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COMPACT HOMOGENEOUS EINSTEIN MANIFOLDS IN CODIMENSION TWO

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Abstract. *In this paper we study compact Riemannian homogeneous submanifolds of Euclidean spaces in codimension 2 for which the metric is Einstein. We prove that they are spheres or product of spheres. We apply this result to study compact cohomogeneity one hypersurfaces whose principal orbits are Einstein manifolds. In the case that they are irreducible manifolds, we conclude that the cohomogeneity one manifold is immersed as a hypersurface of revolution.*

1. INTRODUCTION

A result of Jensen [10] states that a *four-dimensional homogeneous Einstein manifold is symmetric*. A nice proof of this result is given by Derdzinski in [9] for the compact case. A classification of Einstein homogeneous spaces is still an open problem even in dimension 5. Both homogeneous spaces and Einstein submanifolds of Euclidean spaces have only been studied in codimension 1 (see [11], [13], [12] Vol. II, pg. 36). Kobayashi [11] showed that a compact homogeneous hypersurface of $\mathbf{R}^{n+1}$ is congruent to a sphere. In this paper we want to consider compact codimension two homogeneous spaces that are Einstein. One of the motivations for this is the study of cohomogeneity one hypersurfaces of Euclidean spaces.

A Riemannian G-manifold is said to be of *cohomogeneity one* if the group G acts effectively and isometrically with principal orbits of codimension one. In [15], the authors studied compact cohomogeneity one hypersurfaces of Euclidean spaces and related them to hypersurfaces of revolution. A hypersurface of $\mathbf{R}^{n+1}$ is called a *hypersurface of revolution* if it is invariant under the group $SO(n)$ of rotations around a fixed line l of $\mathbf{R}^{n+1}$. They proved that if each principal orbit under the action of G is umbilical in M then M is immersed as a hypersurface of revolution. It is clear that if the principal orbits are isotropy-ireducible then they are umbilical. Since the G-invariant metric of isotropy-irreducible homogeneous spaces in Einstein (see [6] pg. 187), it is then natural to ask if a compact cohomogeneity one hypersurface whose principal orbits are Einstein manifolds is a hypersurface of revolution. Since these orbits are compact codimension two homogeneous spaces we first classify them. We obtain the following result.

Theorem. *Let* $f : M^n \rightarrow \mathbf{R}^{n+2}, n \geq 3$, *be an isometric immersion of a compact Riemannian homogeneous Einstein manifold. Then M is either a sphere or a product of two spheres, each of which is of dimension greater than 1. In the latter case the immersion f is a product of hypersurface immersions.*

In [2], it was proved that if each principal orbit of a compact cohomogeneity one hypersurface has constant sectional curvature then it is umbilical and hence it is immersed as hypersurface of revolution. Combining this result with the Theorem above we conclude immediately the following Corollary.

Corollary 1. *Let $f : M^n \mathbb{R} \rightarrow^{n+1}, n \geq 4$, be an isometric immersion of a compact cohomogeneity one Riemannian G-manifold. If the principal orbits under the action of G are irreducible Einstein manifolds then $f(M^n)$ is a hypersurface of revolution.*

We want to point out that we can remove the assumption on the irreducibility of the orbits in Corollary 1 if for some point on a principal orbit, the type number $\tau(x)$ of the immersion f satisfies $\tau(x) \leq 1$. If $f(M^n)$ is of revolution then the group G is either $S)(n)$ or $SO(n+1)$. Otherwise, if the principal orbits split as a product of spheres, we can show that the immersion f restricted to each principal orbit is rigid. Then we obtain the following.

Corollary 2. *Let $f : M^n \rightarrow \mathbf{R}^{n+1}, n \geq 4$, be an isometric immersion of a compact cohomogeneity one Riemannian G-manifold. If the principal orbits under the action of G are Einstein manifolds then*
(a) any isometry $g \in G$ is induced by an element g' of a group G' of rigid motions of $\mathbf{R}^{n+1}$
(b) the principal orbits of G' are spheres or product of spheres
(c) there is a curve γ in $\mathbf{R}^{n+1}$ such that $f(M) = G'(\gamma)$.

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2. A RIGIDITY RESULT FOR HOMOGENEOUS SUBMANIFOLDS

An isometric immersion $f : M^n \rightarrow \mathbf{R}^N$ is said to be *rigid* if given any other isometric immersion $g : M^n \rightarrow \mathbf{R}^N$, there exists a rigid motion T of $\mathbf{R}^N$, such that $f = T \circ g$. In this case any isometric immersion $g : M^n \rightarrow \mathbf{R}^N$ is also rigid. We point out that the fact that an immersion from a manifold M is rigid does not imply that $f_{|U}$, the restriction of f to a subset U of M, is rigid. Observe also that if $f : M^n \rightarrow \mathbf{R}^N$ is non-rigid there may be $U \subset M$ for which $f_{|U}$ is rigid. The next lemma is crucial for the result of this section.

Lemma 2.1. *Let $f : M^n \rightarrow \mathbf{R}^N, n \geq 2$, be an isometric immersion of a connected manifold. If f is non-rigid then there exists a point p in M and an open neighborhood U of p such that $f_{|U}$ is non-rigid and if $p \in U' \subset U$, $f_{|U'}$ is also non-rigid.*

Proof. Suppose that for each point p in M and every open neighborhood U of p there exists an open set $U' \subset U$ such that $f_{|U'}$ is rigid. We fix a point p and a neighborhood U' of p such that $f_{|U'}$ is rigid. If $g : M^n \rightarrow \mathbf{R}^N$ is any other isometric immersion, there exists a rigid motion T of $\mathbf{R}^N$ such that $f_{|U'} = T \circ g_{|U'}$. We set

$$A = \{x \in M : f(x) = T(g(x))\}$$

Obviously, A is non-empty and closed. Further, since $U' \subset A$, the interior of A, denoted by $int(A)$, is non-empty. Let q be an arbitrary limit point of $Int(A)$. Our assumption implies that there exists an open neighborhood V of q such that $f_{|V}$ is rigid. Therefore there exists some rigid motion S of $\mathbf{R}^N$ such that $f_{|V} = S \circ g_{|V}$. Let $W = V \cap Int(A)$. Clearly W is non-empty and $S = T$ on $g(W)$. Our assumtion implies again that for each point x in W there exists a neighborhood $W' \subset W$ of x such that $f_{|W'}$ is rigid, otherwise x and W would satisfy the lemma. Then $g_{|W'}$ is also rigid which in turn implies that there is no affine subspace of $\mathbf{R}^N$ containing

$g(W')$. This means that $T = S$ everywhere on $\mathbf{R}^N$, q is in A and $V \subset A$, that is, A is open and hence $A = M$. But this contradicts that f is non-rigid.

Proposition 2.2. *Let $f : M^n \to \mathbf{R}^N, n \geq 2$, be an isometric immersion of a homogeneous manifold. Suppose that there exists a point p in M and an open neighborhood U of p such that $f_{|U'}$ is rigid for every $U' \subset U$. Then f is rigid.*

Proof. If f is non-rigid then by Lemma 2.1 there exists $q \in M$ and a open neighborhood W of q such that if $W' \subset W$ then $f_{|W'}$ is non-rigid. Since M is homogeneous there exists an isometry h of M such that $h(q) = p$. Then there exists $U' \subset U$ given by $U' = h(W')$ for some $W' \subset W$. We have then that $f_{|U'}$ is rigid and $f_{|W'}$, is non-rigid. This is clearly a contradiction, since U' and W' are isometric.

Now we apply this proposition to study codimension two isometric immersions of a homogeneous space. Before that, we recall a result of do Carmo-Dajczer in [8] for rigidity of isometric immersions in higher codimensions. Let $f : M^n \to \mathbf{R}^{n+2}$ be an isometric immersion, s an integer, $1 \leq s \leq 2$, and $U^s \subset T_pM^\perp$ a s-dimensional subspace. Let $\pi : T_pM^\perp \to U^s$ be the orthogonal projection. Consider the bilinear from $\alpha_{U^s} : T_pM \times T_pM \to U^s$ given by $\alpha_{U^s} = \pi \circ \alpha$, where α is the second fundamental form of the immersion. Following [8], we call

$$\nu_s(p) = \max\{\dim N(\alpha_{U^s}) : U^s \subset T_pM^\perp\}$$

where $N()$ denotes the nullity space of the enclosed bilinear form. Clearly, $\nu_2(p) = \nu(p)$, the usual relative nullity. Theorem 2.5 of [8] states that if for every p in $M, \nu_s(p) \leq n - (2s + 1)$, for $s = 1, 2$ then f is rigid.

Proposition 2.3. *Let $f : M^n \to \mathbf{R}^{n+2}, n \geq 5$, be an isometric immersion of a homogeneous space such that $\nu(p) = 0$ for every p in M. Then either f is rigid or for every point p in M there exists $\eta \in T_{f(p)}M^\perp$ such that rank $A_\eta \leq 2$.*

Proof. Let us suppose that f is non-rigid. We will show that this implies that $\nu_1(p) \geq n - 2$ for every $p \in M$. This means that there exists $\eta \in T_{f(p)}M^\perp$ such that *rank* $A_\eta \geq 2$, where A_η denotes the Weingarten operator. In fact, assume that there is p in M for which $\nu_1(p) \leq n - 3$. Then there exists an open set U containing p such that $\nu_1(q) \leq n - 3$ for every q in U. Since $n \geq 5$ and $\nu_2(p) = \nu(p) = 0$, it follows from the result of [8] that $f_{|U'}$ is rigid for every $U' \subset U$. But this fact implies by Proposition 2.2 that f is rigid.

3. PROOF OF THE THEOREM

Since M is Einstein, the Ricci curvatures are given by S / n, where S is the scalar curvature. We first remark that $S > 0$. In fact, if $S = 0$, being homogeneous, M is a flat torus T^n (see Corollary 5.6 in [12], Vol. I, p. 251). But $n \geq 3$ and hence T^n cannot be isometrically immersed in $\mathbf{R}^{n+2}$, by a classical result of Tompkins, [17]. Now, if $S < 0$, Corollary 5.4 of [12], Vol. I, pg. 251, implies that the group of isometries of M is finite, contradicting that M is homogeneous. Therefore $S > 0$ and we can conclude that $\nu(p) = 0$, for every p in M.

If $n = 3$, it is well known that a 3-dimensional Einstein manifold has constant sectional curvature. If $n = 4$, as we stated in the introduction, M is symmetric. The classification of compact symmetric spaces of dimension 4 that are Einstein implies that M is either of constant curvature or isometric to the complex projective plane CP^2 or covered by a product of two 2-spheres with the same constant curvature. Hence, in dimensions $n = 3, 4, M$ has nonnegative sectional curvature. Now, since M is in codimension 2 we use the results in [5] to conclude that M is simply connected and CP^2 cannot be imersed in $\mathbf{R}^6$. Therefore M is either isometric to a sphere or splits as product of two 2-spheres of constant curvature.

Now, if $n \geq 5$, we suppose first that f is rigid. Then, if h is an isometry of M, $\bar{f} = f \circ h$ is another isometric immersion of M that being congruent to f, identifies the isometry h with an isometry of $\mathbf{R}^{n+2}$. This means that the group of isometries G acting on M can be realized as a subgroup G' of rigid motions of $\mathbf{R}^{n+2}$. By a result of Cartan (see [12] Vol. II, pg. 111), G' has a fixed point O and hence M is mapped by f into a hypersphere centered at O. Therefore, M is an Einstein hypersurface of a sphere. A result of Ryan in [16], pg. 376, implies then that M is isometric either to a sphere or to a product of two spheres each of which is of dimension greater than 1.

If f is non-rigid we can use Proposition 2.3, since $n \geq 5$ and $\nu(p) = 0$. With the same notation, let ξ be the unit vector orthogonal to η in $T_{f(p)}M^{\perp}$. Observe that if for some point $p \in M$, $rankA_{\eta(p)} \leq 1$, the Gauss equation depends only on A_ξ. Then the same arguments used in [12], Vol. II, pg. 36 for Einstein hypersurfaces of positive scalar curvature can be applied to conclude that $A_{\xi(p)} = \lambda I$ and hence all sectional curvatures are the same at p. By homogeneity, we conclude that M has constant sectional curvature and is isometric to a sphere, since by [5] we conclude again that it is simply conencted.

We were left with the case that $rankA_\eta = 2$ for every point of M. The next proposition shows that in this case M splits in a product of two spheres and one of them is 2-dimensional. Now a result in [1] implies the last assertion of the Theorem. It states that if M is compact and is a product of two manifolds of dimension greater than 1, then f is product of hypersurface immersions.

Proposition 3.1. *Let $f : M^n \to \mathbf{R}^{n+2}, n \geq 5$, be an isometric immersion of an Einstein homogeneous space such that for every point p in M there exists $\eta \in T_{f(p)}M^{\perp}$ such that rank $A_\eta = 2$. Then M is a Riemannian product of two spheres and one of them is 2-dimensional.*

We first prove some preliminary Lemmas.

Lemma 3.2. *Let $f : M^n \to \mathbf{R}^{n+2}, n \geq 5$, be an isometric immersion with the same hypotheses of Proposition 3.1. Then the normal curvature $R^{\perp} = 0$.*

Proof. Let ξ be the unit vector orthogonal to η in $T_{f(p)}M^{\perp}$. Let us assume that η is also a unit vector. We will show that for each point p in M there is an orthonormal basis of T_pM that diagonalizes both Weingarten operators, A_ξ and A_η. The Lemma will follow from the Ricci equation.

Let us consider $\{Y_1, \ldots, Y_n\}$ an orthonormal basis of T_pM and X and Y two arbitrary vectors. Then by the Gauss equation we have

$$Ric(X, Y) = \sum_{i=1}^{n} \langle R(X, Y_i)Y_i, Y\rangle =$$

$$= t_1\langle A_\xi(Y), X\rangle + t_2\langle A_\eta(Y), X\rangle - \langle A_\xi(X), A_\xi(Y)\rangle - \langle A_\eta(X), A_\eta(Y)\rangle$$

where $t_1 = traceA_\xi$ and $t_2 = traceA_\eta$. Now suppose that $A_\xi(Y) = \lambda Y$ and X is orthogonal to Y. Since M is Einstein, $Ric(X, Y) = 0$ and we obtain that

$$0 = t_2\langle A_\eta(Y), X\rangle - \langle A_\eta(Y), A_\eta(X)\rangle = \langle X, A_\eta(t_2Y - A_\eta Y)\rangle$$

for every X orthogonal to Y, implying that $A_\eta(t_2Y - A_\eta(Y)) = \gamma Y$. If $\gamma = 0$ we have $A_\eta(A_\eta(Y)) = t_2A_\eta(Y)$. If $A_\eta(Y) = 0$, we have then that $Y \in KerA_\eta$. If not, $A_\eta(Y)$ is an eigenvector of A_η with corresponding eigenvalue t_2. But this is a contradiction, since $t_2 = traceA_\eta$ and $rankA_\eta = 2$. Now if $\gamma \neq 0$ then $Y \in ImA_\eta$. This shows that whenever Y is an eigenvector of A_ξ, Y is either in $kerA_\eta$ or in ImA_η. Therefore we find an orthonormal basis $\{X_1, \dots, X_n\}$ of T_pM such that $A_\xi(X_i) = \lambda_i X_i$ and $X_i \in KerA_\eta$ for $i \geq 3$. Let us suppose that the matrix of A_η restricted to $span\{X_1, X_2\}$ is

$$\begin{pmatrix} a & c \\ c & b \end{pmatrix}$$

We compute now the Ricci curvatures of M, denoted by Ric, using the Gauss equation. For $i \geq 3$ we have

$$Ric(X_i) = \sum_{j \neq i} \lambda_j \lambda_i \tag{3.3}$$

If all λ_i's, $i \geq 3$, are the same, let us denote them by μ. Then we have

$$Ric(X_1) = \lambda_1\lambda_2 + ab - c^2 + \sum_{i \geq 3} \lambda_1 \lambda_i = \lambda_1\lambda_2 + ab - c^2 + (n-2)\lambda_1\mu$$

$$Ric(X_2) = \lambda_1\lambda_2 + ab - c^2 + \sum_{i \geq 3} \lambda_2 \lambda_i = \lambda_1\lambda_2 + ab - c^2 + (n-2)\lambda_2\mu$$

Since $Ric(X_1) - Ric(X_2) = 0$, we get $(n-2)\mu(\lambda_1 - \lambda_2) = 0$. But $\mu \neq 0$, otherwise $Ric(X_i) = 0$ for $i \geq 3$ which would contradict that the Ricci curvature is positive. Then we conclude that $\lambda_1 = \lambda_2$.

If there are distinct λ_i's, $i \geq 3$, it follows from (3.3) that λ_i must be root of the quadratic equation

$$t^2 - traceA_\xi t + \frac{S}{n} = 0$$

and hence there are only two distinct eigenvalues λ_i, say μ_i and μ_2. They satisfy the relations $\mu_1 + \mu_2 = traceA_\xi$ and $\mu_1\mu_2 = S/n$. Therefore, μ_1 and μ_2 have the same sign. Let m_i be the multiplicity of μ_i. Then

$$\mu_1 + \mu_2 = traceA_\xi = m_1\mu_1 + m_2\mu_2 + \lambda_1 + \lambda_2. \tag{3.4}$$

Now from the expression for $Ric(X_i), i = 1, 2$, we conclude that λ_1 and λ_2 are roots of the quadratic equation

$$t^2 - traceA_\xi t + \frac{S}{n} - (ab - c^2) = 0.$$

Therefore, if $\lambda_1 \neq \lambda_2$, we have again that $\lambda_1 + \lambda_2 = traceA_\xi$ that substiituted in (3.4) gives

$$\lambda_1 + \lambda_2 = m_1\mu_1 + m_2\mu_2 + \lambda_1 + \lambda_2$$

implying that $m_1\mu_1 + m_2\mu_2 = 0$, which is a contradiction, since μ_1 and μ_2 have the same sign. Therefore, we have again $\lambda_1 = \lambda_2$. Let us denote this eigenvalue by λ.

This implies that the eigenvectors of A_η in $span\{X_1, X_2\}$ are also eigenvectors of A_ξ and the lemma is proved.

We point out that only the assumptions M Einstein and codimension 2 are enough to imply that the normal bundle of M is flat. We proved Lemma 3.5 with the hypotheses of Proposition 3.1 because the proof shows that with such hypotheses A_ξ has at most three distinct eigenvalues. In the case of three distinct eigenvalues, it follows from (3.4) that they satisfy the equation

$$(m_1 - 1)\mu_1 + (m_2 - 1)\mu_2 + 2\lambda = 0. \tag{3.5}$$

We will use this fact later.

Lemma 3.6. *With the notation above, the eigenvalues of A_ξ are constant on M.*

Proof. Let Λ^2 be the space of 2-forms defined on the tangent space T_pM. Consider $\mathcal{R} : \Lambda^2 \to \Lambda^2$ the symmetric *curvature operator* given by

$$\langle \mathcal{R}(X \wedge Y), Z \wedge W\rangle = \langle R(X, Y)W, Z\rangle$$

where R denotes the curvature tensor of M. Then if $\{X_1, \ldots, X_n\}$ is an orthonormal basis of the tangent space that diagonalizes both A_ξ and A_η, the 2-forms $X_i \wedge X_j$ are eigenvectors of $\mathcal{R}$. Let δ_1 and δ_2 denote the non-null eigenvalues of A_η. Using the Gauss equation, we conclude from the proof of Lemma 3.2 that the eigenvalues of $\mathcal{R}$ are given by $\lambda^2 + \delta_1\delta_2, \lambda\mu_i, \mu_i^2, i = 1, 2$, and $\mu_1\mu_2$ (recall that μ_1 and μ_2 may be the same). They are the sectional curvatures $K(X_i, X_j)$ of the planes spanned by $\{X_i, X_j\}$. Moreover their multiplicities are constant by homogeneity.

Now we fix p in M. For these tangent vectors X_i's at p we have $\langle R(X_i, X_j)X_k, X_m\rangle = 0$, whenever the set of indices $\{i, j, k, m\}$ has more than two elements. If q is another point in M, since there exists an isometry h of M taking p to q, we have the same equation for the vectors $\bar{X}_i = dh_p(X_i)$. Therefore the 2-forms $\bar{X}_i \wedge \bar{X}_j$ are the eigenvectors of $\mathcal{R}$ at q. Since the eigenvalues are given by the sectional curvatures and $K(X_i, X_j) = K(\bar{X}_i, \bar{X}_j)$, it is easy now to conclude that λ and μ_i are constant.

Lemma 3.7. *With the notation above, ξ and η are parallel sections of the normal bundle $\mathcal{N}(M)$.*

Proof. We first show that ξ and η are differentiable section of $\mathcal{N}(M)$. In fact, let ζ be a differentiable local section of $\mathcal{N}(M)$. Since ζ is linear combination of ξ and η, and the same orthonormal basis diagonalizes all Weingarten operators, A_ζ has eigenvalues of constant multiplicity. This implies by [14] that its eigenvectors can be chosen smoothly. Therefore there exist local vector fields X_i diagonalizing all Weingarten opertors. Observe that η is orthogonal to $\alpha(X_i, X_i)$ for $i \geq 3$ (where α is the second fundamental form) and hence is a differentiable normal vector field. Consequently, ξ is also differentiable.

Now we consider the Codazzi equation

$$\nabla^\perp_{X_i}\alpha(X_j, X_j) - 2\alpha(\nabla_{X_i}X_j, X_j) = \nabla^\perp_{X_j}\alpha(X_i, X_j) - \alpha(\nabla_{X_j}X_i, X_j) - \alpha(X_i, \nabla_{X_j}X_j).$$

Taking inner product to η, we get for $i, j \geq 3$

$$\langle \alpha(X_k, X_j), \nabla^\perp_{X_i}\eta\rangle = 0.$$

Since $\nabla^{\perp}_{X_i}\eta$ is parallel to ξ and $\langle \alpha(X_j,X_j),\xi\rangle \neq 0$ (otherwise we would have zero Ricci curvature), we conclude that $\nabla^{\perp}_{X_i}\eta = 0 = \nabla^{\perp}_{X_i}\xi$, for $i \geq 3$. Now we consider in the Codazzi equation, $i = 1,2$ and $j \geq 3$. Taking inner product to ξ we obtain

$$\langle \nabla_{X_j}X_j, X_i\rangle\langle \alpha(X_j,X_j) - \alpha(X_i,X_i),\xi\rangle = 0,$$

where we used that the fact that the eigenvalues of A_ξ are constant, by the previous lemma. Notice that the proof of Lemma 3.2 shows that the eigenvalues of A_ξ corresponding to X_i and X_j are distinct. Therefore this last equation implies that $\langle \nabla_{X_j}X_j, X_i\rangle = 0$. We apply this fact to the Codazzi equation above but now taking inner product to η. We obtain now

$$\langle \alpha(X_j,X_j), \nabla^{\perp}_{X_i}\eta\rangle = \langle \nabla_{X_j}X_j, X_i\rangle\langle \alpha(X_i,X_i),\eta\rangle = 0$$

and hence $\nabla^{\perp}_{X_i}\eta = 0 = \nabla^{\perp}_{X_i}\xi$, for $i = 1,2$.

Proof of Proposition 3.1. From the previous lemmas and with the same notation, we can define differentiable distributions

$$D = span\{X_1,X_2\} = ImA_\eta$$

$$D^{\perp} = span\{X_3,\ldots,X_n\} = KerA_\eta.$$

They are globally defined and we will show that they are involutive and parallel. For $X_i,X_j \in D^{\perp}$ and $X_k \in D$ we write the Codazzi equation

$$\nabla^{\perp}_{X_i}\alpha(X_k,X_j) - \alpha(\nabla_{X_i}X_k,X_j) - \alpha(X_k,\nabla_{X_i}X_j) =$$

$$= \nabla^{\perp}_{X_k}\alpha(X_i,X_j) - \alpha(\nabla_{X_k}X_i,X_j) - \alpha(X_i,\nabla_{X_k}X_j).$$

Taking inner product to η and using the fact that it is parallel, we are left with

$$\langle \nabla_{X_i}X_j, X_k\rangle\delta_k = 0$$

where δ_k denotes the eigenvalue of A_η which is non-null, since it corresponds to $X_k \in D$. Therefore $\langle \nabla_{X_i}X_j, X_k\rangle = 0$. Now for $X_1,X_2 \in D$ and $X_i \in D^{\perp}$ we use the same procedure, taking inner product of the Codazzi equation to ξ and using the facts that ξ is parallel and the eigenvalues of A_ξ corresponding to X_1 and X_2 are the same. We obtain $\langle \nabla_{X_k}X_m, X_i\rangle = 0$, for $\{k,m\} = \{1,2\}$. Then the distributions are parallel and this implies that M is locally a Riemannian product. Notice that the proof did not depend on the number of distinct eigenvalues of the operator A_ξ.

The local reducibility implies that the sectional curvature $K(X_1,X_i) = \lambda\mu_i = 0$, for $i \geq 3$. Since $\mu_i \neq 0$, we conclude that $\lambda = 0$. Then, if $\mu_1 \neq \mu_2$, we have the equation 3.5 which gives then $m_1 = m_2 = 1$ because $\lambda = 0$ and μ_1 and μ_2 have the same sign. This contradicts our assumption that $n \geq 5$, which in turn implies $m_1 + m_2 \geq 3$. Therefore $\mu_1 = \mu_2 = \mu$.

By the de Rham decomposition theorem we conclude that the universal covering $\tilde{M}$ of M, considered with the covering metric, is a Riemannian product $M_1^2 \times M_2^{n-2}$. By the result in [1], $\tilde{f} = \pi \circ f$, where π is the covering map, is a product of hypersurface immersions. Since

M_1 and M_2 are still homogeneous spaces, we conclude that M_1 is isometric to a 2-dimensional sphere and M_2 is isometric to a $(n-2)$-sphere. Then, the sectional curvatures of $\tilde{M}$ are nonnegative and so are the curvatures of M. Again, by the results in [5] we have that M itself is simply connected and is the Riemannian product $S^2 \times S^{n-2}$.

4. COHOMOGENEITY ONE HYPERSURFACES

Let M be a connected manifold with a Riemannian metric $\langle , \rangle$. Let $Iso(M)$ be the Lie group of all isometries of M with respect to $\langle , \rangle$. Let G be a connected and closed subgroup of $Iso(M)$. We say that M is a *cohomogeneity one Riemannian G-manifold*, if G acts effectively and the principal orbits under the action of G have codimension one.

From the general theory of G-manifolds (see [7] for instance), we get that for compact cohomogeneity one manifolds, the orbit space Ω is a 1-dimensional Hausdorff space homeomorphic to either S^1 or to $[0, \pi]$. Let $p : M \to \Omega$ be the projection onto the orbit space. A point x is called *regular* if $p(x)$ is an interior point of Ω. We will denote the set of all regular point of M by M_{reg}. Notice that M_{reg} is open and dense in M.

Definition 4.1. *A complete geodesic γ on a cohomogeneity one Riemannian manifold is called a* normal geodesic *if crosses each orbit orthogonally.*

The following properties of cohomogeneity one manifolds are known. The reader is referred to [3], [4] for the proof of (a), (b), (c), (d) and to [15] for (e).

Proposition 4.2. *a) A geodesic γ is normal if and only if it is orthogonal to each orbit $\sum_x$ through x in γ.*
b) Each regular point belongs to a unique normal geodesic.
c) The group G acts transitively on the set of normal geodesics.
d) The isotropy subgroup H of a regular point x in γ preserves the normal geodesic pointwise.
e) If $x \in M_{reg}$, then there exists a neighborhood U of x such that U is locally isometric to $((G/H) \times I, g_t + dt^2)$ where I is an open interval of $\mathbf{R}$, and g_t a family of left invariant metrics on G/H, depending smoothly on t.

Compact cohomogeneity one hypersurfaces of Euclidean spaces were studied in [15] and [2]. It was shown in [2] that if $n \geq 4$ and the principal orbits under the action of G have constant sectional curvature then M is immersed in $\mathbf{R}^{n+1}$ as a hypersurface of revolution. This implies immediately Corollary 1 stated in the introduction.

Recall that if $f : M^n \to \mathbf{R}^{n+1}$ is an isometric immersion and ξ is a unit normal vector defined locally on $f(M)$, the *type number* $\tau(x)$ of f at x is defined by the *rank* of A_ξ on T_xM. Suppose now that M is cohomogeneity one G-manifold. If Σ is a principal orbit and for $x \in \Sigma$, $\tau(x) \leq 1$, Proposition 3.2 of [2] implies that $\tau(y) \leq 1$ for every y in the orbit Σ. If G is compact, Σ is a compact homogeneous manifold immersed in codimension 2 such that for each p in M there exists a normal vector ξ satisfying $rank\bar{A}_\xi \leq 1$. Here $\bar{A}_\xi$ is the Weingarten operator of the immersion $\Sigma \to \mathbf{R}^{n+1}$ induced by the Weingarten operator A_ξ of the immersion f. If Σ in Einstein, the proof of our Theorem in the previous section showed that in this case Σ is isometric to a sphere. Since all principal orbits are diffeomorphic to each other, if follows that they are all irreducible manifolds. Therefore if for some $x \in M_{reg}$, $\tau(x) \leq 1, f(M)$ is hypersurface of revolution.

Proof of Corollary 2. The result is obvious if $f(M)$ is a hypersurface of revolution. Now, if the principal orbits are Riemannian products, we have that $f_{|\Sigma}$, f restricted to a principal orbit Σ, is a product of hypersurface immersions. Then each factor of $f_{|\Sigma}$ is an umbilic immersion, since each factor of the orbit Σ is a standard sphere. Therefore each factor of $f_{|\Sigma}$ is rigid. It is not difficult to see that this implies that $f_{|\Sigma}$ is rigid. Since M_{reg} is dense in M, we conclude that f restricted to each orbit of G is rigid and this implies (a) of Corollary 2. Now, the orbits of G' are $f(\Sigma)$ and this implies part (b). Part (c) will follow from Proposition 4.2 (d) for the normal geodesic γ.

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