $\hat{R}$. Recall (cf. [1], Chap. 1, Sec. G), that we may decompose R uniquely into its $O(2n)$-irreducible components: $R = U + Z + W$, where $U = \frac{s}{4n(2n-1)} g \wedge g$, $z = \frac{1}{2(n-1)} \left[(Ric - \frac{s}{2n} g) \wedge g\right]$ (the Ricci-traceless part of R), and W is the Weyl tensor.

Given a real vector space V with inner product $\langle , \rangle$, let $V^{\mathbb{C}} = V \otimes_{\mathbb{R}} \mathbb{C}$ denote the complexification of V. Extend $\langle , \rangle$ to a complex bilinear form on $V^{\mathbb{C}}$. Finally, let $\langle\langle \ , \ \rangle\rangle$ denote the extension of $\langle , \rangle$ to $V^{\mathbb{C}}$ which is linear in the first component and conjugate linear in the second. That is, $\langle\langle z, w \rangle\rangle = \langle z, \bar{w} \rangle$ for $z, w \in V^{\mathbb{C}}$.

We now recall the definition of totally isotropic curvature, which we denote by K^{iso}, introduced in [5].

Definition. A complex vector $z \in V^{\mathbb{C}}$ is *isotropic* if $\langle z, z \rangle = 0$. A complex subspace W of $V^{\mathbb{C}}$ is *totally isotropic* if z is isotropic for all $z \in W$.

For a complex subspace σ of $V^{\mathbb{C}}$ spanned by z and w, let $K(\sigma) = \frac{\langle\langle \hat{R}(z\wedge w), z\wedge w \rangle\rangle}{\|z\wedge w\|^2}$. We note that this number does not depend on the choice of z and w.

Definition. A curvature tensor R has *non-negative (positive) curvature on totally isotropic 2-planes* if $K(\sigma) \geq 0$ (>0) for all totally isotropic complex two-dimensional subspaces of $V^{\mathbb{C}}$.

We note that two well-studied conditions on curvature imply conditions on isotropic curvatures (cf. [5]):

(1) If the curvature tensor R is positive (non-negative), that is if $\langle \hat{R}\alpha, \alpha \rangle > 0$ (≥ 0) for all non-zero $\alpha \in \bigwedge^2$, then K^{iso} is positive (non-negative).

(2) If sectional curvature K is quarter-pinched, that is if $\frac{\Delta}{4} \leq K \leq \Delta$ for some $\Delta > 0$, then $K^{iso} \geq 0$. If K is strictly quarter-pinched, that is if at least one of the inequalities is strict, then $K^{iso} > 0$.

In this paper, we examine a condition, [Ric $\wedge g, W] = 0$, which applies to two important classes of Riemannian manifolds: Einstein and conformally flat. Combined with the concept of curvature on totally isotropic two-planes, this condition invites various diagonalizations of $\hat{R}$ which can be exploited to a certain extent. In particular, [7] introduces an orthonormal basis for two-forms which allows the expression of eigenvalues of the Weitzenböck operator on 2-forms (denoted $\mathcal{R}_2$) as sums of (non-negative multiples of) totally isotropic curvatures.

A derivation of this formula begins with the choice of an orthonormal basis $\{f_i\}_{i=1}^{2n}$ of a vector space V with respect to which an eigenvector α of $\mathcal{R}_2$ may be written as $\alpha = \sum_{i=1}^{n} \mu_k(f_{2i-1} \wedge f_{2i})$, where $1 = \mu_1 \geq \mu_2 \geq \ldots \geq \mu_n \geq 0$. Let $\mathcal{R}_2\alpha = q\alpha$. Using the Weitzenböck formula for $\mathcal{R}_2$ (see for example [9], Chapter 2) one computes as in [8] (Eq. 1.4, page 849):

$$q = \langle \mathcal{R}_2\alpha, f_1 \wedge f_2 \rangle = \sum_{k=2}^{n} \{K_{1,2k} + K_{1,2k-1} + K_{2,2k} + K_{2,2k-1} + 2\mu_k R_{1,2,2,k-1,2k}\}$$

One of the insights of [7], via Lie algebra theory, is that, using the basis $X_{ij} = \frac{1}{\sqrt{2}}[(f_{2i-1} + if_{2i}) \wedge (f_{2j-1} + if_{2j})]$ and $X'_{ij} = \frac{1}{\sqrt{2}}[(f_{2i-1} + if_{2i}) \wedge (f_{2j-1} - if_{2j})]$, the right hand side of this equation is equal to the expression:

$$\sum_{k=2}^{n} \{(1 + \mu_k)\langle\langle \hat{R}X_{1k}, X_{1k} \rangle\rangle + (1 - \mu_k)\langle\langle \hat{R}X'_{1k}, X'_{1k} \rangle\rangle\}$$

which one can confirm by expanding the expression above. Thus we have:

$$q = \langle \mathcal{R}_2\alpha, f_1 \wedge f_2 \rangle = \frac{1}{2}\sum_{k=2}^{n} \{(1 + \mu_k)\langle\langle \hat{R}X_{1k}, X_{ik} \rangle\rangle + (1 - \mu_k)\langle\langle \hat{R}X'_{1k}, X'_{1k} \rangle\rangle\} \tag{1}$$

where $\langle\langle \hat{R}X_{ik}, X_{1k} \rangle\rangle$ and $\langle\langle \hat{R}X'_{1k}, X'_{1k} \rangle\rangle$ represent sectional curvatures on totally isotropic two-planes.

Using the formula $\hat{R} = \frac{1}{2}(Ric \wedge g - \mathcal{R}_2)$, we note that $[Ric \wedge g, W] = 0 \Rightarrow [Ric \wedge g, \hat{R}] = 0 \Rightarrow [Ric \wedge g, \mathcal{R}_2] = 0$. This in turn implies the existence of an orthonormal basis of 2-forms simultaneously diagonalizing $Ric \wedge g$ and $\mathcal{R}_2$. We will use this basis to establish the following consequence of the "sphere theorem" proved in [5]:

Theorem 1. *Let M be a compact, orientable $2n$-dimensional Riemannian manifold without boundary, $n \geq 2$, with curvature tensor R satisfying $[W, Ric \wedge g] = 0$. Then if $q_1(x) + q_2(x)$ is positive, where $q_1(x)$ and $q_2(x)$ are the two smallest eigenvectors of $Ric_x : T_xM \longrightarrow T_xM$, and if $\exists$ functions $b, \delta : M \longrightarrow \mathbb{R}$ such that $\delta(x) > 0$ and $\delta(x)b(x) \leq K^{iso}(T_xM) \leq b(x)$, $\forall x \in M$, $\pi_i(M) = 0$ for $2 \leq i \leq n$. In particular if M is also simply connected, M is homeomorphic to a sphere.*

Next, we turn our attention to Kähler manifolds, and again using [Ric $\wedge g, \mathcal{R}_2] = 0$ as a key step, prove (cf. [8], Theorem B):

Theorem 2. *Let $n \geq 2$ and suppose (M, g) is a compact, connected $2n$-dimensional Kähler manifold satisfying:*

(i) $[Ric \wedge g, W] = 0$
(ii) $K^{iso} \geq 0$
(iii) $H(X, Y) > 0$ *whenever* X, Y, JX *are orthonormal vectors in* T^*M
Then M is biholomorphically isometric to $\mathbb{CP}^n$*, with a multiple of the canonical metric.*

Here H denotes *holomorphic bisectional* and is given by $H(X, Y) = \langle R(X, JX)Y, JY\rangle$ (adopting the opposite sign convention of that in [2]).

Finally, we once again exploit the orthonormal basis used to prove Theorem 1 to obtain explicit formulas for eigenvalues of the Weyl tensor in both the Kähler and non-Kähler case.

We note that the condition $[\mathrm{Ric} \wedge g, W] = 0$ also arises in notions generalizing the Einstein condition. [1] (Chapter 16) gives the following equivalent conditions, all of which (by a result of J.P. Bourguignon-Corollary 16.17, p. 439) imply $[\mathrm{Ric} \wedge g, W] = 0$ on a manifold M:

(1) $d^{\nabla}\,\mathrm{Ric} = 0$ (i.e., Ric is a Codazzi tensor)
(2) $\delta R = 0$ (harmonic curvature)
(3) $n \geq 4 : \delta W = 0$ and scalar curvature constant
(4) $n = 3 : M$ conformally flat and scalar curvature constant

Here d^{∇} Ric denotes the exterior differential (cf. [1], 1.12) of Ricci tensor considered as an element of $\bigwedge^1 M \otimes TM$ (a one-form with values in the tangent bundle). i.e. $(d^{\nabla}\,\mathrm{Ric})(X, Y) = \nabla_X(\mathrm{Ric}Y) - \nabla_Y(\mathrm{Ric}X) - \mathrm{Ric}([X, Y])$. $\delta W \in \bigwedge^2 M \otimes T^*M$ is given by $(\delta W)(X_1, X_2) = -\mathrm{Tr}\,[(Y, Z) \to (\nabla_Y W)(Z, X_1, X_2)]$ (cf. [1], 16.3).

In the same chapter (p. 439), [1] discusses the concept of a *pure curvature operator* (one for which $\hat{R}$ is diagonalizable by simple two-forms-see [4]). In particular, if R is conformally flat, $\hat{R}$ may be diagonalized by the 2-forms $e_i \wedge e_j$, where $\{e_i\}$ is an orthonormal basis diagonalizing the Ricci operator. It is this basis which is used in [3] and in [6].

2. RESULTS

Proof of Theorem 1. We begin by finding bounds on $\hat{R}$ based on bounds on isotropic curvatures and on eigenvalues of the Ricci operator.

Proposition 1. *Let V be a 2n-dimensional vector space with curvature tensor R satisfying:*
(i) $[W, Ric \wedge g] = 0$
(ii) $a \leq K^{iso} \leq b$
(iii) $c \leq Ric \leq d$
Then any eigenvalue k of $\hat{R} : \bigwedge^2 V \to \bigwedge^2 V$ *satisfies* $c - 2b(n-1) \leq k \leq d - 2a(n-1)$.

Proof. Let α be an element of an orthonormal basis which simultaneously diagonalizes Ric $\wedge g$ and $\mathcal{R}_2$. Then $\mathrm{Ric} \wedge g\alpha = (\lambda_i + \lambda_j)\alpha$ (where the $\lambda_i, \lambda_1 \leq \lambda_2 \leq \ldots \leq \lambda_n$, are eigenvalues of the Ricci operator, and $\mathcal{R}_2\alpha = q\alpha$, where q is given in (1). Then $\hat{R}\alpha = \lambda\alpha$, where

$$\lambda = \frac{1}{2}(\lambda_i + \lambda_j) - \frac{1}{4}\sum_{k \geq 2}(1 + \mu_k)\langle\langle \hat{R}X_{1k}, X_{1k}\rangle\rangle + (1 - \mu_k)\langle\langle \hat{R}X'_{1k}, X'_{1k}\rangle\rangle \tag{2}$$

Now apply the bounds on Ric and on K^{iso} to obtain the desired result. □

We note that for $\mathbb{CP}^n$, equipped with the metric giving constant holomorphic sectional curvature 1, $0 \leq K^{iso} \leq \frac{3}{4}$. Hence the above inequality is sharp for $\mathbb{CP}^2$ (the right hand inequality remains sharp for $\mathbb{CP}^n$, for any n).

Corollary. *Under the same hypotheses as in Proposition 1:*

(i) If the smallest two eigenvalues of Ric sum to a positive (non-negative) number, then b must be positive (non-negative).

(ii) If the largest two eigenvalues of Ric sum to a negative (non-positive) number, then a must be negative (non-positive).

Proof.

(i) If the smallest two eigenvalues of Ric sum to a positive number and $b \leq 0$, then $k > 0$ by Proposition 2.5. But then $K^{iso} > 0$, which implies $b > 0$, a contradiction.

(ii) If the largest two eigenvalues of Ric sum to a negative number and $a \geq 0$, then $k < 0$ by Proposition 2.5. But then $K^{iso} < 0$, which implies $a < 0$, a contradiction.

The cases for non-negative and non-positive are similar. □

We now apply that vector space result to a compact manifold to obtain:

Proposition 2. *Suppose M is a compact, orientable 2n-dimensional Riemannian manifold without boundary and curvature tensor R satisfying (pointwise):*

(i) $[W, Ric \wedge g] = 0$

(ii) $\exists$ functions $b, \delta : M \to \mathbb{R}$ such that $\delta(x) > 0$ and $\delta(x)b(x) \leq K^{iso}(T_xM) \leq b(x), \forall x \in M$.

(iii) $\forall x, q_1(x) + q_2(x)$ is positive (non-negative), where $q_1(x)$ and $q_2(x)$ are the two smallest eigenvectors of $Ric_x : T_xM \to T_xM$.

Then K^{iso} is pointwise positive (non-negative).

By Proposition 2, the suppositions of Theorem 1 imply $K^{iso} > 0$, precisely the hypothesis needed in the Micallef-Moore sphere theorem. ■

Proof of Theorem 2.

Lemma 1. *Let V be a 2n-dimensional vector space with orthogonal complex structure J and Kähler curvature R satisfying:*

(i) $K^{iso} \geq 0$

(ii) $H(X, Y) > 0$ whenever X, Y, JX are orthonormal vectors in V

Then $\omega = \sum e_{2i-1} \wedge e_{2i} \in \ker \mathcal{R}_2$ (where $Je_{2i-1} = -e_{2i}$) implies $\ker \mathcal{R}_2 = \mathbb{R}\omega$.

Proof. Since the Kähler form may be written as $\omega = \sum e_{2i-1} \wedge e_{2i}$ with respect to any J-adapted orthonormal basis of V, we may as well assume that $\{e_i, Je_i\}_{i=1}^n$ is an orthonormal basis diagonalizing the Ricci operator. Say $\lambda_i = Ric(e_i, e_i)$. Then:

$$\lambda_i = Ric(e_i, e_i) = \sum_{a=1}^{n} R_{a^*aii^*} = \sum_{a \neq i} R_{a^*aii^*} + K_{ii^*}$$

(where $e_{i*} = Je_i$). But $R_{a^*aii^*} = K_{a^*i} + K_{ai}$ by the Kählerity of R. Therefore we have:

$$\lambda_i + \lambda_j = \sum_{a \neq i} (K_{a^*i} + K_{ai}) + \sum_{a \neq j} (K_{a^*j} + K_{aj}) + (K_{ii^*} + K_{jj^*})$$

By assumption (ii), the first two summands above are strictly positive. One then shows (cf. [8], page 849) that under the assumption $K^{iso} \geq 0$, the last summand is non-negative for $i \neq j$, and hence that for $i \neq j$, $\lambda_i + \lambda_j > 0$. It is this observation which is needed to conclude (through a computation) that an arbitrary element of ker $\mathcal{R}_2$ has no component in the -1 eigenspace of $J : \bigwedge^2 V \to \bigwedge^2$ (cf. [8], Equation 1.12).

Next, using a method found in [2] (Lemma 1, page 229), one shows that any element of the +1 eigenspace of $J : \bigwedge^2 V \to \bigwedge^2$ may be written $\sum \beta_i Jf_i \wedge f_i$, where $\beta_1 \geq \beta_2 \geq \ldots \geq \beta_n$ and $\beta_1 > 0$. Writing an arbitrary element of ker $\mathcal{R}_2$ in this way we conclude through computation (once again using (ii)), that $\beta_i = \beta_1, \forall i$. □

If we assume $[\text{Ric} \wedge g, \mathcal{R}_2] = 0$, in addition to the hypotheses of Lemma 1, the Ric $\wedge g$ preserves ker $\mathcal{R}_2 = \mathbb{R}\omega$. Thus ω is an eigenvector of Ric $\wedge g$. A simple computation now gives:

Lemma 2. *Let V be a $2n$-dimensional vector space with complex structure J and Kähler curvature R. If $\omega = \sum e_{2i-1} \wedge e_{2i}$ is an eigenvector of Ric $\wedge g$, R is Einstein.*

Proof. Using the same J-adapted basis $\{e_i, Je_i\}_{i=1}^n$ diagonalizing the Ricci operator one computes:

$$\sum_{i=1}^{n} ke_i \wedge Je_i = k\omega = Ric \wedge g\omega = 2\sum_{i=1}^{n} \lambda_i e_i \wedge Je_i$$

Therefore $\lambda_1 = \lambda_2 = \ldots = \lambda_n = \frac{k}{2} = \frac{s}{n}$. □

Finally, to complete the proof of Theorem 2, we note that M is a compact Kähler-Einstein manifold satisfying (iii). In [2], Theorem 5, we observe that in their proof of that theorem the hypothesis of positive holomorphic bisectional curvature is only used to assert that $R_{11^*ii^*} > 0$, where $i \geq 2$ (p. 232). But $R_{11^*ii^*} = K_{1i} + K_{1i^*}$, which is positive by our assumption (iii). ■

3. EIGENVALUES OF THE WEYL TENSOR

Suppose α is any two-form in an orthonormal basis simultaneously diagonalizing Ric $\wedge g$ and W. Since eigenvalues of Ric $\wedge g$ have the form $\lambda_i + \lambda_j$ (sums of eigenvalues of the Ricci operator), we assume that Ric $\wedge g\alpha = (\lambda_i + \lambda_j)\alpha$, and that $\hat{W}\alpha = w\alpha$. Then on writing $\alpha = \sum_{k=1}^{n} \mu_k f_{2k-1} \wedge f_{2k}$, where $1 = \mu_1 \geq \mu_2 \geq \ldots \geq \mu_n \geq 0$ and $\{f_i\}_{i=1}^{2n}$ is an orthonormal basis of V, we note that α is an eigenvector of $\hat{R}$ with eigenvalues given by (2), and also by the formula:

$$\begin{aligned} \hat{R}\alpha &= \left\{ \frac{1}{2n-2} \left[Ric - \frac{s}{2(2n-1)} g \right] \wedge g + W \right\} \alpha \\ &= \left\{ \frac{1}{2n-2} \left[\lambda_i + \lambda_j - \frac{s}{(2n-1)} \right] + w \right\} \alpha \end{aligned} \tag{3}$$

Equating (2) and (3) gives:

$$\begin{aligned} w = {} & \frac{n-2}{2(n-1)}(\lambda_i + \lambda_j) \\ & + \frac{s}{(2n-1)(2n-2)} \\ & - \frac{1}{4}\sum_{k\geq 2}\{(1+\mu_k)\langle\langle \hat{R}X_{1k}, X_{1k}\rangle\rangle + (1-\mu_k)\langle\langle \hat{R}X'_{1k}, X'_{1k}\rangle\rangle\} \end{aligned} \tag{4}$$

Exploiting this equation, we obtain (cf. [7], Proposition 2.5):

Proposition 3. *Let V be a 2n-dimensional vector space with curvature tensor R satisfying:*
(i) $[Ric \wedge g, W] = 0$
(ii) $a \leq K^{iso} \leq b$
(iii) $c \leq Ric \leq d$

Then if $d \leq (2n-1)a$, *or if* $(2n-1)b \leq c$, *R is conformally flat. In particular, if* $a = d = 0$, *or if* $b = c = 0$, *R is flat.*

Proof. By (4) and the hypotheses above:

$$\begin{aligned} w = {} & \frac{n-2}{2(n-1)}(r_i + r_j) \\ & + \frac{s}{(2n-1)(2n-2)} \\ & - \frac{1}{4}\sum_{l\geq 2}\{(1+\mu_l)\langle\langle \hat{R}X_{1l}, X_{1l}\rangle\rangle + (1-\mu_l)\langle\langle \hat{R}X'_{1l}, X'_{1l}\rangle\rangle\} \\ \leq {} & \frac{n-2}{n-1}d + \frac{2nd}{(2n-1)(2n-2)} - 2a(n-1) \\ = {} & \frac{2(n-1)}{2n-1}d - 2a(n-1) \leq 0 \end{aligned}$$

Thus since W has trace zero, we must have $W = 0$. In the second case we obtain $w \geq 0$, and so similarly, $W = 0$.

If $a = 0$, then $s \geq 0$ by [7], Proposition 2.5. But $d = 0$ implies eigenvalues of Ric are non-positive. Hence $\text{Ric} \equiv 0$ and $s = 0$, and so $R = 0$ □

Note that in the four-dimensional case (4) gives:

$$w = \frac{s}{6} - \frac{1}{4}\{(1+\mu_l)\langle\langle \hat{R}X_{12}, X_{12}\rangle\rangle + (1-\mu_l)\langle\langle \hat{R}X'_{12}, X'_{12}\rangle\rangle\}$$

We use this formula to show that the signature integrand of a compact, oriented 4-manifold without boundary is determined by curvatures on totally isotropic two-planes. Recall that the signature of a compact, oriented 4-manifold is given by $\tau(M) = \frac{1}{12\pi^2}\sum_M(|W^+|^2 - |W^-|^2)\mu_g$.

Here W^+ and W^- are the parts of W acting on the $+1$ and -1 eigenspaces of the Hodge *-operator, respectively (cf. [1]). We will list the eigenvalues of W as follows:

Eigenvalues of W^+	Eigenvalues of W^-
$\frac{s}{6} - q_1$	$\frac{s}{6} - q_4$
$\frac{s}{6} - q_2$	$\frac{s}{6} - q_5$
$\frac{s}{6} - q_3$	$\frac{s}{6} - q_6$

Note that $\frac{s}{2} - (q_1 + q_2 + q_3) = TrW^+ = 0$ and $\frac{s}{2} - (q_4 + q_5 + q_6) = TrW^- = 0$.

Proposition 4. *Let (M, g) be a compact oriented Riemannian 4-manifold without boundary. Then:*

(i) The 4-form representing the signature class of M is completely determined by curvature on isotropic two-planes.

(ii) If $a \leq K^{iso} \leq b, \forall x \in M$, then $|\tau(M)| \leq \frac{2}{\pi^2}(b - a)\max(|a|, |b|)$ Vol (M).

(iii) The upper bounds in (ii) are invariant under dilations of the metric.

Proof. (i) For each point $x \in M$:

$$\begin{aligned} &|W^+(x)|^2 - |W^-(x)|^2 \\ &= \sum_{i=1}^{3}\left(\frac{s(x)}{6} - q_i(x)\right)^2 - \sum_{i=4}^{6}\left(\frac{s(x)}{6} - q_i(x)\right)^2 \\ &= \sum_{i=1}^{3}(q_i(x))^2 - \frac{s(x)}{3}\left(\sum_{i=1}^{3} q_i(x) - \sum_{i=4}^{6} q_i(x)\right) - \sum_{i=4}^{6}(q_i(x))^2 \\ &= \sum_{i=1}^{3}(q_i(x) - q_{i+3}(x))(q_i(x) + q_{i+3}(x)) \end{aligned} \tag{5}$$

Since each $q_i(x)$ is made up of sums of multiples of isotropic curvatures, (i) follows from the formula $\tau(M) = \frac{1}{12\pi^2}\sum_M(|W^+|^2 - |W^-|^2)\mu_g$.

(ii) Note that as in the proof of Proposition 1, we have $2a \leq q_i(x) \leq 2b, \forall x \in M$. Thus $|q_i(x) - q_j(x)| \leq 2(b - a)$ and $|q_i(x) + q_j(x)| \leq 4\max(|a|, |b|)$, and the final line in (5) is bounded above in absolute value by 24 $(b - a)\max(|a|, |b|)$. Then on integrating we obtain:

$$\tau(M) = \frac{1}{12\pi^2}\int_M (|W^+(x)|^2 - |W^-(x)|^2)\mu_g \leq \frac{2}{\pi^2}(b - a)\max(|a|, |b|)Vol(M)$$

(iii) Given $t > 0$, and an m-dimensional manifold (M, g), replacement of g by tg has the following effects on the volume and curvatures on totally isotropic two-planes of M:

(i) $\mathrm{Vol}_{tg}(M) = t^{m/2}Vol_g(M)$.

(ii) $K^{iso}_{tg} = \frac{1}{t}K^{iso}_g$

Since M, is a compact manifold, we may assume that the bounds a and b on K^{iso} are actually achieved, and so for (M, tg), the right hand side of the inequalities in (ii) change to (a constant multiple of) $\frac{1}{t}(b - a)\,\frac{1}{t}\max(|a|, |b|)\,t^2$ Vol $(M) = (b - a)\max(|a|, |b|)$ Vol (M). □

We note that as a consequence of this theorem, any compact, oriented, Riemannian 4-manifold without boundary having constant curvature on totally isotropic two-planes must have signature zero.

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