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SECOND ORDER METRIC SINGULARITIES

JENS CHR. LARSEN

Abstract. *In a previous paper* [21] *the author proved existence and uniqueness of geodesics through second order metric singularities tangent to weakly radial and strongly radial vectors respectively. In this paper we prove that a C^1 pregeodesic through a second order metric singularity can be reparametrized to one of these geodesics. So we have found all C^1 pregeodesics through the second order metric singularity.*

1. INTRODUCTION

In Riemannian geometry it has become fashionable to consider limits of sequences of Riemannian manifolds. For instance you can consider limits of Riemannian manifolds that are convergent in the Gromov Hausdorff metric, see [3, 4, 5, 6, 7, 8, 9]. The limit need not be a Riemannian manifold. Actually the limit space need not be a smooth manifold.

There are other notions of limits. For example you can consider sequences (M, g_i) of Riemannian manifolds, where $\{g_i\}$ is a convergent sequence with a smooth limit g. The limit is then a smooth, symmetric two tensor field. However it need not be positive definite. In this paper we shall be interested in the case where g has isolated metric singularities. We shall impose nondegeneracy conditions on the metric singularities.

To this end let (M, g) denote a smooth manifold with a smooth symmetric two tensor g. Let

$$\Xi = \{q \in M | g_q \text{ is degenerate } \}$$

denote the singular set. Given a singular point $p \in \Xi$ let (U, ϕ) denote a chart around p with coordinates $(x_1, \ldots, x_n)$ and define

$$f = \det \{g_{ij}\}$$

We are going to assume that $df_p = 0$, $\operatorname{rank} g_p = n - 1$ and

$$\frac{\partial^2 f}{\partial x_i \, \partial x_j}(\phi(p)) \text{ is definite}$$

It follows that p is an isolated singular point. Let

$$X : V \to TV \qquad V = U \backslash \{f = 0\}$$

denote the geodesic spray and let

$$\tilde{X} : U \to TU$$

denote the smooth extension of fX to U. We shall assume that

$$\tilde{X}(v) = 0 \quad \forall v \in T_pM$$

Recall from [21] the definition of the two metric invariants $\lambda_1(w)$ and $\lambda_2(w)$. We computed them in a chart and found

$$\lambda_1(w) = -\frac{1}{f_2(w)} \frac{\partial}{\partial x_1} \tilde{\Gamma}^1_{ij} w^i w^j$$
$$\lambda_2(w) = \frac{2}{f_2(w)} \frac{\partial}{\partial x_l} \tilde{\Gamma}^1_{i1} w^l w^i$$

Here $\tilde{\Gamma}^k_{ij}$ is the unique smooth extension of $f\Gamma^k_{ij}$. We are going to assume that

$$\lambda_2(w) < 1 \quad \lambda_2(w) < -\lambda_1(w).$$

A $p \in \Xi$ is called a second order metric singularity. Existence and uniqueness of geodesics for second order metric singularities was proven in [21]. Here we introduced the notion of weakly radial and strongly radial vectors and proved that there are geodesics tangent to such directions.

In this paper we shall prove that a C^1 pregeodesic can be reparameterized to one of the above mentioned geodesics. Furthermore we show how to parallel translate vector fields along the geodesics, beyond the metric singularity.

Finally we prove a Gauss Lemma for the exponential map of a second order metric singularity.

Metric singularities have attracted attention in physics, see [14, 16, 17, 18, 19, 29].

2. EXISTENCE AND UNIQUENESS OF GEODESICS

Define

$$G : \mathbb{R}^n \longrightarrow \mathbb{R}^n$$
$$(v_1, \ldots, v_n) \longmapsto (v_1, v_1v_2, \ldots, v_1v_n)$$

Fix an orthonormal basis $w_1, \ldots, w_n$ at the second order metric singularity p with w_1 isotropic. The restriction of G to $\{v_1 \neq 0\} = H$ also denoted

$$G : H \to H$$

has an inverse

$$G^{-1} : H \to H$$
$$(v_1, \ldots, v_n) \longmapsto (v_1, v_2 / v_1, \ldots, v_n / v_1)$$

Two curves

$$\beta_i : I \to M \qquad i = 1, 2$$

with $\beta_i'(0) = w_1$ are resolvent tangent provided

$$\frac{d}{dt}(G^{-1} \circ \phi \circ \beta_1)(0) = \frac{d}{dt}(G^{-1} \circ \phi \circ \beta_2)(0)$$

in one and hence any chart (U, ϕ) adapted to $w_1, \ldots, w_n$, that is

$$\underset{i}{\partial}(p) = w_i.$$

Resolvent tangential is an equivalence relation. An equivalence class is called a resolvent tangent. The set of equivalence classes is denoted

$$T_p(M, \Xi)$$

We have an injective mapping

$$\begin{aligned} T_\Xi \phi : T_p(M, \Xi) &\to \mathbb{R}^n \\ [\beta] &\mapsto \frac{d}{dt}(G^{-1} \circ \phi \circ \beta)(0) \end{aligned}$$

Existence and uniqueness theorem for geodesics. Given $u \in T_p(M, \Xi)$ then there exists a unique geodesic

$$\gamma : I \backslash \{0\} \to M \backslash \Xi$$

such that $\gamma \circ \tau$ has resolvent tangent u where

$$\begin{aligned} \tau(t) &= 2^{-\frac{4}{3}} v_1^{\frac{4}{3}} t^2 \\ v &= (v_1, \ldots, v_n) = T_\Xi \phi(u) \end{aligned}$$

Proof. Existence. According to Theorem 9.4 of [21] there exists a geodesic

$$\gamma : I \backslash \{0\} \to M \backslash \Xi$$

such that for every chart (U, ϕ) adapted to $w_1, \ldots, w_n$, $\gamma' \circ \tau$ is resolvent smooth with resolvent tangent u_* where

$$T_\Xi \phi^*(u_*) = (v, y) = ((v_1, \ldots, 0), (2^{-\frac{1}{3}} v_1^{\frac{1}{3}}, 2v_2, \ldots, 2v_n))$$

Recall that this means that

$$\Phi \circ \gamma' \circ \tau$$

is smooth where

$$\Phi(v, y) = (v, \frac{1}{y_1}(1, \ldots, y_n))$$

and

$$\frac{d}{dt}(\Phi \circ \gamma' \circ \tau_*)(0) = (v, y)$$

Define

$$\beta(t) = \gamma \circ \tau(t)$$

with

$$\beta'(t) = \gamma' \circ \tau(t)\tau'(t)$$

We find that

$$\beta(t) = (v_1 t, \frac{1}{2} v_1 y_2 t^2, \ldots, \frac{1}{2} v_1 y_n t^2) + \ldots$$

Hence

$$\frac{d}{dt}(G^{-1} \circ \beta)(0) = (v_1, \frac{1}{2} y_2, \ldots, \frac{1}{2} y_n) = (v_1, \ldots, v_n)$$

Uniqueness. Reverse the process above and apply the uniqueness statement of [21] Theorem 9.4.

3. PARALLEL TRANSLATION ALONG GEODESICS

Consider now a geodesic

$$\gamma : I \backslash \{0\} \to M \backslash \Xi$$

such that $\gamma' \circ \tau$ is resolvent smooth with resolvent tangent

$$u \in T_p^G(\Xi, TM)$$

where

$$\tau(t) = \frac{1}{4} \frac{v_1^2}{y_1^2} t^2 \operatorname{sgn} t$$

$$(v, y) = T_\Xi \phi^*(u)$$

see [21].

Define $\beta = \gamma \circ \tau$.

Given $v \in T_p M$ let Z denote a smooth vector field along β with $Z(0) = v$. Define a linear map

$$L(v) = \lim_{t \to 0} (t Z'(t))$$

and

$$\Lambda_p = \ker L$$

Notice that

$$\dim \Lambda_p = n - 1$$

see [21].

Proposition 3.1 Given $v \in \Lambda_p$ then there exists a unique smooth parallel vector field along β such that

$$X(0) = v$$

Proof. Let

$$\tilde{\Gamma}_{ij}^{k}(x) = \Omega_{ij}^{kl}(x)x_l$$

denote the unique smooth extension of $f\Gamma_{ij}^{k}$ to a neighbourhood of 0. Then

$$A_j^k(t) = -\frac{t}{f \circ \gamma(t)}\tilde{\Gamma}_{ij}^{k} \circ \beta(t)\beta_i'(t)$$

is smooth at 0. Define

$$B(Z, \theta) = \begin{pmatrix} A_j^k(\theta)Z^j e_k \\ \theta \end{pmatrix}$$

Then $(v, 0)$, $v \in \Lambda_p$ is a singular point for B and

$$DB_{(v,0)} = \begin{pmatrix} -1 & * & \dots & * \\ 0 & 0 & \dots & 0 \\ . & . & \dots & . \\ 0 & 0 & \dots & 1 \end{pmatrix}$$

So B has a one dimensional unstable manifold

$$W^u(v, 0) = \{\rho(t) = (\rho_1(t), t) | t \in J\}$$

$W^u(v, 0)$ is invariant for B so

$$\rho'(t) = \frac{1}{k(t)}B(\rho(t))$$

It follows that $k(t) = t$ and then

$$\rho_1'(t) = -\frac{1}{f \circ \beta(t)}\tilde{\Gamma}_{ij}^{k} \circ \beta(t)\beta_i'(t)\rho_1^j(t)e_k$$

So $t \mapsto \rho_1(t)$ is the local representative of a parallel vector field along β with $\rho_1(0) = v$.

It is unique by uniqueness of stable and unstable manifolds.

4. UNIQUENESS OF GEODESICS

Let

$$\gamma :]0, c[\to M \backslash \Xi$$

denote a geodesic in $M\backslash\Xi$. We assume that it can be reparameterized via

$$\eta :]0, d[\to]0, c[$$

to a curve $\delta = \gamma \circ \eta$ which is the restriction of a regular curve on a neighbourhood of 0. In other words δ is a C^1 pregeodesic through p.

Then we have the following theorem.

Theorem 4.1 Either
(i) γ is the restriction of a smooth curve

$$\beta :]a,b[\to M \qquad b > 0$$

such that $\beta_{|]a,b[\setminus\{0\}}$ are geodesics and

$$\beta'(0) \in T_pM$$

is weakly radial.
Or
(ii) $\gamma \circ \tau,\ \tau(t) = kt^2, k > 0$ is the restriction of a smooth curve

$$\beta :]a,b[\to M \quad b > 0$$

such that $\beta \circ \tau^{-1}_{|]a,b[\setminus\{0\}}$ are geodesics and

$$\beta'(0) \in T_pM$$

is strongly radial.

Proof. Let h denote a flat Riemannian metric on an open neighbourhood U of p. Define

$$s(t) = \int_0^t \sqrt{h(\gamma', \gamma')} ds$$

Define

$$\beta(t) = \gamma \circ s^{-1}(t) \quad t \in \text{ Im } s =]0, c[$$

and

$$\rho_1(t) = \beta(t) / t$$
$$\rho_2(t) = \beta'(t)$$
$$\rho_3(t) = t$$

Let

$$Y(x,y,t) = -\sum_{k=1}^{n} \Gamma_{ij}^{k}(tx) y_i y_j e_k$$

$$\tilde{\Gamma}_{ij}^{k}(x) = \sum_{l=1}^{n} \Omega_{ij}^{kl}(x) x_l$$

$$f(x) = \sum_{i,j=1}^{n} f_{ij}(x) x_i x_j$$

$$f(tx) = t^2 \sum_{i,j=1}^{n} f_{ij}(tx) x_i x_j = t^2 h(t,x)$$

in a chart $U, \phi)$ centered at the origin $\phi(p) = 0$ and adapted to $w_1, \ldots, w_n$. Here $e_1, \ldots, e_n$ is the canonical basis in $\mathbb{R}^n$. Then

$$
\begin{aligned}
Y(x, y, t) &= -\frac{1}{f(tx)} \tilde{\Gamma}_{ij}^{k}(tx) y_i y_j e_k \\
&= -\frac{1}{t^2 h(t, x)} \Omega_{ij}^{kl} t x_l y_i y_j e_k \\
&= -\frac{1}{t} \frac{1}{h(t, x)} \Omega_{ij}^{kl}(tx) x_l y_i y_j e_k \\
&= \frac{1}{t} \tilde{Y}(x, y, t)
\end{aligned}
$$

Now

$$
\begin{aligned}
\rho_1'(t) &= \beta'(t) / t - \beta(t) / t^2 \\
&= \frac{1}{t}(-\rho_1(t) + \rho_2(t))
\end{aligned}
$$

and

$$
\begin{aligned}
\rho_2'(t) &= (\gamma \circ s^{-1})''(t) \\
&= (\gamma'(s^{-1}(t)) \frac{1}{s'(s^{-1}(t))})' \\
&= \gamma''(s^{-1}(t)) \frac{1}{s'(s^{-1}(t))^2} - \gamma'(s^{-1}(t)) \frac{1}{s'(s^{-1}(t))^3} s''(s^{-1}(t))
\end{aligned}
$$

Using

$$
\begin{aligned}
\gamma''(s^{-1}(t)) \frac{1}{s'(s^{-1}(t))^2} &= Y(\rho_1(t), \rho_2(t), t) \\
s''(s^{-1}(t)) &= \frac{1}{t} s'(s^{-1}(t))^2 h(\tilde{Y}(\rho(t)), \rho_2(t)) \\
h(\rho_2(t), \rho_2(t)) &= h(\gamma'(s^{-1}(t), \gamma'(s^{-1}(t))) / s'(s^{-1}(t))^2 = 1
\end{aligned}
$$

we find that

$$
\begin{aligned}
\rho_2'(t) &= \frac{1}{t}(\tilde{Y}(\rho(t)) - \rho_2(t) h(\tilde{Y}(\rho(t)), \rho_2(t)) / h(\rho_2(t), \rho_2(t))) \\
\rho_3'(t) &= \rho_3(t) / t = 1
\end{aligned}
$$

Define the vector field

$$
Z : \phi(U) \times \mathbb{R}^n \backslash \{0\} \times I \to \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}
$$

$$
Z(x, y, \theta) = \begin{pmatrix} -x + y \\ \tilde{Y}(x, y, \theta) - y h(\tilde{Y}(x, y, \theta), y) / h(y, y) \\ \theta \end{pmatrix}
$$

We see that

$$\rho'(t) = \frac{1}{t} Z(\rho(t))$$

So

$$\xi(t) = \rho(e^t) = (\xi_1(t), \xi_2(t), \xi_3(t))$$

is an integral curve of Z since

$$\xi'(t) = \frac{1}{e^t} Z(\rho(e^t))e^t = Z(\xi(t))$$

Now

$$(\gamma \circ \eta)'(t) \to v \in T_pM \backslash \{0\}$$

So

$$\beta'(t) \to v \,/\, ||v|| \quad t \to 0$$

It follows that

$$\rho_1(t) \to v \,/\, ||v||$$
$$\rho_2(t) \to v \,/\, ||v||$$

as $t \to 0$. It is then immediate that

$$\xi(t) \to (v \,/\, ||v||, v \,/\, ||v||, 0) = z = (z_*, z_*, 0) \quad t \to -\infty$$

Now z is a singular point for Z. Otherwise we have a contradiction with the flow box theorem. z_* is then a weakly radial or strongly radial vector. z_* is weakly radial if

$$\tilde{Y}(z_*, z_*, 0) = 0$$

If

$$\tilde{Y}(z_*, z_*, 0) \neq 0$$

then

$$\tilde{Y}(z_*, z_*, 0) = \tilde{Y}^1(z_*, z_*, 0)e_1$$

So

$$z_* = \pm e_1$$

and z_* is strongly radial. We are going to need to compute two linearizations namely

$$DZ_z$$

when z_* is weakly radial and when z_* is strongly radial. Suppose z_* is weakly radial. Then

$$DZ_z = \begin{pmatrix} -1 & \dots & 0 & 1 & \dots & 0 & 0 \\ . & \dots & . & . & \dots & . & . \\ 0 & \dots & -1 & 0 & \dots & -1 & 0 \\ a_1 & \dots & a_n & b_1 & \dots & b_n & c_1 \\ . & \dots & . & . & \dots & . & . \\ 0 & \dots & 0 & 0 & \dots & 0 & c_n \\ 0 & \dots & 0 & 0 & \dots & 0 & 1 \end{pmatrix}$$

assuming $z_* = e_2$.

Now

$$a_1 = \lambda_1 \qquad b_1 = \lambda_2$$

Since $\lambda_2 < 1,\ \lambda_2 < -\lambda_1,\ 1$ is an eigenvalue of geometric multiplicity 1 and all other eigenvalues of DZ_z have real parts ≤ 0. In fact there are n negative eigenvalues counted with multiplicity and 0 is an eigenvalue of geometric multiplicity $n-1$. Define

$$F(x) = Y^1(x, x, 0)$$

Then

$$\frac{\partial F}{\partial x_1}(z) = \frac{\partial Y^1}{\partial x_1}(z, z, 0) + \frac{\partial Y^1}{\partial y_1}(z, z, 0) = \lambda_1(z) + \lambda_2(z) \neq 0$$

So the unit length weakly radial vectors constitute an $n-2$ dimensional submanifold V of T_pM. This submanifold consists of singular points for Z. These singular points are normally hyperbolic meaning that DZ_z is nonsingular on a complement to T_zV in T_zN where

$$N = \{(x, y, \theta) | \theta x \in \phi(U) \quad ||y|| = 1\}$$

Let ϕ denote the flow of the restriction of Z to N. The unstable manifold

$$W^u(z_*, z_*, 0) = \{x \in N | \phi_t(x) \to (z_*, z_*, 0) \quad t \to -\infty\}$$

is then a onedimensional manifold in N, see [11] p. 39.

So there exists

$$\rho_*(t) = (\rho_*^1(t), \rho_*^2(t), \rho_*^3(t))$$
$$\rho_*^3(t) = t \qquad t \in I$$

such that locally

$$W^u(z_*, z_*, 0) = \{\rho_*(t) | t \in I\}$$

We have seen that

$$\xi(t) \in W^u(z, z, 0)$$

So

$$\rho(t) \in W^u(z, z, 0)$$

By uniqueness of unstable manifolds

$$\rho(t) = \rho_*(t)$$

near $t = 0$. So

$$\beta_*(t) = t\rho_*^1(t) = t\rho_1(t) = \beta(t)$$

is a pregeodesic. This is due to the fact that

$$\rho_*'(t) = \frac{1}{t} Z(\rho_*(t))$$

So

$$\beta''_*(t) = k(t)\beta'_*(t)$$

for some smooth function k. From this it follows that β_* is a pregeodesic. But then γ is the unique geodesic with

$$\gamma'(0)$$

weakly radial.

It remains to consider the case where z_* is strongly radial. Here

$$DZ_z = \begin{pmatrix} -1 & \dots & 0 & 1 & 0 & \dots & 0 & 0 \\ . & \dots & . & . & . & \dots & . & . \\ 0 & \dots & -1 & 0 & 0 & \dots & 1 & 0 \\ 0 & \dots & 0 & 0 & 0 & \dots & 0 & 0 \\ 0 & \dots & 0 & 0 & 1 & \dots & 0 & 0 \\ . & \dots & . & . & . & \dots & . & . \\ 0 & \dots & . & . & . & \dots & 1 & 0 \\ 0 & \dots & . & . & . & \dots & 0 & 1 \end{pmatrix}$$

where $z_* = \pm e_1$.

We have used that

$$\frac{\partial Y^k}{\partial \theta} = \frac{1}{h(0,e_1)} \frac{\partial}{\partial u_1} \Omega^{k1}_{11}(0) = 0 \qquad k \geq 2$$

see [21].

We see that there is an n dimensional stable manifold and an n dimensional unstable manifold W^u. We shall now show that solutions on the stable manifold give rise to pregeodesics β through p with

$$\beta'(0)$$

strongly radial. The projection

$$\pi : W^u \longrightarrow \mathbb{R}^n$$
$$(x, y, \theta) \mapsto (y_2, \dots, y_n, \theta)$$

has a local inverse

$$\rho : \mathcal{W} \rightarrow W^u \subset \mathbb{R}^{2n+1}$$

Define the vector field

$$W(x) = \pi \circ Z \circ \rho(x) \qquad x \in \mathcal{W}$$

The origin is a singular point with

$$DW_0 = \text{ id}$$

So we can blow up using the blow up maps

$$F : \mathbb{R}^n \setminus \{0\} \rightarrow E = \{||x|| > 1\}$$
$$G : \{0 < ||x|| < 1\} = I \rightarrow I$$

The blown up vector field is

$$V(x) = \frac{1}{||x||-1}\begin{cases} DF(W(F^{-1}(x))) & ||x||>1 \\ DG(W(G^{-1}(x))) & 0<||x||<1 \end{cases}$$

It is the restriction of a smooth vector field also denoted V on a neighbourhood of S^{n-1}. Let Φ^V denote the flow of this vector field. It gives rise to the reparameterization function

$$\tau_{t_0}(t) = \tau(t) = \int_{t_0}^{t} \frac{1}{||\Phi_z^V(s)||-1} ds$$
$$t_0 \in \mathbb{D}(\Phi_z^V) \quad z \in S^{n-1}\backslash\{z_n = 0\}$$

Now

$$\xi_{t_0}(t) = \rho \circ F^{-1} \circ \Phi_z^V \circ \tau_{t_0}^{-1}(t)$$
$$\xi_{-t_0}(t) = \rho \circ F^{-1} \circ \Phi_z^V \circ \tau_{-t_0}^{-1}(t)$$

are then integral curves for Z.

Define

$$\rho_*(t) = \begin{cases} \xi_{t_0}(\ln t) & t>0 \\ \xi_{-t_0}(\ln(-t)) & t<0 \end{cases}$$

We shall show that

$$t \mapsto \tau^{-1}(\ln t) \quad t>0$$

is the restriction of a smooth function μ_* on a neighbourhood of 0. But we can write

$$\frac{1}{||\Phi_z^V(s)||-1} = \frac{1}{s} + k_1(s)$$

for some smooth function k_1 on a neighbourhood of 0. So

$$\tau(t) = \ln t / t_0 + \int_{t_0}^{t} k_1(s) ds$$

So

$$\pm \exp \circ \tau(t) = (t / t_0) \exp(\int_{t_0}^{t} k_1(s) ds) = \mu_*^{-1}(t)$$

The inverse function theorem defines μ_*. So ρ_* is a Z invariant smooth curve with

$$\begin{aligned} \rho_*'(0) &= D\rho \circ DF^{-1}(V(z))c \\ &= D\rho(z)c \\ &= \begin{pmatrix} 0 \\ \frac{1}{2}z_2 \\ \cdot \\ \cdot \\ \frac{1}{2}z_n \\ 0 \\ z_2 \\ \cdot \\ \cdot \\ z_n \\ z_{n+1} \end{pmatrix} c \quad c \in \mathbb{R}\backslash\{0\} \end{aligned}$$

Since $z_{n+1} \neq 0$ we can reparameterize ρ_* to a smooth curve $\bar{\rho} = (\bar{\rho}_1, \bar{\rho}_2, \bar{\rho}_3)$ with

$$\bar{\rho}_3(t) = t$$

By Z invariance of $\bar{\rho}$ we have

$$\bar{\rho}'(t) = \frac{1}{t} Z(\bar{\rho}(t))$$

We have then previously seen that

$$\beta(t) = t\bar{\rho}_1(t)$$

is a pregeodesic. If we arc length parametrize β we get a geodesic. We shall do so now. To this end define

$$s(t) = \int_0^t \sqrt{g(\beta', \beta')(s)}ds \,/\, \sqrt{\lambda}$$

where

$$g(\beta', \beta') = \lambda s^2 + \ldots \quad \lambda > 0$$

To see that $\lambda > 0$ let

$$\beta(s) = (w_1 s, w_2 s^2, \ldots, w_n s^2) + \ldots$$

and compute

$$\frac{1}{2}\frac{\partial^2}{\partial s^2} g(\beta', \beta')_{|s=0} = \frac{1}{2}\frac{\partial^2 g_{11}}{\partial x_1^2}(0) + 2\sum_{i=2}^{n} \frac{\partial g_{1i}}{\partial x_1}(0) 2w_i + \sum_{i=2}^{n} g_{ii} 4 w_i^2$$

Letting $h_{1i}^1 = \frac{\partial g_{1i}}{\partial x_1}(0)$ we get

$$\begin{aligned}\lambda &= \frac{1}{2}\frac{\partial^2 g_{11}}{\partial x_1^2}(0) - \sum_{i=2}^{n} (h_{1i}^1)^2 / g_{ii} + \sum_{i=2}^{n} 4 g_{ii}(w_i + h_{1i}^1 / 2g_{ii})^2 \\ &= \frac{1}{g_{22} \ldots g_{nn}} \frac{1}{2}\frac{\partial^2 f}{\partial x_1^2} + \sum_{i=2}^{n} 4 g_{ii}(w_i + h_{1i}^1 / 2g_{ii})^2 > 0\end{aligned}$$

since

$$\{\frac{\partial^2 f}{\partial x_i \partial x_j}(p)\}$$

is assumed positive definite.

So

$$s(t) = \begin{cases} \int_0^t sl(s)ds & t > 0 \\ -\int_0^t sl(s)ds & t \leq 0 \end{cases}$$

for some smooth function l with

$$l(0) = 1$$

Now $\beta \circ s^{-1}$ is a curve with constant velocity. β being a pregeodesic $\beta \circ s^{-1}$ is then a geodesic.

Notice that we can reparametrize ξ and hence ρ to $\bar{\rho}$. It follows that $\beta \circ s^{-1}$ is an affine reparametrization of the original γ that we started with.

Given $v = (v_1, \ldots, v_n),\ v_1 > 0$ define

$$\gamma(t) = \beta \circ s^{-1}((2^{\frac{2}{3}} v_1^{\frac{1}{3}})^2 \frac{1}{2} t)$$

Choosing z suitable this is the geodesic that we seek. In fact

$$\begin{aligned}\gamma \circ \tau_*(t) &= \beta \circ s^{-1}((2^{\frac{2}{3}} v_1^{\frac{1}{3}})^2 \frac{1}{2} \tau_*(t)) \\ &= \beta \circ s^{-1}(\tau_*(2^{\frac{2}{3}} v_1^{\frac{1}{3}} \frac{1}{\sqrt{2}} t))\end{aligned}$$

and

$$\frac{d}{dt}(G^{-1} \circ \beta)(0) = (1, \frac{1}{2}\frac{z_2}{z_{n+1}}, \ldots, \frac{1}{2}\frac{z_n}{z_{n+1}})$$

Now

$$s(v) = \begin{cases} v^2 l_1(v) & v \geq 0 \\ -v^2 l_1(v) & v < 0 \end{cases}$$

for some smooth function l_1 with $l_1(0) = \frac{1}{2}$. Write

$$\begin{aligned}\tau_*(t) &= k_\pm t^2 = 2^{-\frac{4}{3}} v_1^{\frac{4}{3}} t^2 \operatorname{sgn} t \\ k &= k_+ = -k_-\end{aligned}$$

So

$$\tau_*^{-1}(t) = \pm \int_0^t \frac{1}{2} |s|^{-\frac{1}{2}} ds \frac{1}{\sqrt{k}}$$

Substitute $s = \pm s(v) = v^2 l_1(v)$ to get

$$\tau_*^{-1}(t) = \int_0^{s^{-1}(t)} \frac{1}{2} \frac{2 l_1(v) + v l_1'(v)}{\sqrt{l_1(v)}} dv$$

So

$$\tau_*^{-1} \circ s(t) = \int_0^t \frac{1}{2} \frac{2 l_1(v) + v l_1'(v)}{\sqrt{l_1(v)}} dv \,/\, \sqrt{k}$$

which is smooth and invertible. Now

$$\frac{d}{dt}(s^{-1} \circ \tau_*)(0) = \sqrt{k l_1(0)} \,/\, \frac{1}{2} 2 l_1(0) = \sqrt{2}\sqrt{k}$$

So finally

$$\frac{d}{dt}(G^{-1} \circ \gamma \circ \tau_*)(0) = (1, \frac{1}{2}\frac{z_2}{z_{n+1}}, \ldots, \frac{1}{2}\frac{z_n}{z_{n+1}}) \sqrt{2k} 2^{\frac{2}{3}} v_1^{\frac{1}{3}} \frac{1}{\sqrt{2}} = v_1 (1, \frac{1}{2}\frac{z_2}{z_{n+1}}, \ldots, \frac{1}{2}\frac{z_n}{z_{n+1}})$$

and the Theorem follows.

5. THE GAUSS LEMMA

We shall start this section by defining the exponential map

$$\exp_p : \mathbb{D}_p \subset T_p(M, \Xi) \to M$$

To this end let

$$\tau_v(s) = 2^{-\frac{4}{3}} v_1^{\frac{4}{3}} s^2 \text{ sgn } s$$

where

$$v \in T_p(M, \Xi) \quad v_1 > 0$$

Let γ_v denote the unique maximal geodesic γ such that

$$\gamma \circ \tau_v$$

has resolvent tangent v. Let $\mathbb{D}_p$ denote the set of resolvent vectors v such that 1 is in the domain of definition of γ_v. Define

$$\exp_p(v) = \gamma_v(1)$$

Lemma 5.1. $\exp_p$ is smooth.

Proof. We have seen that

$$(t, v) \mapsto \gamma_v(\tau_v(t)) = H(t, v)$$

is smooth. Define for $t > 0$

$$\tau_v(t) = k(v_1)t^2$$

Then for

$$t = \frac{1}{\sqrt{k(v_1)}}$$

we have $\tau_v(t) = 1$. So

$$v \mapsto H(\frac{1}{\sqrt{k(v_1)}}, v) = \gamma_v(1)$$

is smooth.

Lemma 5.2. $\gamma_v(t) = \exp_p(t^{\frac{3}{2}} v)$, $t \geq 0$, $t \in \mathbb{D}(\gamma_v)$.

Proof. Define

$$\beta(s) = \gamma_v(ts)$$
$$\tau_*(s) = 2^{-\frac{4}{3}} (t^{\frac{3}{2}} v_1)^{\frac{4}{3}} s^2 \text{ sign } s$$

Then

$$t\tau_*(s) = \tau_v(s)t^{\frac{3}{2}\frac{4}{3}+1} = \tau_v(s)t^3 = \tau_v(st^{\frac{3}{2}})$$

Taking local representatives we see that

$$\begin{aligned}&(G^{-1} \circ \beta \circ \tau_*)(s)\\ &= G^{-1}(\gamma_v(t\tau_*(s)))\\ &= G^{-1} \circ \gamma_v \circ \tau_v(st^{\frac{3}{2}})\end{aligned}$$

So

$$\frac{d}{ds}(G^{-1} \circ \beta \circ \tau_*)(0) = \frac{d}{ds}(G^{-1} \circ \gamma_v \circ \tau_v)(0)t^{\frac{3}{2}} = t^{\frac{3}{2}}v$$

This shows that

$$\beta = \gamma_{t^{\frac{3}{2}}v}$$

So

$$\beta(1) = \gamma_{t^{\frac{3}{2}}v}(1) = \exp_p(t^{\frac{3}{2}}v) = \gamma_v(t)$$

Define a scalar product

$$h : T_p(M,\Xi) \times T_p(M,\Xi) \to \mathbb{R}$$

on $T_p(M,\Xi)$ by the formula

$$h(z,z) = \frac{1}{2}\frac{\partial^2 g_{11}}{\partial x_1^2}(0)z_1^2 + 4\sum_{i=2}^{n}\frac{\partial g_{1i}}{\partial x_1}(0)z_iz_1 + 4\sum_{i=2}^{n} g_{ii}z_i^2$$

We have seen that

$$h(v,v) \neq 0$$

whenever $v_1 \neq 0$.

The Gauss Lemma. Let $v \in \mathbb{D}_p$ and $w_v \in T_vT_p(M,\Xi)$. Then

$$3\langle d\exp_p(v_v), d\exp_p(w_v)\rangle = \frac{\langle \gamma_v', \gamma_v'\rangle}{h(v,v)}(2h(v,w) - \frac{8}{3}h(v,v)\frac{w_1}{v_1})$$

Proof. Define a curve w by

$$\begin{aligned}w &: \mathbb{R} \to T_p(M,\Xi)\\ w(s) &= v + sw\end{aligned}$$

The tangent vector to w at $s = 0$ is w_v. Furthermore define

$$x(t,s) = \gamma_{w(s)}(t) = \exp_p(t^{\frac{3}{2}}w(s))$$

Define

$$\tau_*(t,s) = 2^{-\frac{4}{3}} w_1(s)^{\frac{4}{3}} t^2 \operatorname{sign} t$$
$$z(t,s) = x(\tau_*(t,s), s)$$

Then $G^{-1} \circ z_s$ is smooth at $t = 0$ and

$$\frac{d}{dt}(G^{-1} \circ z_s)(0) = w(s)$$

Now for $t > 0$

$$\begin{aligned} g(z_t, z_t)(t,s) &= \langle x_t, x_t\rangle(s)(\frac{\partial \tau_*}{\partial t}(t,s))^2 \\ &= k(s)(2^{-\frac{4}{3}} w_1(s)^{\frac{4}{3}} 2t)^2 \\ &= k(s) w_1(s)^{\frac{8}{3}} 2^{-\frac{2}{3}} t^2 \end{aligned}$$

Derivating $t \mapsto g(z_t, z_t)(t,s)$ for fixed s we find

$$g(z_t, z_t) = \sum_{i,j} g_{ij} \circ z(t,s) z_t^i z_t^j = t^2 h(w(s), w(s))$$

We have found the relation

$$h(w(s), w(s)) = \langle x_t, x_t\rangle(s) w_1(s)^{\frac{8}{3}} 2^{-\frac{2}{3}}$$

We can differentiate once with respect to s to get

$$\begin{aligned} 2h(v,w) &= \frac{\partial}{\partial s}((\langle x_t, x_t\rangle)(s) v_1^{\frac{8}{3}} 2^{-\frac{2}{3}} + \langle x_t, x_t\rangle(s)\frac{8}{3} w_1(s)^{\frac{5}{3}} w_1'(0) 2^{-\frac{2}{3}} \\ &= \frac{\partial}{\partial s}\langle x_t, x_t\rangle(s)\frac{h(v,v)}{\langle\gamma_v', \gamma_v'\rangle} + \frac{8}{3} h(v,v) w_1 / v_1 \end{aligned}$$

Now

$$\frac{\partial}{\partial s}\langle x_t, x_t\rangle = 2\frac{\partial}{\partial t}\langle x_s, x_t\rangle$$

which is independent of t so

$$\frac{\partial}{\partial s}\langle x_t, x_t\rangle(1,0) = 2\langle x_s, x_t\rangle(1,0) = 2\frac{3}{2}\langle d\exp_p(w_v), d\exp_p(v_v)\rangle$$

Combining the last two derivations the formula in the Gauss Lemma results.

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Department of Mathematics
University of Copenhagen
DK - 2100 Copenhagen Ø
DENMARK