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REMARKS ON MAXIMA OF REAL RANDOM SEQUENCES

R. GIULIANO ANTONINI

Let $(X_n)_{n\geq 1}$ be a sequence of real random variables, and put

$$Y_n = \max(X_1, \dots, X_n) - \sqrt{2\log n}.$$

It is well known that, if X_n are independent and $N(0,1)$-distributed, then (Y_n) converges to zero almost everywhere as $n \to \infty$ (see, for instance, [4] pp. 198-199).

The aim of this brief note is to prove two simple lemmas that will allow us to deduce that (Y_n) exhibits the same behaviour if the tail distribution of X_n (namely the function $x \to \mathbb{P}(X_n > x)$) satisfies suitable "exponential-type" conditions.

We point out that the assumptions of Lemma 1 are satisfied, in particular, in the sub-Gaussian case. We are indebted to prof. G. Letta for having greatly simplified our original proof of Lemma 2.

Our first lemma is concerned with the behaviour of $Y_n^+ = \max(Y_n, 0)$ as $n \to \infty$.

Lemma 1. *Assume that, for every n and all $x > 0$, we have*

$$\mathbb{P}(X_n > x) \leq \exp(-\frac{x^2}{2}).$$

Then the sequence (Y_n^+) converges a.s. to zero.

Proof. Fix $\epsilon > 0$: we have to prove that $\mathbb{P}(Y_n^+ > \epsilon \text{ i.o.}) = 0$. Put $Z_n = X_n - \sqrt{2\log n}$; then it is easy to see that

$$\{Y_n^+ > \epsilon \text{ i.o.}\} = \{Z_n^+ > \epsilon \text{ i.o.}\}.$$

We shall prove, using Borel-Cantelli lemma, that the last event above has probability zero. In fact

$$\mathbb{P}(Z_n^+ > \epsilon) = \mathbb{P}(Z_n > \epsilon) = \mathbb{P}(X_n > \sqrt{2\log n} + \epsilon)$$

$$\leq \exp\left(-\frac{1}{2}(\sqrt{2\log n} + \epsilon)^2\right) = \exp\left(-\frac{\epsilon^2}{2}\right) n^{-1} \exp\left(-\epsilon\sqrt{2\log n}\right).$$

The last expression above is the general term of a convergent series, since it behaves like $\int_1^{+\infty} x^{-1} \exp(-\epsilon\sqrt{2\log x})dx = \int_0^{+\infty} t\exp(-\epsilon t)dt < +\infty$.

Remark 1. *As we have pointed out, there is an important case covered by lemma 1, namely the sub-Gaussian case.*

A real random variable X is said to be *sub-Gaussian* if there exists a positive number a such that, for all real t, $\mathbb{E}[\exp(tX)] \leq \exp\left(\frac{a^2t^2}{2}\right)$. In this case the number

$$\tau(X) = \inf\left\{a \geq 0 : \mathbb{E}[\exp(tX)] \exp\left(\frac{a^2t^2}{2}\right) \text{ for every } t \in \mathbb{R}\right\}$$

is called the *Gaussian standard* of X (see [1] for an introduction to sub-Gaussian variables). If X is a sub-Gaussian variable with Gaussian standard $\tau(X)$, then for all positive x and t we have

$$\mathbb{P}(X>x) \le \mathbb{E}[e^{tX}e^{-tx}] \le \exp\left(\frac{1}{2}\tau^2(x)t^2 - tx\right).$$

By choosing $t = \frac{x}{\tau^2(X)}$, we have $\mathbb{P}(X>x) \le \exp(-\frac{x^2}{2\tau^2(X)})$

Clearly the assumptions of lemma 1 are satisfied if the variables X_n are sub-Gaussian with $\tau(X_n) \le 1$ for every n.

We now examine the behaviour of $Y_n^- = \max((-Y_n), 0)$.

Lemma 2. *Assume that X_n are independent and there exists a number $C>0$ such that, for every n and all $x>0$, we have*

$$\mathbb{P}(X_n<x) \le \exp\left(-Ce^{-\frac{x^2}{2}}\right).$$

Then the sequence Y_n^- converges a.s. to zero.

Proof. We shall prove that, again by applying the Borel-Cantelli lemma, for every $\epsilon>0$, $\mathbb{P}(Y_n^- > \epsilon \text{ i.o.}) = 0$. In fact

$$\mathbb{P}(Y_n^- > \epsilon) = \mathbb{P}(Y_n < -\epsilon) = \mathbb{P}\left(\max(X_1,\dots,X_n) < \sqrt{2\log n} - \epsilon\right)$$

$$\le \exp\left[-nC\exp\left(-\frac{1}{2}(\sqrt{2logn} - \epsilon)^2\right)\right] = \exp\left(-c\exp(\epsilon(\epsilon\sqrt{2logn})\right)$$

(with $c = C\exp(-\frac{\epsilon^2}{2})$). The last term above is the general term of a convergent series since it behaves like

$$\int_1^{+\infty} \exp\left(-c\exp(\epsilon\sqrt{2\log x})\right)dx = \int_0^{+\infty} t\exp\left(-ce^{\epsilon t} + \frac{t^2}{2}\right)dt \le +\infty$$

Remark 2. *The condition imposed in Lemma 2 is on the distribution function of X_n; but it is easy to see that it is implied by an "exponential type" condition on the tail distribution of X_n, considered in [3] p. 493.*

If in fact there exists a positive C such that, for every n and all $x>0$, $\mathbb{P}(X_n \ge x) \ge$ $\ge C\exp(-\frac{x^2}{2})$, then, by recalling that $t \le \exp(t-1)$, we have

$$\mathbb{P}(X_n<x) \le \exp(-\mathbb{P}(X_n \ge x)) \le \exp\left(-Ce^{-\frac{x^2}{2}}\right).$$

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R. GIULIANO ANTONINI
Dipartimento di Matematica
Via Buonarroti, 2
I-56127 Pisa
ITALIA