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CONTRIBUTIONS TO THE THEORY OF BOUNDEDNESS IN UNIFORM SPACES AND TOPOLOGICAL GROUPS

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Abstract. *First, we discuss the behavior of boundedness in uniform spaces with respect to subspaces, projective limits, and suprema in relation to precompactness. A special uniformly isomorphic embedding of an arbitrary uniform space in a bounded uniform space is presented and examined in 2.6. Hejcman's characterization (by B-conservativity) of uniform spaces in which boundedness can be tested by a single pseudometric is proved in a new way, see 3.13, using a version 3.1 of the metrization lemma. We comment briefly on boundedness in topological vector spaces. In topological groups we investigate a hierarchy of partly new notions of boundedness, strongly interrelated among themselves, and exhibit various situations in which certain of these notions coincide. "Boundedness respecting subspaces" of a uniform space prove useful. Many examples illustrate and complement the general theory, see, e.g., Example 6.4.*

Key Words: bounded uniform space, bounded topological group, infrabounded topological group, B-conserving, pseudocomponent, boundedness respecting subspace, ASIN-group

0. INTRODUCTION

In this paper we continue the study of boundedness in uniform spaces and topological groups, as initiated by Hejcman [13] and Atkin [1]. In the first three sections, after reviewing and developing basic materia1 for background and later use, we examine thoroughly the behavior of boundedness with respect to subspaces, projective limits, and suprema in relation to precompactness. From Section 1 we mention in particular Example 1.10 of an unbounded projective limit of bounded uniform spaces, and 1.13, 1.14, 1.15 on maximality properties of the finest precompact compatible uniformity on a Tychonoff space. 1.16 and 1.17 are examples of two bounded uniformities on a set which induce equal topologia but have unbounded supremum. Typically, boundedness is more difficult to handle than precompactness and gives rise to specific concepts like "boundedness respecting subspace" Y of a uniform space X in the sense that any bounded set $A \subset Y$ is bounded in Y. For corresponding results, see 2.2, 2.3, 2.4, 3.5, and Section 6 for "infraboundedness respecting". Theorem 2.6 contains the known fact (see Isbe11 [16], p. 20, no. 21) that every uniform space has a uniformly isomorphic embedding into a bounded uniform space. Our construction has the advantage of being well compatible with other structure on the space, like previously known analogous constructions for topologica1 vector spaces and topological groups. The third section deals with the connections between boundedness, uniformly continuous functions, and pseudometrics. In 3.13 and 3.14 Hejcman's characterization (by B-conservativity) of uniform spaces in which boundedness can be tested by a single pseudometric is proved in a new way, using a version 3.1 of the metrization lemma. A somewhat expository short fourth section on boundedness in topologica1 vector spaces mainly illustrata the materia1 of the preceding sections. In the final sections 5 and 6 we treat boundedness in topological groups. By 5.6, every topological group has a topologically isomorphic embedding into an $\mathcal{L} \vee \mathcal{R}$-bounded topologica1 group. In addition to boundedness with respect to the four natural uniformities of the group, we consider the notions bibounded, infrabounded, and strongly infrabounded (see 5.1 and 6.1) which are natural from the roles they play and strongly interrelated. In the hierarchy which these notions form (see 6.3), any two of them are different except possibly for the pair $\mathcal{L} \vee \mathcal{R}$-bounded and bibounded. In this context the most difficult example 6.4, essentially due to Uspenskij, is that of a (strongly) infrabounded, hut not $\mathcal{L} \vee \mathcal{R}$-bounded group. We exhibit various situations in which certain of our notions coincide. E.g., special results are obtained for ASIN-groups (see 6.14 through 6.19) and for groups with open $\mathcal{L}$-pseudocomponent (see 6.29, 6.30, 6.31). The general theory is illustrated by many examples. A number of questions remain open. Little could be said about $L \vee \mathcal{R}$-boundedness. In a subsequent paper we will treat invariant pseudometrics in relation to boundedness in topological groups, extending Hejcman's work on abelian groups.

In regard to terminology and notation we remark: A uniformity V on a set X will be understood as a set of vicinities. The topology induced by V will also be called the V-topology. For $A \subset$ X, $\mathcal{V}|A$ denotes the restriction of V to A. $A \subset$ X is called uniformly discrete if $\mathcal{V}|A$ is discrete. A uniformity on a topologica1 space $(X, \mathcal{T})$ is called ($\mathcal{T}$-)compatible if it induces $\mathcal{T}$. For $V \in V$, V^0 denotes the diagonal of X. We let $\mathbb{N}$ denote the set of non-negative integers and put $\mathbb{N}' := \mathbb{N} \setminus \{O\}$. For a topologica1 space $(X, \mathcal{T})$ and $a \in$ X, $\mathcal{U}_a(X, \mathcal{T})$ denotes the neighborhood filter of a.

1. BOUNDEDNESS AND PRECOMPACTNES

Definition and Remark 1.1. *Let $(X, \mathcal{V})$ be a uniform space and $A \subset X$. A is called* **precompact**, *if for every vicinity $V \in V$ there exists $F \subset X$ finite such that $A \subset V[F]$. A is called* **bounded,** *if for every vicinity $V \in V$ there exist $F \subset X$ finite and $n \in \mathbb{N}$ such that A C $V^n[F]$. In both cases F can be chosen as a subset of A, see* [13], 1.5. *If A c $Y \subset X$, we call A* **bounded in Y** *if A is bounded with respect to the uniform space (Y, V Y). If we consider several uniformities on X we use the terms "$\mathcal{V}$-boundedness" and "$\mathcal{V}$-precompactness" for distinction. If the whole space is bounded or precompact w. r.t. a given uniformity, we call the uniformity bounded resp. precompact.*

Example 1.2. Let X be a seminormed real or complex vector space. (A seminorm is a function $x \mapsto \|x\|$ of X in $\mathbb{R}$ such that, for all $x, y \in$ X and scalars λ, one has $\|x\| \geq 0$, $\|\lambda x\| = |\lambda| \|x\|$ and $x + y\| \leq \|x\| + \|y\|$.) The seminorm induces the pseudometric $(x, y) \mapsto \|x - y\|$ on X which in turn generata the "standard" uniformity V of X. A set $A \subset$ X is V-bounded in X iff $\sup\{\|a\| : a \in A\}$ is finite.

Boundedness in the context of topological vector spaces will be discussed in Section 4.

Remark 1.3. (1) For a subset A of a uniform space the following implications are obvious: A is compact $\Rightarrow A$ is precompact $\Rightarrow A$ is bounded. Subsets and finite unions of precompact (resp. bounded) sets are precompact (resp. bounded). Also the closure of a precompact (resp. bounded) set in the topology induced by the uniformity is again precompact (resp. bounded), see [13], Theorem 1.9.

(2) The precompactness of a subset A depends entirely on the relative uniformity on A, i.e. A is precompact iff it is precompact with respect to $(A, \mathcal{V}|A)$. This is easily proved by means of the formula $(V \cap (A \times A))[F] = A \cap V[F]$, for $F \subset A$ and $V \in \mathcal{V}$.

As a consequence, precompact sets are bounded in themselves.

(3) If $A \subset Y \subset X$ and A is bounded in Y, then it is also bounded in X. In contrast to the case of precompact sets. the converse is generally not true, as the following example by Atkin shows (cf. [1], Ex. 1.8 b)) : Let A be an infinite orthonormal subset of a Hilbert space X. uniformized as in 1.2. Then A is bounded in X, but the relative uniformity on A is discrete, so that A is not bounded in itself. As a consequence, A is not precompact. From this example one obtains a bounded uniform space X' with a closed and open bounded subset A' that is not bounded in itself by putting $X' := \{x \in X : \|x\| < 2$ und $\|x - a\| \neq \frac{1}{2}$ for all $a \in A\}$ and $A' := \{x \in X : \|x - a\| < \frac{1}{2}$ for some $a \in A\}$.

(4) One easily proves that finite unions of bounded in themselves subsets are bounded in themselves. The next fact, which is essentially (2.1) of [1], has a slightly more technical proof: For $Z \subset Y \subset X$ with Z dense in Y, one has: Z is bounded in itself iff Y is bounded in itself.

(5) On every completely regular space X there exists a compatible precompact uniformity, e.g. the uniformity induced by the Stone-Čech-compactification. This shows that generally the topology does not yield sufficient information concerning precompactness.

(6) Plainly, finite sums of bounded uniform spaces are bounded.

Proposition 1.4. *Let $(X, \mathcal{V})$ be a uniform space and $A \subset X$. Then A is bounded (resp. precompact) iff every countable discrete $B \subset A$ is bounded (resp. precompact, or, equivalently, finite).*

Proof. The condition is necessary by 1.3(1). Now let A be unbounded. Then there exists a vicinity V such that for any finite $F \subset A$ and, for any $n \in \mathbb{N}'$, $A \not\subset V^n[F]$. Take any $a_0 \in A$ and choose inductively $a_n \in A$ with $a_n \notin V^n[\{a_0, \ldots, a_{n-1}\}]$ (for $n > 0$). Then the set $B := \{a_n : n \in \mathbb{N}\}$ is not bounded: Suppose it is. Then there exist $m, n \in \mathbb{N}'$ with $B \subset V^n[\{a_0, \ldots, a_m\}]$. We can assume $m = n - 1$ (if necessary, increase either m or n), so that especially $a_n \in V^n[\{a_0, \ldots, a_{n-1}\}]$, which contrudicts the choice of a_n.

If A is not precompact, there is a vicinity V with $A \not\subset V[F]$, for all finite $F \subset A$. Then choose inductively $a_n \in X$, $n \in \mathbb{N}$ with $a_n \notin V[\{a_0, \ldots, a_{n-1}\}]$. Then $B := \{a_n : n \in N\}$ is not precompact.

It remains to show that B is uniformly discrete. Pick a symmetric $W \in \mathcal{V}$ with $W^2 \subset V$. For $i \neq k$ we have $a_i \notin V[a_k]$ (in both cases), and therefore $W[a_i] \cap W[a_k] = \emptyset$. □

We omit the proof of the following known

Corollary 1.5. *(a) A subset A of a uniform space is precompact iff every uniformly discrete $B \subset A$ is finite.*

(b) A countably compact space is precompact.

More generally than 1.5 (b), every pseudocompact subset of a uniform space is precompact, see [9], Problem 8.5.10 or [9], Chapter IX,§1, Exercise 21.

Proposition 1.6. *Let $f : X \to Y$ be a uniformly continuous mapping between uniform spaces.*

If $A \subset X$ is bounded (bounded in itself or precompact, respectively), then $f(A) \subset Y$ has the same property.

Proof. Elementary, cf. [13], **Theorem** 1.10. □

Proposition 1.7. *Let $(f_i)_{i\in I}$ be a family of maps $f_i : X \to X_i$, let $\mathcal{V}_i$ be a uniformity on X_i for $i \in I$, and let $\mathcal{V}$ be the initial uniformity with respect to $(f_i)_{i\in I}$. A subset $A \subset X$ is precompact in $(X, \mathcal{V})$ iff $f_i(A)$ is $\mathcal{V}_i$-precompact for each $i \in I$.*

Proof. See [4], Chapter II, §4.2, Proposition 3.

Since the relative uniformity is the initial uniformity with respect to the inclusion map, we know from example 1.3 (4). that the analogue for bounded sets does not hold in general. However. it has been shown to be true for product spaces and certain projective limits:

Proposition 1.8. *Let X be the product of a family $(X_i)_{i\in I}$ of uniform spaces, endowed with the product uniformity. For $i \in I$, let $\pi_i : X \to X_i$ be the projection. Then for all $A \subset X$: A is bounded in X iff $\forall i \in I$: $\pi_i(A)$ is bounded in X_i.*

Proof. See [13], **Theorem 1.11.** □

Proposition 1.9. *Let $(X, \mathcal{V})$ be the projective limit of a projective system $(X_i, \mathcal{V}_i, f_{ik}, I)$ of uniform spaces $(X_i, \mathcal{V}_i)$ with directed index set I in the sense of [4], Chapter II, §2.7. ("inverse system" in the terminology of [9], Exercise 8.2.B). Assume that, for each $i \in I$, the canonical map $f_i : X \to X_i$ is surjective. Then a subset A of X is $\mathcal{V}$-bounded (resp. bounded in itself) iff $f_i(A)$ is $\mathcal{V}_i$-bounded (resp. bounded in itself), for each $i \in I$.*

Proof. See [1], Lemma 2.3. For the part concerning "bounded in itself" the surjectivity of the maps f_i is not needed. □

That the assumption of surjectivity of the maps f_i is not superfluous is shown by

Example 1.10. of an unbounded projective limit of a decreasing sequence of bounded uniform spaces. Let Y be a countable Hausdorff, dense in itself and bounded uniform space with an unbounded subspace X (2.6 and 2.7 (5), (1) and (0) yield such Y and X). Writing Y \ X as $\{y_n : n \in \mathbb{N}\}$, we obtain a decreasing sequence of dense open subsets $Y \setminus \{y_0, \dots, y_n\}$ which are bounded (in themselves) by 1.3 (4). Their intersection X is the desired projective limit.

One could obtain un example with closed subspaces if, in the above, Y could be chosen in addition locally precompact. (However we do not know whether such a Y exists.) The unbounded in itself subspace X may be assumed to be closed because of 1.3 (2). Then one can choose, for each $n \in$ N, an open precompact neighborhood $V_n \subset Y \setminus X$ of y_n, so the subspaces $Y \setminus \bigcup_{i=0}^{n} V_i$, $n \in$ N, are closed, and they are bounded in themselves by 3.7 (which we anticipate for this construction) and they have again as their projective limit the unbounded space X.

The following notion of V-component, helpful in the discussion of boundedness, has already been used by Bourbaki ([4], Chapter 11, §4, Exercise 7) and Atkin ([1], 1.3).

Definition and Remark 1.11. *Let X be a uniform space and V a symmetric vicinity. Then $\hat{V} := \bigcup_{n\in\mathbb{N}} V^n$ defines an equivalence relation on X. The equivalence classes modulo $\hat{V}$ are called the V*-**components**. *They are open and therefore closed.*

The **pseudocomponent** *of $x \in$ X is defined as the intersection of all V-components containing* *x. The pseudocomponent of x is a superset of its quasicomponent (defined as the intersection of all clopen sets containing x) which in turn contains its connected component. Plainly, the pseudocomponents are closed, and they form a partition of X.*

The equivalence "$(a) \Leftrightarrow (b)$" of the following lemma has been shown by Atkin

Lemma 1.12. *Let $(X, \mathcal{V})$ be a uniform space. For $A \subset X$ the following are equivalent:*
(a) A is bounded.
(b) For all symmetric vicinities V, A meets only finitely many V-components; and for every V-component W and every $x \in W$ there exists $n \in \mathbb{N}$ with $A \cap W \subset V^n[x]$.
(c) For all symmetric vicinities V. A meets only finitely many V-components; and, for every V-component W, $W \cap A$ is bounded in W.

Proof. For "$(a) \Leftrightarrow (b)$" see [1], Lemma 1.4
"$(a) \Rightarrow (c)$": Let A be a bounded set and W be a V-component. Because of $(a) \Leftrightarrow (b)$, it remains to show that $A \cap W$ is bounded in W. Let U be a symmetric vicinity, $U \subset V$. Then we have $A \cap W \subset U^n[F]$, with suitable $n \in \mathbb{N}$ and finite $F \subset A \cap W$. Since $U \subset V$ and W is a V-component, we have $(U \cap (W \times W))^n = U^n \cap (W \times W)$, so that $A \cap W \subset (U \cap (W \times W))^n[F]$. Thus $A \cap W$ is bounded in W.
"$(c) \Rightarrow (a)$": (c) implies that A is the finite union of bounded sets and hence bounded. □

Proposition 1.9 applies in particular to the supremum of a directed set of uniformities on a given set X. (Here the directedness of the set of uniformities is crucial, see examples 1.16 through 1.18.) In particular, by Zorn's lemma, for any set $\mathcal{A}$ of subsets of a set X there exist uniformities V which are maximal with respect to the property "each $A \in \mathcal{A}$ is $\mathcal{V}$-bounded". We call these uniformities $\mathcal{A}$-maximal. If there exist two distinct $\mathcal{A}$-maximal uniformities V and W for the same set A then some $A \in \mathcal{A}$ is not bounded for the supremum $\mathcal{V} \vee \mathcal{W}$. This remark applies in particular to the case $\mathcal{A} = \{X\}$. (Conversely, if V and W are two uniformities on X for which each $A \in \mathcal{A}$ is bounded, but not every $A \in \mathcal{A}$ is $\mathcal{V} \vee \mathcal{W}$-bounded, then any $\mathcal{A}$-maximal uniformity $\mathcal{V}' \supset \mathcal{V}$ is distinct from any $\mathcal{A}$-maximal uniformity $\mathcal{W}' \supset \mathcal{W}$.)

We now give some results concerning the maximality of the finest precompact compatible uniformity on Tychonoff spaces.

Proposition 1.13. *The finest precompact uniformity W on a set X is a maximal element in the set of all bounded uniformities on X. In particular, W is maximal in the set of all bounded compatible uniformities on* **X** *endowed with the discrete topology.*

Proof. Suppose there is a bounded uniformity $V \supsetneq W$ *on X*. Then V is not precompact. Hence, by 1.5 (a), X has an infinite V-discrete subset *A*. *On* the other hand $N := (A \times A) \cup ((X \setminus A) \times (X \setminus A))$ generates a precompact uniformity 2 on X; so $N \in 2 \subset \mathcal{W} \subset V$, and A is an N-component. But then, by 1.12, $(a) \Rightarrow (c)$, A is $\mathcal{V}$-bounded in itself, which contradicts the $\mathcal{V}$-discreteness. □

The following is an example that the proposition does not hold for arbitrary Tychonoff spaces.

Example 1.14. of a Tychonoff space X whose finest compatible precompact uniformity W is not maximal in the set of all compatible bounded uniformities on X. In a Hilbert space with norm $\| \ \|$ and infinite orthonormal system $(e_\iota)_{\iota \in I}$, let $X := \bigcup_{\iota \in I} [0, 1] e_\iota$ be equipped with the usual topology induced by $\|\,\|$. The usual compatible uniformity $\mathcal{V}$ on X is clearly bounded. $\mathcal{V}$ is not comparable with the finest precompact compatible uniformity $\mathcal{W}$: $\mathcal{V}$ is not coarser than $\mathcal{W}$ since the set $\{e_\iota : \iota \in I\}$ is not precompact: it is not finer than $\mathcal{W}$ since it is easy to exhibit bounded continuous real functions that are not $\mathcal{V}$-uniformly continuous, but these functions are $\mathcal{W}$-uniformly continuous. $\mathcal{W}$ is the initial uniformity with respect to all bounded continuous functions $X \to \mathbb{R}$, see [11], 15 I and J. Let $U \in \mathcal{V} \vee \mathcal{W}$. For the boundedness of $\mathcal{V} \vee \mathcal{W}$ we prove that $X = U^n[0]$ for some $n \in \mathbb{N}'$. Putting $f_1(x) := \|x\|$ $(x \in X)$, there are some more bounded real functions $f_2, \ldots, f_r$ on X and an $\epsilon > 0$. such that

$$U \supset \{(x,y) \in X \times X : \|x - y\| < \epsilon \text{ and } |f_\rho(x) - f_\rho(y)| < \epsilon \text{ for } 1 \leq \rho \leq r\}$$

We cover $J := \bigcup_{\rho=1}^{r} f_\rho(X)$ by finitely many open intervals $I_1, \ldots, I_m$ of lengths at most ϵ. Let $\iota \in I$. Since the m^r products $I_{\mu_1} \times \ldots \times I_{\mu_r}$ with $\mu_1, \ldots, \mu_r \in \{1, \ldots m\}$ cover J^r, the m^r open sets

$$\{x \in [0,1]e_\iota : (f_1(x), \ldots, f_r(x)) \in I_{\mu_1} \times \ldots \times I_{\mu_r}\}$$

cover $[0,1]e_\iota$. Therefore, and since $[0,1]e_\iota$ is connected, we can index these sets as $X_{\iota,1}, \ldots, X_{\iota,m^r}$ in such a way that $0 \in X_{\iota,1}$, and such that each non-empty $X_{\iota,\tau}$ with $2 \leq \tau \leq m^r$ intersects some $X_{\iota,\sigma}$ with $1 \leq \sigma < \tau$. Now, for $\tau \geq 2$, one has for all $x, y \in X_{\iota,\tau}$, $|f_\rho(x) - f_\rho(y)| < \epsilon$ for all $1 \leq \rho \leq r$, und $\|x - y\| = |f_1(x) - f_1(y)| < \epsilon$, whence $(x,y) \in U$. It follows that $X_{\iota,\tau} \subset U[X_{\iota,\sigma}]$ for some $\sigma < \tau$, and we obtain $X = U^{m^r}[0]$

Remarks 1.15. (1) $\mathcal{V} \vee \mathcal{W}$ is complete since it is compatible and $\mathcal{V}$ is complete.
(2) The dense subspace $X_0 := X \setminus \{0\}$ of $(X, \mathcal{V} \vee \mathcal{W})$ is locally compact (in fact, the topological sum of the spaces $]0,1]e_\iota$) and it is bounded. but not precompact. Therefore, by 1.20 below. it is not uniformly locally precompact. Further, a slight adaptation of the arguments from the example shows for the locally compact space X_0: The uniformity $\mathcal{V}_0 := \mathcal{V}|X_0$ is not comparable with the finest precompact compatible uniformity $\mathcal{W}_0$ on X_0, and $\mathcal{V}_0 \vee \mathcal{W}_0$ is again bounded. Indeed, instead of considering $U^n[0]$ with $U \in \mathcal{V}$, one shows that, for every $U_0 \in \mathcal{V}_0 \vee \mathcal{W}_0$, there is $p \in \mathbb{N}$ such that for all $\iota \in I$, $U_0^p[e_\iota] =]0,1]e_\iota$. Since $U_0 = U \cap (X_0 \times X_0)$ for some $U \in \mathcal{V}$, and $U[0]$ is a neighborhood of 0 in X. it follows that $X_0 \subset U_0^{p+1}[e_\iota]$, for any $\iota \in I$.
(3) We do not know whether $\mathcal{V} \vee \mathcal{W}$ is maximal or even the finest uniformity in the set of all bounded compatible uniformities on X.
(4) Perhaps every bounded, non-precompact compatible $\mathcal{R}$ on X is the supremum of $\mathcal{V}$ and the finest precompact uniformity $\mathcal{P} \subset \mathcal{R}$. ($\mathcal{P}$ is the initial uniformity with respect to all bounded $\mathcal{R}$-uniformly continuous functions $X \to \mathbb{R}$.) This would imply that $\mathcal{V}$ is the smallest of all bounded, non-precompact compatible uniformities, and that $\mathcal{V} \vee \mathcal{W}$ is the finest bounded compatible uniformity (cf. (3)).

Example 1.16. of two bounded metrizable uniformities $\mathcal{V}$ and $\mathcal{W}$ on a set X which induce equal topologies and have unbounded supremum. In the Banach space $l^p_{\mathbb{Z}}$ with $1 \leq p \leq \infty$ consider the unit vectors $e_k := (\delta_{ki})_{i \in \mathbb{Z}}$, $k \in \mathbb{Z}$, and define $X := \bigcup_{k \in \mathbb{Z}} ([0, \frac{1}{2}[\cup]\frac{1}{2}, 1]) e_k$ with its usual topology. Let $\mathcal{V}$ be the metrizable uniformity on X induced by the standard uniformity of $l^p_{\mathbb{Z}}$. $\mathcal{V}$ is easily seen to be bounded. Let $\phi : X \to X$ be the bijection that leaves the points of $\bigcup_{k \in \mathbb{Z}} [0, \frac{1}{2}[\cdot e_k$ fixed and sends $(x_i)_{i \in \mathbb{Z}}$ into $(x_{i-1})_{i \in \mathbb{Z}}$, for $\|x\| > \frac{1}{2}$. Let

W be the uniformity on X for which $\phi : (X, \mathcal{V}) \to (X, \mathcal{W})$ is a uniform equivalence; so W is also hounded. Clearly, $\mathcal{V}$ and $\mathcal{W}$ are compatihle. To show that V V W is not bounded consider the vicinities $M := \{(x, y) \in X \times X : \|x\ y\| < \frac{1}{2}\} \in \mathcal{V}$ and N $:= (\phi \times \phi)(M) \in \mathcal{W}$, so $M \cap$ N $\in V$ v $\mathcal{W}$. It suffices to show that the M n N component of e_k is equal to $]\frac{1}{2}, 1]\, e_k$, since this implies that X has infinitely many $M \cap N$-components and hence cannot be hounded, by Lemma 1.12. In fact we will show $(*)(M \cap N)^n[e_k] =]\frac{1}{2}, 1] \cdot e_k$, for all $n \geq 1$. By definition one has $M[e_k] = N[e_k] =]\frac{1}{2}, 1]\, e_k$, which proves $(*)$ for $n = 1$. The induction step amounts to proving $(M \cap N)[\lambda e_k] \subset]\,\frac{1}{2}, 1]\ e_k$, for $\frac{1}{2} < \lambda \leq 1$. Suppose that there exists $y \in (M \cap N)[\lambda e_k] \setminus \left(]\frac{1}{2}, 1]\, e_k\right)$. Then $y \in M[\lambda e_k]$ implies thut

$$(**) \qquad y \in]0, \frac{1}{2}[\cdot e_k$$

since points in $[0, 1] \cdot e_j$ are at distance $> \frac{1}{2}$ from λe_k for $j \neq k$. Therefore $y \in N[\lambda e_k]$ implies that $y = \phi^{-1}(y) \in M[\phi^{-1}(\lambda e_k)] = M[\lambda e_{k-1}]$. It follows that $y \in]0, 1[\cdot e_{k-1}$, in contradiction to $(**)$.

Remembering that the supremum of two precompact uniformities is precompact. the following example is of interest.

Example 1.17. of two uniformities V and W on a set X. inducing equal topologies, such that V is hounded and $\mathcal{W}$ is precompact and such that $\mathcal{V} \vee$ W is unhounded. Lct (X, V) be a bounded uniform space which has an open and closed subset A that is not $\mathcal{V}$-bounded in itself (cf. end of 1.3 (3), where $\mathcal{V}$ is metrizahle). Let $\mathcal{W}_A$ be the precompact uniformity on X with basis {N}, where N $:= (A \times A)$ U $((X \setminus A) \times (X \setminus A))$. The $\mathcal{W}_A$-topology $\{\emptyset, X, A, X \setminus A\}$ is coarser than the $\mathcal{V}$-topology $\mathcal{T}$. Hence $\mathcal{W}_A$ is coarser than the finest $\mathcal{T}$-compatible precompact uniformity W (the precompact uniformity $W \vee \mathcal{W}_A$ is $\mathcal{T}$-compatible, hence equal to $\mathcal{W}$). To prove that $\mathcal{V} \vee \mathcal{W}$ is unbounded we show that A is $\mathcal{V}$ V $\mathcal{W}_A$-unbounded. As A is not $\mathcal{V}$-bounded in itself, it is also not V V $\mathcal{W}_A$-bounded in itself. But A is an N-component, hence, hy 1.12. "$(a) \Rightarrow (c)$", A is not $\mathcal{V}$ V $\mathcal{W}_A$-bounded in X.

An example with metrizahle $\mathcal{V}$ and W can he ohtained similarly if $\mathcal{V}$ is metrizable (cf. end of 1.3 (4)) and if, in addition to A as above, there exists a metrizahle precompact uniformity W' inducing the $\mathcal{V}$-topology on X: One may then take $W := W' \vee \mathcal{W}_A$. E.g., if the metrizable $\mathcal{V}$-topology has a countable basis $\mathcal{B}$ consisting of closed and open sets, then the sets $(B \times B)$ U $((X \setminus B) \times (X \setminus B))$ with $B \in \mathcal{B}$ generate such a uniformity $\mathcal{W}'$.

In hoth examples X is not connected; we do not know a connected counterexample. But we mention that we were able to construct a connected Tychonoff space X, two compatible metrizahle uniformities $\mathcal{V}$ and W and a subset $A \subset$ X which is $\mathcal{V}$- as well as W-bounded, but not V V $\mathcal{W}$-bounded.

Example 1.18. of two bounded uniformities $\mathcal{V}$ and W on a set X such that the W-topoloyy is compact and $\mathcal{V}$ V W is unhounded. Let $(X, \mathcal{V})$ he any bounded uniform space with an unbounded subspace $(A, \mathcal{V}|A)$, and let W he any uniformity for which A and X \ A *are* compact. Then an argument very similar to that of the preceding example yields that $\mathcal{V} \vee W$ is unbounded.

Definition 1.19. *A uniform space* $(X, \mathcal{V})$ *is called* **uniformly locally compact** (**uniformly locally precompact** *resp.* **uniformly locally bounded**) *if there exists* $V \in \mathcal{V}$ *such that, for all*

$x \in X$, $V[x]$ is compact (precompact resp. bounded). A vicinity $V \in \mathcal{V}$ is called **B-conserving** *if $V[A]$ is bounded for each bounded $A \subset X$, or, equivalently, if $V^n[x]$ is bounded for ench $x \in X$ and $n \in \mathbb{N}$. The uniformity $\mathcal{V}$ and the space $(X, \mathcal{V})$ are called* **B-conservative** *if there is a B-conserving $V \in \mathcal{V}$.*

The concept B-conserving has been introduced and studied by Hejcman in [14], Def. 3, und in [15]. That B-conservative is strictly stronger than uniformly locally bounded is shown in [14], Example. In particular, B-conservative is used to characterize uniform spaces in which boundedness can be tested by a single pseudometric, see [15], Theorem 1 and our Theorem 3.13.

The next proposition gives some easily found classes of uniform spaces in which boundedness and precompactness are the same. See 2.4 fora characterisation of this property.

Proposition 1.20. *Let $(X, \mathcal{V})$ be a uniform space fulfilling one of the following conditions:*
(a) x is uniformly locally precompact.
(b) $\mathcal{V}$ has a basis consisting of transitive relations.
Then every bounded subset of X is precompact. If X is even uniformly locally compact, then every bounded subset is relatively compact.

Proof. For case (a), see [13], Theorem 1.18. Case (b) is clear, since transitivity of V means $V^n = V$. for all $n \geq 1$, and because it suffices to consider a basis of $\mathcal{V}$. The last statement follows easily from case (a). □

Remarks 1.21. (1) Condition (b) of the previous proposition is equivalent to
(b') $\mathcal{V}$ has a basis consisting of equivalence relations,

since, for any transitive reflexive relation V, $V \cap V^{-1}$ is an equivalence relation.
(2) There are locally compact (even discrete) Tychonoff spaces which have compatible uniformities which are not uniformly locally precompact. For instance, take an infinite set X and a sequence of subsets $(U_n)_{n\in\mathbb{N}}$ with $U_n \supset U_{n+1}$ and such that $U_n \setminus U_{n+1}$ is infinite for every $n \in \mathbb{N}$. Suppose further that $\bigcap_{n\in\mathbb{N}} U_n = \emptyset$. Then the sequence $(V_n)_{n\in\mathbb{N}}$, defined by $V_n := \Delta_X \cup (U_n \times U_n)$ is the basis of a uniformity $\mathcal{V}$ which induces the discrete topology on X. But $\mathcal{V}$ is not uniformly locally precompact: To sec this, it is sufficient to observe that for every $n \in \mathbb{N}$ and every $F \subset X$ finite we have $V_{n+1}[F] \subset U_{n+1} \cup F$, in particular for every $x \in U_n \setminus U_{n+1}$ we have $V_n[x] = U_n \not\subset V_{n+1}[F]$.

Since $\mathcal{V}$ has a basis consisting of equivalenze relations, we see (by (1)) that $\mathcal{V}$ is not even uniformly locally bounded.
(3) Fora further example of non-locally bounded compatible uniformities on a discrete space let X be an uncountable set. Wc equip X with the initial uniformity $\mathcal{V}$ with respect to all functions $X \rightarrow \mathbb{R}$ (with $\mathbb{R}$ carrying the usual uniformity); the V-topology is discrete. We show: All bounded subsets of X are finite and $(X, \mathcal{V})$ is not uniformly locally bounded. If $A \subset X$ is infinite, any function $X \rightarrow \mathbb{R}$ which is unbounded on A shows that A is unbounded, by 1.6. To prove the second assertion note that the sets of the form

$$V = \{(x,y) \in X \times X : \sup\{|f_i(x) - f_i(y)| : i = 1,..,n\} < 1\}$$

where $f_1, ,f_n$ are arbitrary real functions on X, form a basis of $\mathcal{V}$. For any such basic vicinity V there is an uncountable set $Y \subset X$ such that

$$\forall i \in \{1,.., n\} \exists k \in \mathbb{Z} \forall y \in Y : f_i(y) \in [k, k+1[$$

Hence $Y \times Y \subset V$, so that, for each $y \in Y$, $V[y]$ is infinite, i.e. unbounded.
(4) There exist bounded, non-precompact uniform spaces with discrete topology, see 3.9. Because of 1.20, they cannot be uniformly locally precompact. Note that a locally compact space is uniformly locally compact for some compatible uniformity iff it is paracompact, see [18], Chap. 6, Problem T (e).

The last proposition of this section concerns the behavior of boundedness with respect to special quotients.

Proposition 1.22. *Let $(X, \mathcal{V})$ be a uniform space and R an equivalence relation on X which is* **compatible with V** *(in the sense of [25], 4.10), i.e.*

$$\forall V \in \mathcal{V} \exists U \in \mathcal{V} : R \circ U \subset V \circ R.$$

Suppose all equivalence classes modulo R are bounded (resp. precompact). If we endow X / R with the quotient uniformity then for the quotient map q and any subset A of X we have the equivalence:
A is bounded in X (resp. precompact) iff q(A) is bounded in X / R (resp. precompact).

Proof. We shall only prove the statement for bounded sets, since the proof for precompact sets is completely analogous (see also [25], 12.15). The "only if" part is clear by the uniform continuity of q. Now let A be a subset of X with $q(A)$ bounded and let V be any vicinity on X. By the assumption on R there exists a vicinity U with $R \circ U \subset V \circ R$. We have $(q \times q)(U) \in \mathcal{V} / \mathcal{R}$ by [25], 4.10, whence

$$q(A) \subset ((q \times q)(U))^n [\tilde{F}]$$

with $n \in \mathbb{N}$ and finite $\tilde{F} \subset q(A)$. Choose a finite $F \subset A$ with $q(F) = \tilde{F}$. Then one verifies easily

$$q(A) \subset ((q \times q)(U))^n[q(F)] = ((q \times q)(U^n)[q(F)] \subset q(U^n \circ R[F]),$$

which entails

$$A \subset q^{-1}(q(A)) \subset q^{-1}q((U^n \circ R)[F]) = (R \circ U^n \circ R)[F] \subset (V^n \circ R)[F],$$

due the choice of U (and to $R^2 = R$). By the assumption on R, $R[F]$ is bounded, so that $R[F] \subset V^m[G]$, with a finite set G and $m \in \mathbb{N}$. Thus $A \subset V^{m+n}[G]$. □

2. BOUNDEDNESS IN SUBSPACES

As was noted in 1.3(1), precompactness of a subset is a property of its relative uniformity, whereas boundedness in $Y \subset X$ may depend on the subspace Y. In this section we will discuss this point in greater detail. For this purpose, we give the following definition:

Definition 2.1. *Let $(X, \mathcal{V})$ be a uniform space. We say that a subspace $Y \subset X$* **respects boundedness** *(or, for short,* **is a b.r. subspace***) if it induces the same notion of boundedness as the whole space, i.e. if a subset of Y is bounded in Y if (and only if) it is bounded in X.*

Trivially, a bounded subset of X respects boundedness iff it is bounded in itself. If Y is a b.r. subspace of X and Z is a b.r. subspace of Y, Z is a b.r. subspace of X.

Regarding b.r. subspaces we prove now:

Proposition 2.2. *Let $(X, \mathcal{V})$ be a uniform space.*
(i) A subspace $Y \subset X$ respects boundedness if it fulfills one of the following conditions
(a) Y is a union of V-components, for some symmetric $V \in \mathcal{V}$.
(b) Y is dense in X.

(ii) All subspaces of X respect boundedness if every bounded subset is precompact.
(iii) If Y_1 and Y_2 are b.r. subspaces of X, so is $Y_1 \cup Y_2$.

Proof. To (i). Case (a) is clear by Lemma 1.12 $(c) \Leftrightarrow (a)$, using the fact that for two symmetric vicinities $U, V \in \mathcal{V}, U \subset V$, every U-component is contained in a V-component. For case (b), see [13], Theorem 1.20.
To (ii). The hypothesis and 1.3(2) imply that every bounded subset of X is bounded in itself. Hence 1.3(3) yields the result.
To (iii). Let $A \subset Y_1 \cup Y_2$ be bounded in X. Then $A \cap Y_i$ is bounded in X for $i = 1, 2$, hence bounded in $Y_1 \cup Y_2$, by 1.3 (2). Therefore $A = (A \cap Y_1) \cup (A \cap Y_2)$ is bounded in $Y_1 \cup Y_2$ by 1.3(1). □

Proposition 2.3. *Let $(X, \mathcal{V})$ be a uniform space, $Z \subset Y \subset X$, Z dense in Y. If* **Y** *respects boundedness, then so does* **Z**. *If $\mathcal{V}|Z$ is pseudometrizable, the converse also holds.*

Proof. The first statement is a consequence of Proposition 2.2(i)(b) and the transitivity of the b.r.-property. For the converse in case that $\mathcal{V}|Z$ is pseudometrizable. we observe that also $\mathcal{V}|Y$ is pseudometrizable, since the closure of a countable basis of $\mathcal{V}|Z$ in $Y \times Y$ is a basis of $\mathcal{V}|$ Y. We must show: If $A \subset Y$ is V-bounded. it is $\mathcal{V}|Y$-bounded. By 1.4. we may assume without loss of generality that A is countable, $A = \{a_n : n \in N\}$. Choose a basis $(W_n)_{n\in\mathbb{N}}$ of $\mathcal{V}|Y$ with $W_n = W_n^{-1}$ and $W_{n+1}^2 \subset W_n$ for all n. Choose $b_n \in W_n[a_n] \cap Z$, for $n \in$ N. Let $V \in \mathcal{V}$. There are $E \subset A$ finite and $p \in$ N such that $A \subset V^p[E]$, and there is $m \in \mathbb{N}$ such that $W_m \subset V \cap (Y \times Y)$. Consequently $\{b_n : n \geq m\} \subset W_m[A] \subset V^{p+1}[E]$. Hence $\{b_n : n \in \mathbb{N}\}$ is $\mathcal{V}$-bounded and therefore, by assumption, $\mathcal{V}|Z$-bounded. So there are $F \subset Z$ finite and $q \in \mathbb{N}$ such thnt $\{b_n : n \in \mathbb{N}\} \subset (V \cap (Z \times Z))^q[F]$. Now we obtain $\{a_n : n \geq m\} \subset W_m[\{b_n : n \geq m\}] \subset (V \cap (Y \times Y))^{q+1}[F]$. This shows that A is $\mathcal{V}|Y$-bounded. □

We would not be surprised to see a counterexample for non-pseudometrizable Z.
The following proposition yields the converse of Proposition 2.2 (ii):

Proposition 2.4. *For a uniform space X the following are equivalent:*
(a) Every subspace respects boundedness.
(b) Every bounded subset of **X** *is bounded in itself.*
(c) Every bounded uniformly discrete subset of X is finite.
(d) Every bounded subset of X is precompact.
(e) Every bounded countable, uniformly discrete subset of X is precompact

Proof. "$(a) \Rightarrow (b)$": Obvious.
"$(b) \Rightarrow (c)$": Every uniformly discrete subset which is bounded in itself is finite.
"$(c) \Rightarrow (d)$": If A is bounded and $B \subset A$ is uniformly discrete. then B is bounded and therefore finite. hy (c). By 1.5 (a), this implies that A is precompact.

"$(d) \Rightarrow (e)$": Obvious.

"$(e) \Rightarrow (a)$": Suppose. (a) does not hold, i.e. there is $A \subset Y \subset X$ whith A bounded in X but not in Y. Then, by 1.3 (2) and (4). A is not precompact and hence contains a countable uniformly discrete subset B which is not precompact, by 1.4. As a subset of a bounded set, B is bounded, which contradicts (e). □

Corollary 2.5. *For a bounded uniform space X the following are equivalent:*

(a) Every subspace respects boundedness.

(b) Every subspace is bounded in itself.

(c) X is precompact.

The following theorem, illustrating the dependence of boundedness on the subspace, is a known consequence of Isbell [16], p. 20, no. 21, combined with 1.8 and the fact that unit balls of normed spaces are bounded in their natural uniformities, see 4.3. We give here a different proof, by a construction that involves convergence in measure, and (with some modifications) has been used in various contexts: Hartman and Mycielski embedded topological groups into connected ones (see [12]), Peck and Porta used it for the construction of dual-less topological vector spaces (see [22], [23]). We have not seen this construction in the context of uniform spaces, and it will be needed also in the proofs of 4.7 and 5.6.

Theorem 2.6. *Every uniform space has a uniformly isomorphic embedding into a bounded uniform space.*

Proof. Let $(X, \mathcal{V})$ be a uniform space. Let $B(X)$ be the set of all maps $f : [0,1[\to X$ which are continuous on the right and piecewise constant in the sense that, for some $t_0, t_1, t_n \in [0, 1]$ with $0 = t_0 < t_1 < < t_n = 1, f|[t_{i-1}, t_i[$ is constant for $1 \leq i \leq n$.

For $f, g \in B(X)$ let $\langle f, g\rangle : [0, 1[\to X \times X$. $\langle f, g\rangle(t) := (f(t), g(t))$. Let λ be the Lebesgue measure on $[0, 1[$. For any open $V \subset X \times X$ and $f, g \in B(X)$ the set $\langle f, g\rangle^{-1}(V)$ is clearly a Borel set. Thus we can define, for any open $V \in \mathcal{V}$ and any $\epsilon > 0$,

$$N(V, \epsilon) := \{(f, g) \in B(X) \times B(X) : \lambda(\langle f, g\rangle^{-1}(V)) > 1 - \epsilon\}.$$

We will show that the set $C := \{N(V, \epsilon) : V \in \mathcal{V} \text{ open}, \epsilon > 0\}$ is a basis of a uniformity $\mathcal{W}$ on $B(X)$. such that $(B(X), \mathcal{W})$ is bounded and the canonical embedding $\phi : X \to B(X)$. defined by $\phi(x)(t) := x$ for $0 \leq t < 1$, is a uniform isomorphism onto its image.

For $U, V \in \mathcal{V}$ open and $\epsilon, \delta > 0$ we have $N(U \cap V, \min(\epsilon, \delta)) \subset N(U, \epsilon) \cap N(V, \delta)$. Therefore C is a filter basis.

Now let $V \in \mathcal{V}$ be open and $\epsilon > 0$. For any $f \in B(X)$ we have $\langle f, f\rangle(t) = (f(t), f(t)) \in V$, for all $t \in [0, 1[$. Therefore $\lambda(\langle f, f\rangle^{-1}(V)) = 1$ and $(f, f) \in N(V, \epsilon)$.

Now let $f_1, f_2, f_3 \in B(X)$ with $(f_1, f_2) \in N(V, \epsilon) \ni (f_2, f_3)$. We have

$$\langle f_1, f_3\rangle^{-1}(V^2) \supset \langle f_1, f_2\rangle^{-1}(V) \cap \langle f_2, f_3\rangle^{-1}(V),$$

whence

$$\begin{aligned} \lambda(\langle f_1, f_2\rangle^{-1}(V^2)) &\geq \lambda(\langle f_1, f_2\rangle^{-1}(V) \cap \langle f_2, f_3\rangle^{-1}(V)) \\ &\geq \lambda(\langle f_1, f_2\rangle^{-1}(V)) + \lambda(\langle f_2, f_3\rangle^{-1}(V)) - 1 \geq 1 - 2\epsilon \end{aligned}$$

From this we obtain $N(V, \epsilon) \circ N(V, \epsilon) \subset N(V \circ V, 2\epsilon)$. Finally, $N(V, \epsilon)^{-1} = N(V^{-1}, \epsilon)$ is easily checked. Since $\mathcal{V}$ has a basis consisting of open vicinities we conclude that C is a basis of a uniform structure W on $B(X)$.

Next we show that the space $(B(X), \mathcal{W})$ is bounded. Let $U \in \mathcal{V}$ be open and $\epsilon > 0$. Take any $g \in B(X)$ and choose $n \in \mathbb{N}'$ with $1/n < \epsilon$. For any $f \in B(X)$ define $f_0, , f_n \in B(X)$ as follows: $f_0 := g$, and for i between 1 and n let

$$f_i(t) := \begin{cases} f(t) & : \quad 0 \leq t \leq i/n \\ g(t) & : \quad \text{otherwise} \end{cases}$$

For $0 \leq i < n$ the maps f_i and f_{i+1} agree on the set $[0, 1[\setminus[i/n, (i+1)/n[$, whence $\lambda(\langle f_i, f_{i+1}\rangle^{-1}(U)) \geq 1 - 1/n > 1 - \epsilon$. Since $f = f_n$, we have $f \in N(U, \epsilon)^n[g]$, and since g and n were chosen indepently of f, this implies $X = N(U, \epsilon)^n[g]$.

ϕ is uniformly isomorphic onto its image due to the equetion

$$N(U. \text{Ei} \cap (\phi(X) \times \phi(X)) = (\phi \times \phi)(U).$$

Remarks 2.7. (Using the notation of the preceding proof:)

(0) If X has at least two points, B(X) is dense in itself.

(1) If X is Hausdorff, $B(X)$ is easily seen to be Hausdorff as well. Moreover, in this case $\phi(X)$ is closed in B(X): Let $f \in B(X) \setminus \phi(X)$. We exhibit a neighborhood of f which is disjoint to $\phi(X)$. There are $r, s \in [0, 1[$ and a symmetric $V \in V$ such that $(f(r), f(s)) \notin V^2$. $\epsilon := \min\{\lambda(f^{-1}(f(\{t\}))) : t \in [0, 1[\}$ is positive. We show that $N(V, \epsilon) \cap \phi(X) = \emptyset$. Let $g \in \phi(X)$ and puti := $g(0)$. Then $(x, f(r)) \notin V$ or $(x, f(s)) \notin v$, say $(x, f(s)) \notin v$. SO $\langle f, g\rangle^{-1}(X \times X \setminus V) \supset f^{-1}(f(s))$, whence $\lambda(\langle f, g\rangle^{-1}(X \times X \setminus V)) \geq \epsilon$ and $g \notin N(V, \epsilon)$.

(2) $B(X)$ is pseudometrizable iff X is: If $B(X)$ is pseudometrizable, then also its "subspace" X. On the other hand, X is pseudometrizable iff U has a countable base $\mathcal{B}$, in which case $\{N(U, 1/n) : U \in \mathcal{B}, \setminus \in \mathbb{N}\}$ is a countable base of V.

(3) $B(X)$ is pathwise connected and locally pathwise connected: For f, $g \in B(X)$ and $s \in [0,1]$ let f_s : $[0, 1[\rightarrow$ x,

$$f_s(t) := \begin{cases} f(t) & : \quad 0 \leq t < s \\ g(t) & : \quad \text{otherwise} \end{cases}$$

Then $f_s \in B(X)$, and $s \mapsto f_s$ is a continuous path connecting f and g, due to $\lambda(\langle f_r, f_s\rangle^{-1}(U)) \geq 1\ s + r$ for $0 \leq r < s \leq 1$. $U \in \mathcal{U}$. By a similar argument, we conclude $(f, f_s) \in N(U, \epsilon)$ from $(f, g) \in N(U, \epsilon)$, whence $B(X)$ is locally pathwise connected.

(4) A much bigger bounded uniform space containing $B(X)$ as a subspace is obtained very similarly by putting on $Y := X^{[0,1[}$ (the set of all maps f : $[0, 1[\rightarrow$ X) the uniformity $\mathcal{W}^*$ with the basis $\{N^*(V, \epsilon) : V \in \mathcal{V}, \epsilon > 0\}$ where $N^*(V, \epsilon) := \{(f, g) \in B(X)'' \times B^*(X) : \lambda^*(\langle f, g\rangle^{-1}(X \times X \setminus V)) < \epsilon\}$, λ^* denoting outer Lebesgue measure. W' may he called the uniformity of convergence in Lebesgue measure, cf. [8], p. 104, Def. 6 (where X is a Banach space and maps f, g which are equal almost everywhere are identified).

(5) If X is countable, we can construct a countable dense subspace Y with $\phi(X) \subset Y \subset B(X)$ hy admitting only the functions with rational points of discontinuity. This subspace is again

bounded in itself, by 2.2, $(i)(b)$. Using similar arguments, it is easy to show that B(X) is separable if X is.

Excluding the case of a trivial $\mathcal{V}$ we prove

Proposition 2.8. *If the uniform space $(X, \mathcal{V})$ does not carry the coarsest uniformity then $B(X)$ is neither precompact nor complete.*

Proof. For the first assertion, let $V = V^{-1} \in \mathcal{V}$, V open, be such that $V^2 \neq X \times X$, and let $F \subset B(X)$ be finite. We show that $N(V, \frac{1}{2})[F] \neq B(X)$. There are $t_0, t_1, \ldots, t_n \in [0, 1]$, $0 = t_0 < t_1 < \ldots < t_n = 1$, such that each $f \in F$ is constant on $[t_{i-1}, t_i[$, for $1 \leq i \leq n$. Choose $(a, b) \in (X \times X) \setminus V^2$. Then, for all $f \in F$ and $0 \leq t < 1$, V'(t), $a) \notin V$ or $(f(t), b) \notin V$. Putting $s_i := \frac{1}{2}(t_{i-1} + t_i)$ for $1 \leq i \leq n$ and defining $g \in$ **B(X)** by g(r) := a for $t \in \bigcup_{i=1}^{n}[t_{i-1}, s_i[$ and $g(t) := b$ for $t \in \bigcup_{i=1}^{n}[s_i, t_i[$, one obtains $\lambda(\langle f, g\rangle^{-1}(V)) \leq \frac{1}{2}$ for all $f \in$ F, hence $g \notin N(V, \frac{1}{2})[F]$.

For non-completeness choose $a, b \in X$, $(a, b) \notin \bigcap \mathcal{V}$, and a sequence $t_0 < t_1 < t_2 <$ in $[0, 1[$, and define $f_n \in B(X)$ for $n \in \mathbb{N}$ by

$$f_n(t) := \begin{cases} a & : \quad \text{for } t \in \bigcup_{k=0}^{n}[t_{2k}, t_{2k+1}[\\ b & : \quad \text{otherwise} \end{cases}$$

It is not hard to show (indirectly) that (f_n) is a Cauchy sequence without a limit in H(X). □

3. BOUNDED SETS, UNIFORMLY CONTINUOUS FUNCTIONS AND PSEUDOMETRICS

In this section we study the relations between boundedness and uniformly continuous functions and pseudometrics. We start with a metrization lemma. It differs from the standard formulation in that it allows the construction of unbounded pseudometrics, which is very useful for our purposes. This additional feature has already been realized in the case of topological groups (in [27], Theorem 6.2). Also the inclusion (c) is sharper than usual.

Theorem 3.1. (Metrization Lemma) *Ler X be a set and $(U_n)_{n\in\mathbb{Z}}$ a sequence of subsets of $X \times X$ satisfying $\bigcup_{n\in\mathbb{Z}} U_n = X \times X$ and, for all $n \in \mathbb{Z}$, $\Delta_X := \{(x, x) : x \in X\} \subset U_n$. Furthermore assume that the sequence has one of the following two properties:*

(i) $\forall n \in \mathbb{Z} : U_n^3 \subset U_{n+1}$

(ii) $\forall n, m \in \mathbb{Z} : U_n \circ U_m = U_m \circ U_n$ and $U_n^2 \subset U_{n+1}$

Then $d : X \times X \to \mathbb{R}$, defined by

$$d(x, y) := \inf\left\{\sum_{i=1}^{k} 2^{n_i} : k \in \mathbb{N}', (x, y) \in U_{n_1} \circ U_{n_2} \circ \circ U_{n_k}\right\}$$

has the following properties:

(a) $\forall x, y \in X : d(x, y) \geq 0$ and $d(x, x) = 0$

(b) $\forall x, y, z \in X : d(x, z) \leq d(x, y) + d(y, z)$

(c) $\forall n \in \mathbb{Z} : \{(x, y) \in X \times X : d(x, y) < 2^n\} \subset U_n \subset \{(x, y) \in X \times X : d(x, y) \leq 2^n\}$

If all U_n are symmetric, d is symmetric, which implies (together with (a) and (b)) that d is a pseudometric. In this case, (c) implies that d is a metric iff $\bigcap_{n\in\mathbb{Z}} U_n = \Delta_X$. If, moreover, X is a uniform space and $(U_n)_{n\in\mathbb{Z}}$ consists of symmetric vicinities, the second inclusion of (c) implies that d is a uniformly continuous pseudometric.

Proof. d is well-defined because of $X \times X = \bigcup_{n\in\mathbb{N}} U_n$, so that $d(x,y)$ is the infimum of a nonempty set of non-negative reals. Properties (n) and (6) are easily checked, as well as the symmetry of d in case the U_n are symmetric. The only critical point in the proof of the remaining assertions is the first inclusion in (c), for which we first prove the following auxiliary statement

$$(A_k) \qquad \forall n, n_1, \ldots, n_k \in \mathbb{Z} : \sum_{1\le i\le k} 2^{n_i} < 2^n \Rightarrow U_{n_1} \circ \ldots \circ U_{n_k} \subset U_n,$$

for all $k \in \mathbb{N}'$. (A_k) is proved via induction over k:
First, suppose (i) holds. (A_1) means that the sequence $(U_n)_{n\in\mathbb{Z}}$ increases witb n, which it does: $U_n \subset U_n^3 \subset U_{n+1}$ Now suppose that (A_i) holds for all $i \le k$. Let $(n_i)_{1\le i\le k+1} \in \mathbb{Z}^{k+1}$ and $a := \sum_{1\le i\le k+1} 2^{n_i}$. Let $n \in \mathbb{Z}$ with $a < 2^n$. Putting

$$j := \max\left\{h \in \mathbb{N}' : h \le k+1, \sum_{1<i<h-1} 2^{n_i} \le a/2\right\}$$

we have $\sum_{1\le i\le j-1} 2^{n_i} \le a/2 < 2^{n-1}$ and $\sum_{j+1\le i\le k+1} 2^{n_i} < a/2 < 2^{n-1}$, whcncc by induction hypothesis: $U_{n_1} \circ \circ U_{n_{j-1}} \subset U_{n-1}$ and $U_{n_{j+1}} \circ \circ U_{n_{k+1}} \subset U_{n-1}$, where the product of the empty sequence of vicinities means Δ_X. From $2^{n_j} < 2^n$ follows $n_j \le n - 1$, and thus $U_{n_j} \subset U_{n-1}$ Summarizing, we obtain

$$\left(U_{n_1} \circ \ldots \circ U_{n_{j-1}}\right) \circ U_{n_j} \circ \left(U_{n_{j+1}} \circ \ldots \circ U_{n_{k+1}}\right) \subset U_{n-1} \circ U_{n-1} \circ U_{n-1} \subset U_n$$

as desired. Now let (ii) hold:
Again, (A_1) is clear. Let (A_i) hold, for every $i \le k$. Let $(n_i)_{1<i<k+1} \in \mathbb{Z}^{k+1}$, and define a as above. Take $n \in \mathbb{Z}$ with $a < 2^n$. Since the U_n commute we can further assume $n_1 \le n_2 \le \le n_{k+1}$. In case that $n_i = n_{i+1}$ (for some i) we put $m_j := n_j$ (for $j <$ i), $m_i = n_i + 1$ and $m_j := n_{j+1}$ (for $i < j \le$ k). Obviously $\sum_{1< <k} 2^{m_j} = a$, and by (ii) we get $U_{n_i} \circ U_{n_{i+1}} \subset U_{m_i}$, whence

$$U_{n_1} \circ \circ U_{n_{k+1}} \subset U_{m_1} \circ \circ U_{m_k} \subset U_n,$$

with the last inclusion due to the induction hypothesis. In the case of $n_1 < n_2 < < n_{k+1}$ we have $\sum_{1<i<k} 2^{n_i} < 2 \cdot 2^{n_k} \le 2^{n_{k+1}}$, so that the induction hypothesis (together with $n_{k+1} + 1 \le n$) yields

$$U_{n_1} \circ \circ U_{n_k} \circ U_{n_{k+1}} \subset U_n.$$

We can now finish the proof of the first inclusion of(c): Let $(x,y) \in X \times X$ with $d(x,y) < 2^n$. By the definition of d, this means that there are $x = x_0, x_1, , x_k = y$ and $n_1, , n_k \in \mathbb{Z}$

satisfying $(x_{i-1}, x_i) \in U_{n_i}$ and $\sum_{1<i<k} 2^{n_i} < 2^n$. So we have $(x, y) \in U_{n_k} \circ \circ U_{n_1} \subset U_n$, by (A_k). ☐

Remarks 3.2. (1) If we are given a sequence $(U_n)_{n\in\mathbb{Z}}$ of equivalence relations, $U_n \subset U_{n+1}$ clearly implies (i) and the resulting pseudometric is ultrametric. On the other hand, Sor every ultrapseudometric d, the sets $\{(x,y): d(x,y) \leq c\}$ and $\{(x,y): d(x,y) < c\}$ are equivalence relations, for any $c > 0$.
(2) The pseudometric d constructed in the metrization lemma is the greatest pseudometric satisfying the second inclusion in (c), as can easily be seen.
(3) If a group G acts on X on the left and the sequence $(U_n)_{n\in\mathbb{N}}$ satisfies

$$\forall n \in \mathbb{N} \forall g \in G \forall (x, y) \in U_n : (g.x, g.y) \in U_n,$$

"U_n is G-**invariant**", then also d is invariant under the action of G. which means $d(g.x, g.y) = d(x, y)$ for all $g \in$ G and $x, y \in$ X. The same can be stated for right group action, with the obvious changes.

The following theorem was show by Hejcman ([13], Theorem 1.12 and 1.14) and also by Atkin ([1], Thcorcm 2.4). Probably the earliest reference for (h) $\Leftrightarrow$ (c) is [2], where Atsuji proved the equivalence for metric spaces (Theorem 2). His proof of the equivalence for uniform spaces can be found in [3], Theorem 7.

Theorem 3.3. *Let X be a uniform space and A ⊂ X. The following are equivalent:*
(a) A is bounded.
(b) Every uniformly continuous real function on X is bounded on A.
(c) A has finite diameter with respect to every uniformly continuous pseudometric on X.

Proof. See [13], Theorems 1.12 and 1.14. ☐

Remarks 3.4. (1) If a pseudometric d induces the uniform structure of X and d' is a uniformly continuous pseudometric such that a subset A does not have a finite d'-diameter, then $d + d'$ is a pseudometric which induces the uniformity of X, and A has no finite $d + d'$-diameter. Hence, if X is pseudometrizable, (c) can be replaced by

(c_1) A has finite diameter with respect to every pseudometric inducing the uniform structure of X.

By an analogous argument (c) can be replaced by

(c_2) A has finite diameter with respect to every uniformly continuous metric on X. if there exists a uniformly continuous metric on X, and by

(c_3) A has finite diameter with respect to every metric inducing the uniform structure of X, if X is metrizable.

(2) The previous theorem allows us to generalize proposition 2.2: If $Y \subset$ X is dense or the union of V-components (for a symmetric viciniry V), it is possible to extend every uniformly continuous function Y $\to \mathbb{R}$ to a uniformly continuous function on X. For dense subsets, this is well-known, cf. [4], Chapter II, $3.6, Section 6, Theorem 2 ($\mathbb{R}$ being complete): if Y is the union of V-components, any extension which is constant outside Y is a uniformly continuous function on X. Now we can show the more general result:
If Y is a subspace of X such that every uniformly continuous real function on Y has a uniformly continuous extension over X then Y respects boundedness. Indeed if $A \subset Y$ is

not bounded in Y there is a uniformly continuous $f : Y \to \mathbb{R}$ which is unbounded on A. f has an extension over X, so that A is not bounded in X. More generally, a slight elaboration of this argument shows: A subspace Y of X respects boundedness if each unbounded, non-negative, uniformly continuous function on Y is dominated by (the restriction to Y of) some non-negative, uniformly continuous function on X.

It should be noted that the converse does not hold, i.e. there exist b.r. subspaces $Y \subset X$ and uniformly continuous functions on Y which do not extend to X. As an example, take $\mathbb{Z} \subset \mathbb{R}$ and the function $n \mapsto n^2$.

The extendability of uniformly continuous real functions from a subspace of a uniform space over the whole space is studied in [21]. Uniform spaces X with the property that, for every $Y \subset X$, every uniformly continuous function $Y \to \mathbb{R}$ has a uniformly continuous extension over X have been studied under the name of RE spaces in [7]. The class of RE spaces includes the fine spaces and it is closed under completion and projective limits.

We use the last remark to show

Proposition 3.5. In every *uniform space, the complement of any precompact subset is a b.r. subspace.*

Proof. In view of (2) of the last remark it is enough to prove the following lemma which in case of a metrizable X is due to Levy and Rice, see the remark after Proposition 4.4. of [21]. □

Lemma 3.6. *Let $(X, \mathcal{V})$ be a uniform space und $Y \subset X$ with $X \setminus Y$ precompact. Then every uniformly continuous function $j : Y \to \mathbb{R}$ has a uniformly continuous extension over X.*

Proof. The general case is easily reduced to the case that X is Hausdorff and complete and Y is closed. The $Y \cap X \setminus Y = \mathrm{Fr}\,(X \setminus Y)$ (the boundary of $X \setminus Y$). As $X \setminus Y$ is compact the continuous restriction $f|\,\mathrm{Fr}(X \setminus Y)$ has a continuous extension g over $X \setminus Y$. Since X is the union of the two closed sets Y and $X \setminus Y$, the functions f and g define a continuous extension h off over X. We finish by showing that h is uniformly continuous. Supposing the contrary, there exist $\epsilon > 0$ and, for each $V \in \mathcal{V}$, a pair $(a_V, b_V) \in V$ such that

$$(*) \qquad |h(a_V) - h(b_V)| > \epsilon$$

Since f is uniformly continuous there exists a $V_0 \in \mathcal{V}$ such that $|f(a) - f(b)| \leq \epsilon$ for all $(a, b) \in V_0 \cap (Y \times Y)$. So $(a_V, b_V) \notin Y \times Y$ for all V from $\mathcal{V}_0 := \{V \in V : V \subset V_0\}$. We may assume that $a_V \in X \setminus Y$ for all $V \in \mathcal{V}_0$. The net $(a_V)_{V \in \mathcal{V}_0}$ has an accumulation point c in the compact set $X \setminus Y$. For every $V \in \mathcal{V}_0$ we choose $W(V) \in \mathcal{V}$ with $W(V) \subset V$ such that $(a_{W(V)}, c) \in V$ and

$$(**) \qquad |h(a_{W(V)}) - h(c)| < \frac{\epsilon}{2}.$$

As $(a_{W(V)}, b_{W(V)}) \in W(V) \subset V$ we obtain $(b_{W(V)}, c) \in V\circ V^{-1}$, hence the net $(b_{W(V)})_{V \in \mathcal{V}_0}$ converges to c. Because of the continuity of h, $h(b_{W(V)})$ converges to $h(c)$. Because of $(**)$, this implies a contradiction to $(*)$. □

Because of a remark before 2.2, Proposition 3.5 implies

Corollary **3.7.** *In every bounded uniform space the complement of any precompact subset is bounded in itself.*

We are now able to construct bounded, non-precompact compatible uniformities on locally compact, in particular on discrete spaces, see 3.9 through 3.11. The first step is the following simple observation whose proof we omit.

Lemma 3.8. *Let* κ *be an infinite cardinal and* $(e_\iota)_{\iota<\kappa}$ *an orthonormal set in a Hilbert space. The spaces* $X_\kappa := \bigcup_{\iota<\kappa}[0,1]e_\iota$ *and* $Y_\kappa := \bigcup_{\iota<\kappa}([0,1]\cap\mathbb{Q})e_\iota$ *have cardinalities* $\max(\kappa, 2^{\aleph_0})$ *and* κ, *respectively. Equipped with their usual metrizable uniformities (induced by the Hilbert space norm),* X_κ, Y_κ *and* $X_\kappa \setminus \{0\}$ *are bounded and non-precompact. The topological space* $X_\kappa \setminus \{0\}$ *is locally compact, in fact, the sum of the locally compact spaces* $]0,1]e_\iota$.

In the following theorem, for a locally compact space X, let $\mathcal{P}$ denote the coarsest precompact compatible uniformity on X. $\mathcal{P}$ is also the coarsest compatible uniformity and the initial uniformity with respect to all continuous real functions on X with compact support, see [5], Chap. IX, §1, Exercise 15.

Theorem *3.9. Let* $(X, \mathcal{T})$ *be a locally compact space and* $\mathcal{V}$ *a bounded, non-precompact uniformity on the set* X *with* $\mathcal{V}$*-topology coarser than* $\mathcal{T}$. *Then* $\mathcal{V} \vee \mathcal{P}$ *is bounded, non-precompact and compatible.*

Proof. Obviously, $\mathcal{V}\vee\mathcal{P}$ is non-precompact and compatible. Its boundedness is proved, in view of the description of $\mathcal{P}$ before the theorem and in view of 1.9, when we show: For every finite set F of continuous real functions on X with compact supports, denoting by $\mathcal{Q}$ the initial uniformity with respect to F, $\mathcal{V}\vee\mathcal{Q}$ is bounded. Let S be the compact union of the supports of the functions $f\in F$. Since X is $\mathcal{V}$-bounded, $X\setminus S$ is $\mathcal{V}$-bounded in itself, by Corollary 3.7. Since $\mathcal{Q}|(X\setminus S)$ is the coarsest uniformity, $X\setminus S$ is also bounded for $(\mathcal{V}\vee\mathcal{Q})|(X\setminus S)$ and hence $\mathcal{V}\vee\mathcal{Q}$-bounded. Since S is also $\mathcal{V}\vee\mathcal{Q}$-bounded, it follows that $(X, \mathcal{V}\vee\mathcal{Q})$ is bounded. □

Corollary 3.10. *For every infinite cardinal* κ, *the locally compact space* $X_\kappa\setminus\{0\}$ *from Lemma 3.8 admits a bounded, non-precompact uniformity.*

Proof. Apply the theorem to the locally compact space $X_\kappa\setminus\{0\}$ and the usual uniformity on it. □

Corollary 3.11. *Every infinite discrete topological space admits a bounded, non-precompact uniformity, which, for countable X may moreover be taken metrizable.*

Proof. Let $\kappa := |X|$. For the first part, apply 3.9 to the discretely topologized set Y_κ from 3.8 and the usual uniformity on it. If κ is countable note that the usual uniformity on Y_κ as well as $\mathcal{P}$ are now metrizable. □

Theorem 3.3 together with the metrization lemma 3.1 gives us the following characterization of boundedness:

Theorem 3.12. *Let X be a uniform space and* $A\subset X$. *Then A is bounded iff, for all families* $(V_n)_{n\in\mathbb{N}}$ *of symmetric vicinities satisfying* $V_n^2\subset V_{n+1}$ *(for all* $n\in\mathbb{N}$*) and* $\bigcup_{n\in\mathbb{N}} V_n = X\times X$, *there exists an* $n\in\mathbb{N}$ *with* $A\times A\subset V_n$.

Proof. By 3.3, if A is bounded, it has finite diameter with respect to the uniformly continuous pseudometric constructed from the family $(U_n)_{n\in\mathbb{Z}}$, defined by $U_n := V_{2n}$ $(n\in\mathbb{N})$ and

$(U_n)_{n<0}$ chosen inductively according to condition (i) of the metrization lemma. By (c) of 3.1, we get $A \times A \subset U_n = V_{2n}$, for some $n \in \mathbb{N}$.
If A is not bounded, 3.3 yields a uniformly continuous pseudometric d on X such that A does not have finite d-diameter. Then $(V_n)_{n\in\mathbb{N}}$, defined by $V_n := \{(x,y) \in X : d(x,y) < 2^n\}$ $(n \in \mathbb{N})$, is the desired family of vicinities. □

The metrization lemma can also be applied to the question, when boundedness can be tested by a single pseudometric. Such spaces were named "B-simple" in [15]. Theorem 1 of [15] contains the equivalences of(n) through (d) of the following theorem. Nonetheless we give a full proof since we believe that the enhanced metrization lemma allows for a more transparent proof, which is already contained in the thesis [IO] of one of the authors.

Theorem 3.13. *For a uniform space (X, V), the following are equivalent:*
(a) There exists a uniformly continuous pseudometric d on X such that $\forall A \subset X$ (A is bounded iff A has finite d-diameter).
(b) There exists a uniformly continuous real function f on X such that $\forall A \subset X$ (A is bounded iff f is bounded on A).
(c) X *is B-conservative and* σ- bounded, *i.e. X is the countable union of bounded sets.*
(d) There exists a symmetric B-conserving $V \in \mathcal{V}$ such that X has only countably many V-components.
(e) There exists a family $(U_n)_{n\in\mathbb{N}}$ of symmetric vicinities satisfying $\bigcup_{n\in\mathbb{N}} U_n = X \times X$ and $U_n^2 \subset U_{n+1}$, such that $\forall A \subset X$ (A is bounded iff $A \times A \subset U_n$ for some $n \in \mathbb{N}$).

Proof. "$(a) \Rightarrow (b)$": Choose any $y \in$ X and let $f(x) := d(x, y)$.
"$(b) \Rightarrow (a)$": Let $d(x,y) := |f(x) - f(y)|$.
"$(b) \Rightarrow (c)$": $X = \bigcup_{n\in\mathbb{N}} f^{-1}([-n, n])$ is a countable union of bounded sets, by the nssumption on f. Let $V := \{(x,y) : |f(x) - f(y)| < 1\}$. V is a (symmetric) vicinity, since f is uniformly continuous. We have $V^n \subset \{(x,y) : |f(x) - f(y)| < n\}$, whence $V^n[x]$ is bounded, by the assumption on f (for all $x \in$ X and $n \in \mathbb{N}$).
"$(c) \Rightarrow (d)$": If X is the union of countably many bounded sets, each of the bounded sets meets only finitely many V-components (by 1.12). therefore X can have only countably many V-components.
"$(d) \Rightarrow (c)$": For any $x \in$ X, the V-component of x equals $\bigcup_{n\in\mathbb{N}} V^n[x]$, hence it is σ-bounded. Then X is the countable union of σ-bounded sets, thus it is σ-bounded.
"$(c) \Rightarrow$ (el": By assumption we have $X = \bigcup_{n\in\mathbb{N}} B_n$, with B_n bounded. We can assume $B_n \subset B_{n+1}$ for all n. Let $U_0 = U_0^{-1} \in V$ be B-conserving, and for $n \geq 0$ define inductively $U_{n+1} := U_n^2 \cup (B_n \times B_n)$. Then $\bigcup_{n\in\mathbb{N}} U_n \supset \bigcup_{n\in\mathbb{N}} B_n \times B_n = X \times X$. Moreover, each U_n is a symmetric vicinity satisfying, by definition, $U_n^2 \subset U_{n+1}$ The "only if" part of the equivalence is true by 3.12, so it remains to show the "if" part. For this, we first prove

$$(*) \qquad \forall n \in \mathbb{N}, \forall m \in \mathbb{N}' : \forall x \in X : U_n^m[x] \text{ is bounded.}$$

For $n = 0$ $(*)$ holds (for all m) by the assumption on U_0. Now let $n \geq 0$ and suppose, thar $(*)$ holds for n and all $m \in \mathbb{N}'$. For $m = 1$ we have $U_{n+1}^m[x] = (U_n^2 \cup (B_n \times B_n))[x] \subset U_n^2[x] \cup B_n$, whence $U_{n+1}[x]$ is bounded by assumption on n and B_n. If $(*)$ holds for $n+1$ and $m \geq 1$, we have

$$U_{n+1}^{m+1}[x] = U_{n+1}[U_{n+1}^m[x]] \subset U_{n+1}[U_n^k[F]]$$

for some finite F and $k \in \mathbb{N}$, by the hypothesis on m. But

$$\begin{aligned} U_{n+1}[U_n^k[F]] &= (U_n^2 \cup (B_n \times B_n))[U_n^k[F]] \\ &= U_n^{k+2}[F] \cup \underbrace{(B_n \times B_n)[U_n^k[F]]}_{\subset B_n} \end{aligned}$$

is bounded by assumption on n, which ends the induction.
Now from $A \times A \subset U_n$ for some $n \in \mathbb{N}$ follows $A \subset U_n[a]$ for any $a \in A$, which implies by $(*)$ that A is bounded.
"$(e) \Rightarrow (a)$": Apply the metrization lemma to the family $(U_{2n})_{n \in \mathbb{N}}$, with a suitable choice of vicinities for negative indica. □

Remarks 3.14. (1) If the space X is uniformly locally precompact the statements in the preceding theorem are equivalent to

(f) X is σ-precompact.

Indeed, there is a vicinity V such that for all $x \in X$ and all $n \geq 1$, $V^n[x]$ is precompact (by [13], Lemma 1.17). Hence (f) implies 3.13 (c). On the other hand, since in every uniformly locally precompact space bounded sets are precompact by Proposition 1.20, 3.13 (c) clearly implies (f).

A similar result has been obtained by Potter, see [24].

(2) Every space X fulfilling the equivalent conditions of theorem 3.13 has a fundamental sequence (B_n) of bounded sets in the sense that the sets B_n are bounded und each bounded set is contained in some B_n. Simply fix $x \in X$ and let $B_n := \{y \in X : d(x, y) < n\}$, where d is the pseudometric from 3.13 (a).

(3) If $(X, \mathcal{V})$ is a uniform space and d is a pseudometric as in 3.13(a), then $\mathcal{V}$ and the uniformity $\mathcal{V}_d$ induced by d yield equal collections of bounded sets, even though they do not necessarily coincide. (Clearly, every $\mathcal{V}$-bounded set is $\mathcal{V}_d$-bounded since $\mathcal{V}_d$ is coarser. On the other hand, every $\mathcal{V}_d$-bounded subset has finite d-diameter, hence it is $\mathcal{V}$-bounded, by the assumption on d.) For a further instance, let $\mathcal{V}_1$ be the initial uniformity on X with respect to all $\mathcal{V}$-uniformly continuous real functions. Plainly $\mathcal{V}_1 \subset \mathcal{V} \wedge \mathcal{C}$, where $\mathcal{C}$ is the initial uniformity with respect to all continuous real functions on X. Clearly $\mathcal{V}_1 \neq \mathcal{C}$ iff there is a continuous, but not uniformly continuous real function on X. We do not have an example for $\mathcal{V}_1 \neq \mathcal{V} \wedge \mathcal{C}$. $\mathcal{V}_1$ induces the same topology on X as $\mathcal{V}$, by [9], 8.5.7 (a). To produce an example for $\mathcal{V}_1 \neq \mathcal{V}$, let $\mathcal{V}$ be discrete. Then $\mathcal{V}_1 = \mathcal{C}$. Further, $\mathcal{C} = \mathcal{V}$ iff $|X| \leq \aleph_0$, by [11], 15.23 (b). So $\mathcal{V}_1 = \mathcal{C} \neq \mathcal{V}$ for uncountable discrete $(X, \mathcal{V})$.

A second example for $\mathcal{V}_1 \neq \mathcal{V}$ is any bounded but not precompact space $(X, \mathcal{V})$ since in this situation every $\mathcal{V}$-uniformly continuous $f : X \longrightarrow \mathbb{R}$ is bounded so that $\mathcal{V}_1$ is precompact, hence different from $\mathcal{V}$. In order to give a sufficient condition for $\mathcal{V}_1 = \mathcal{V} \wedge \mathcal{C}$, denote by $\mathcal{V}_0$ the initial uniformity on X with respect to all bounded $\mathcal{V}$-uniformly continuous real functions. $\mathcal{V}_0$ is the finest precompact uniformity on X that is coarser than $\mathcal{V}$, see [9], 8.5.7 for the case that $\mathcal{V}$ is Hausdorff. We have $\mathcal{V}_0 \subset \mathcal{V}_1 \subset \mathcal{V} \wedge \mathcal{C}$ generally, so if $\mathcal{V} \wedge \mathcal{C}$ is precompact then $\mathcal{V} \wedge \mathcal{C} \subset \mathcal{V}_0$, hence we obtain equality.

For a further discussion of Theorem 3.13 in the context of topological vector spaces, the reader is referred to 4.5

4. BOUNDEDNESS IN TOPOLOGICAL VECTOR SPACES

In the context of topological vector spaces the concept of boundedness (defined somewhat differently, see below) plays an important role. Before we proceed to general topological groups, we take a closer look at what some of the results of the first three sections mean in the context of topological vector spaces.

We recall some of the basic definitions and facts. Let $(X, \mathcal{T})$ denote a real topological vector space (TVS, for short), which for simplicity will always be assumed to be Hausdorff. "The" uniformity $\mathcal{V}$ of $(X, \mathcal{T})$ is defined by the basis of vicinities

$$N_U := \{(x, y) : x - y \in U\}$$

where U ranges over a neighborhood base at zero. $\mathcal{V}$ induces $\mathcal{T}$. In the following, uniform concepts in $(X, \mathcal{T})$ like boundedness or precompactness, will refer to $\mathcal{V}$. The topology of a TVS is metrizable iff its uniformity is metrizable. In the theory of TVS, $\mathcal{T}$**-boundedness** is defined as follows: $A \subset X$ is $\mathcal{T}$-bounded iff for every neighborhood U of 0 there exists $\rho > 0$ such that $A \subset \rho U$. Clearly, every $\mathcal{T}$-bounded subset is bounded. For locally convex TVS the converse was show by Atkin:

Lemma 4.1. *In every locally convex space the bounded sets coincide with the $\mathcal{T}$-bounded sets.*

Proof. See [1], 1.7. □

However, for general TVS this is not true: There exist non-trivial bounded TVS (see 4.7 below), but a $\mathcal{T}$-bounded TVS is easily seen to be trivial.

In the beginning of our discussion we will deal mainly with locally convex spaces. There is the following positive result about the behavior of boundedness with respect to initial uniformities.

Proposition 4.2. *Let $f_i : X \to (X_i, \mathcal{T}_i)$, $i \in I$, be a family of linear maps of a vector space X (over $\mathbb{R}$ or $\mathbb{C}$) into locally convex spaces $(X_i, \mathcal{T}_i)$. Let $\mathcal{V}_i$ be the uniformity of $(X_i, \mathcal{T}_i)$, $i \in I$, and let $\mathcal{V}$ be the initial uniformity on X with respect to the maps $f_i : X \to (X_i, \mathcal{V}_i)$. Then a set $A \subset X$ is $\mathcal{V}$-bounded iff $f_i(A)$ is $\mathcal{V}_i$-bounded for each $i \in I$.*

Proof. One verifies that $\mathcal{V}$ is the uniformity for the (locally convex) initial topology $\mathcal{T}$ on X with respect to the maps $f_i : X \to (X_i, \mathcal{T}_i)$. For the non-trivial direction of the proposition consider $A \subset X$ such that $f_i(A)$ is $\mathcal{V}_i$-bounded, for each $i \in I$, i.e., $\mathcal{T}_i$-bounded. This implies that A is $\mathcal{T}$-bounded, see [17], 2.4.3. Hence A is $\mathcal{V}$-bounded, by 4.1.

The next result concerns the b.r. subsets. It is a slight generalization of [1] (1.8). The proof given there works also for this case.

Proposition 4.3. *Let $(X, \mathcal{T})$ be a locally convex TVS. If $Y \subset X$ is convex (or only star-like in the sense that, for some $c \in Y$ one has $c + \lambda(Y - c) \subset Y$ for $0 \le \lambda \le 1$), then it respects boundedness. In particular Y is bounded iff it is bounded in itself.*

We now turn to the relation between boundedness and uniformly continuous real functions (resp. pseudometrics), as it was exhibited in the Theorems 3.3 and 3.13. As might be hoped, in the context of locally convex TVS one needs only consider continuous linear functionals and seminorms:

Theorem 4.4. *Let* $(X, 7)$ *be a locally convex TVS. For a subset A the following are equivalent:*
(a) A is bounded.
(b) Every continuous seminorm on X is bounded on A.
(c) For some set S **of** *seminorms which induces 7, every* $p \in S$ *is bounded on A.*
(d) Every continuous linear functional on X is bounded on A.

Proof. $(a) \Rightarrow (b)$, since every continuous seminorm is uniformly continuous for the uniformity of X. $(b) \Rightarrow (c)$ is clear. For $(c) \Rightarrow (d)$, we may assume the family of seminorms to be directed (including sums does not affect condition (c)), hence a linear functional is continuous iff it is continuous w.r.t. some p_i, whence (d) follows. For $(d) \Rightarrow (a)$ see [19], §20, 11.(1). □

As a consequence we see that a subset is bounded iff its closed convex hull is bounded in itself. (This is already clear from the well-known fact that the $\mathcal{T}$-bounded sets have the analogous property and that they coincide with the bounded sets.) Note also that two different locally convex topologia on a vector space X have the same bounded sets, if their topologica1 duals coincide. In particular, if $A \subset$ X is bounded, it is S-bounded for the weak topology S of X. In $(X, \mathcal{S})$ every bounded set is precompact, see [19], §20, 9.3. Therefore, by 2.4, $(d) \Leftrightarrow (a)$, every subspace of $(X, \mathcal{S})$ respects boundedness, cf. 4.3. Further, 2.4, (d) is satisfied in every Schwartz space as well as in every semi-Monte1 space, where every bounded subset is even relatively compact by definition, see [17], 10.4.3 and 11.5.

For locally convex TVS, an analogue of Theorem 3.13 can be formulated as follows:

Theorem *4.5. Let (X, 7) De a locally convex TVS. The following are equivalent:*
(a) 3.13 *(a) holds.*
(b) 3.13 *(a) holds,* wirh the *pseudometric arising from a norm.*
(c) There exists a bounded neighborhood U of 0.
(d) 7 is normable.
(e) 7 is metrizable and X has a fundamental sequence **of** *bounded sets.*

Proof. $(d) \Rightarrow (b)$ was stated in 1.2. $(b) \Rightarrow (a) \Rightarrow (c)$ is obvious. $(c) \Leftrightarrow (d)$ is due to Kolmogoroff, see [19], §15, 10.(4). $(d) \Rightarrow (e)$ is again obvious, whereas (e) $\Rightarrow$ *(d)* follows from [19], $29. 1.(2). □

Remark 4.6. The equivalence of (a) and (c) holds in every TVS as is clear from 3.13, $(a) \Leftrightarrow (d)$, see Hejcman [15], Corollary 2 of Theorem 3.

The condition of metrizability in 4.5 (e) cannot be dropped since there exist non-metrirable locally convex TVS with fundamental sequences of bounded sets:

(a) Let Xbe a TVS of countably infinite dimension, carrying the finest locally convex topology. X has a fundamental sequence of bounded sets since its bounded sets are finite dimensiona1 and relatively compact by [19], §18, 5.(6). X is not metrizable by 4.5 and because it is not normable. This example is a special case of [19], §29, 1.(8).

(b) Consider a normed infinite dimensiona1 vector space $(X, \| \|)$. Its weak topology S is locally convex and strictly coarser than the norm topology $\mathcal{R}$ since every S-neighborhood of 0 contains an infinite-dimensional subspace. Let $\mathcal{T}$ be any locally convex topology such that S $\subset \mathcal{T} \subsetneq \mathcal{R}$. Then $\mathcal{T}$ is not metrizable since otherwise $\mathcal{R}$ and $\mathcal{T}$ would coincide with the Mackey topology, see [19], §21, 5.(3). Since the duals of all topologia involved coincide, and since there exists a fundamental sequence of bounded sets for $\mathcal{R}$, we have found a fundamental sequence of bounded sets for the nonmetrizable topology $\mathcal{T}$. This shows again that the metrizability conditi”” ın 4.5(e) cannot be dropped. – Also the strong duals ot metrizable, non-normable locally convex spaces are non-metrizable locally convex spaces with a fundamental sequence of bounded sets, see [19], $29, 1.(S).

We turn now to non-locally convex TVS and some pathologies occurring in connection with them.

Theorem 4.7. *Every TVS $(X, \mathcal{T})$ has a topologically isomorphic embedding into a bounded TVS $B(X)$.*

Proof. We embed the uniform space $(X, \mathcal{V})$ int” the bounded uniform space $(B(X), \mathcal{W})$ as in the proof of 2.6 and observe that $B(X)$ has a natural vector space structure. The proof that the $\mathcal{W}$-topology is a vector space topology can be found in [22], § 1. Note that B(X) is a linear subspace of the vector space considered there, and that the bigger space is easily shown t” be bounded as well. Clearly $\mathcal{W}$ is the uniform structure belonging to the $\mathcal{W}$-topology. □

Theorem 4.1 will be used for the construction of Example 6.3 l.

Remarks 4.8. (1) This example shows that 4.3 does not hold for non-locally convex spaces: If we embed un unbounded TVS X int” $B(X)$, X is also bounded in $B(X)$ but not in itself (observing that the restriction of the uniformity of $B(X)$ to X is the same as the uniformity on X).

(2) The construction of $B(X)$ has been used in [23] for the construction of dual-less vector spaces. Indeed every bounded vector space has trivial dual, since nontrivial continuous linear functionals are uniformly continuous and unbounded. However not every dual-less vectoı space is bounded, as the example of the spaces $l^p_{\mathbb{N}}$, for $0 < p < 1$ shows: By [19], §15, 9.(9) they have trivial dual. However, the quasinorm $\| \|_p$ associated t” $l^p_{\mathbb{N}}$ is uniformly continuous and unbounded. This also serves as a counterexample for 4.4 for non-locally convex TVS.

We end this section with an example showing that the supremum of finitely many bounded vector space topologies need not be bounded. It is due t” Peck and Porta, see [23], Theorem 8.2).

Theorem 4.9. *Let E be a vector space of dimension $\geq 2^{\aleph_0}$. The finest vector space topology on E can be obtained as the supremum of at most the topologically isomorphic and bounded vector space topologies.*

Proof. [23], Theorem 8.2) is stated for dual-less instead of uniformly bounded TVS, but the proof uses the construction of $B(X)$ and thus yields in fact bounded uniformities. □

A closer inspection of the proof of [23], Theorem B.2) shows that of the three bounded topologie? τ_1, τ_2, τ_3 constructed there, already $\tau_1 \vee \tau_3$ is not bounded.

Quite different properties of bounded sets in topological vector spaces are investigated in a recent paper [6] of Burke and Todorcevic.

5. BOUNDEDNESS IN TOPOLOGICAL CROUPS

In this section we consider the natural uniformities on topological groups and discuss boundedness with respect to these uniformities. We let $\mathcal{L}$ denote the left uniformity and $\mathcal{R}$ denote the right uniformity, $\mathcal{L} \wedge \mathcal{R}$ and $\mathcal{L} \vee \mathcal{R}$ are called the lower respectively upper uniformity; e denotes generally the neutral element of a group, and $\mathcal{U}_e$ the filter of neighborhoods of e.

Recall that the basic elements of the standard bases of $\mathcal{L}, \mathcal{R}$, L $\vee \mathcal{R}, \mathcal{L} \wedge \mathcal{R}$ are, respectively, given by

$$U_{\mathcal{L}} := \{(x,y) : y \in xU\}, \qquad U_{\mathcal{R}} := \{(x,y) : y \in Ux\}$$
$$U_{\mathcal{L}\vee\mathcal{R}} := \{(x,y) : y \in xU \cap Ux\}, \qquad U_{\mathcal{L}\wedge\mathcal{R}} := \{(x,y) : y \in UxU\},$$

with U running through $\mathcal{U}_e$. For a survey of these uniformities the reader is referred to [25], Chapter 2.

Remark and Definition 5.1. *For a topological group X and a subset A we have the following equivalences:*

$$A\ \mathcal{L}\text{-bounded} \Leftrightarrow \forall U \in \mathcal{U}_e \exists n \in \mathbb{N} \exists \text{ finite } F \subset A : A \subset FU^n$$
$$A\ \mathcal{R}\text{-bounded} \Leftrightarrow \forall U \in \mathcal{U}_e \exists n \in \mathbb{N} \exists \text{ finite } F \subset A : A \subset U^n F$$
$$A\ \mathcal{L} \wedge \mathcal{R}\text{-bounded} \Leftrightarrow \forall U \in \mathcal{U}_e \exists n \in \mathbb{N} \exists \text{ finite } F \subset A : A \subset U^n F U^n$$

A is $\mathcal{L}$-bounded iff A is $\mathcal{R}$-bounded, thus for symmetric subsets $\mathcal{L}$- and $\mathcal{R}$-boundedness are the same. A is $\mathcal{L} \wedge \mathcal{R}$- respectively $\mathcal{L} \vee \mathcal{R}$-bounded iff A^{-1} is.
We call A bibounded, *if it is both $\mathcal{L}$- and $\mathcal{R}$-bounded. Bibounded topological groups are simply called* bounded.

Remarks 5.2. (1) If X is a hounded topologica1 group, every open subgroup has finite index.
(2) Plainly every $\mathcal{L} \vee \mathcal{R}$-bounded set is hihounded, but we do not know whether the converse holds. We do not even know a hounded topologica1 group that is not $L \vee \mathcal{R}$-bounded. In contrast, $\mathcal{L} \vee \mathcal{R}$-precompactness is equivalent to $\mathcal{L}$- and $\mathcal{R}$-precompactness, because of 1.7.
(3) Every $\mathcal{L}$- (or $\mathcal{R}$-) bounded open submonoid of a topological group is $\mathcal{L}$- (or $\mathcal{R}$-) bounded in itself. This is clear from the definitions and Remark 1.1

Example 5.3. Let *The* an infinite set and X the group of bijections $T \to T$, endowed with the topology of pointwise convergence w.r.t. the discrete topology on T. Then X is a topologica1 group. It has a neighhorhood base at unity consisting of suhgroups, hence each of the five kinds of houndedness in X coincide with the corresponding kind of precompactness. It is $\mathcal{L} \wedge \mathcal{R}$-precompact (hy [25], example 9.14), but not $\mathcal{L}$-precompact, since the open subgroup consisting of all the elements leaving a given $t \in T$ fixed has no finite index in X. Also, X has an infinite $\mathcal{L}$-uniformly discrete suhgroup, namely the cyclic groups of all shifts on an injective sequence $(t_n)_{n\in\mathbb{N}}$. Furthermore there exists an $\mathcal{L}$-precompact subset of X which is not $\mathcal{R}$-precompact and thus not $\mathcal{R}$-bounded and so not hihounded (Cf. [25], Exercise 8.1).

Remark 5.4. For every $U = U^{-1} \in \mathcal{U}_e$ the corresponding basic vicinities V with respect to $\mathcal{L}, \mathcal{R}, L$ AR yield, for a point $x \in$ X, the V-components $x\langle U\rangle, \langle U\rangle x, \langle U\rangle x\langle U\rangle$, respectively, as one verifies easily. Hence the pseudocomponents of e for these uniformities are all equal to the intersection of all open subgroups of X, which is a closed norma1 subgroup *P;* and the pseudocomponents of the elements of X are the cosets of P. The pseudocomponent of e is open iff it is the smallest open subgroup of X. We doubt that any of this extends to $\mathcal{L} \vee \mathcal{R}$.

Remark 5.5. On the analogy of Section 2 we can raise the question, for which subgroups H of a topological group and subsets $A \subset H$, A bounded in some sense in X, *A is* already

bounded in H in the same sense. (Note that the converse is always true.) For U-boundedness ($\mathcal{U} \in \{\mathcal{C}, \mathcal{R}, \mathcal{L} \wedge \mathcal{R}, \mathcal{L} \vee \mathcal{R}\}$), there is an ambiguity conceming "boundedness in H": We can consider the relative uniformity $\mathcal{U}|H$ or the corresponding canonica1 uniformity $\mathcal{U}_H$ of the topologica1 group H. If $\mathcal{U} = \mathcal{C} \wedge \mathcal{R}$, these uniformities may differ, even for closed norma1 subgroups (Cf. [25], 3.25). However, if we restrict ourselves to open or dense subgroups, then the uniformities coincide, by [25], 3.24, which enables us to transfer 2.2 to topologica1 groups: Let $\mathcal{U}$ be any of the canonica1 uniformities. If H is an open subgroup, it is a V-component, V being the vicinity associated to the e-neighborhood H (cf. [25], Chapter 2, formula (2), (3), (10), (14)). Hence every subset of H that is bounded in X is also bounded in H, by 2.2. If H is dense, this implication is immediate from 2.2.
Generally the implication is false, as Theorem 5.6 below shows. For a further example let X be a Hilbert space (more precisely, its additive group) and $A \subset X$ an infinite orthonormal set. In X all canonica1 uniformities coincide with the uniformity induced by the metric on X, whence A is bounded in X. The subgroup H generated by A is discrete, whence A is not bounded in H.

The following is the analogue to 2.6 for topologica1 groups. The construction is due to Hartman and Mycielski ([12]), and, independently, to S. Dierolf (oral communication from the seventies).

Theorem 5.6. *Every topological group has a topologically isomorphic embedding into an $\mathcal{L} \vee \mathcal{R}$-bounded topological group.*

Proof. Let X be a topologica1 group and let, as in the case of uniform spaces, B(X) be the set of all maps $f : [0,1[\to X$ which are continuous on the right and piecewise constant. B(X) carries the canonica1 pointwise group structure. For $\epsilon > 0$ and $U \subset X$ an open neighborhood of e, we define

$$N(U, \epsilon) := \{f \in B(X) : \lambda(f^{-1}(U)) > 1 \ \epsilon\},$$

λ being the Lebesgue measure on $[0,1[$. Then the set $C := \{N(U,\epsilon) : U \in \mathcal{U}_e \text{ open}, \epsilon > 0\}$ is a neighborhood base of a group topology on X. Due to [25], Proposition 1.21, the proof amounts to verifying the following four statements:
(a) C is a filter base.
(b) $\forall P \in C \exists Q \in C : Q^2 \subset P$.
(c) $\forall P \in C \exists Q \in C : Q^{-1} \subset P$.
(d) $\forall P \in C \forall g \in B(X) \exists Q \in C : gQg^{-1} \subset P$.
(a) through (c) can be shown in quite the same way as in the proof of 2.6. For the proof of (d), let $U \in \mathcal{U}_e$ and $\epsilon > 0$. For $g \in B(X)$ choose $W \in \mathcal{U}_e$ such that, for all $t \in [0, 1[$, $g(t)Wg(t)^{-1} \subset U$. This is possible due to the finiteness of $g([0,1[)$. Then for any $f \in N(W, \epsilon)$ we have $f^{-1}(W) \subset (gfg^{-1})^{-1}(U)$, which implies $gfg^{-1} \in N(U, \epsilon)$.
For the proof of the $\mathcal{L} \vee \mathcal{R}$-boundedness of B(X), let $U \in \mathcal{U}_e$ be open and $\epsilon > 0$. Choose $n \in \mathbb{N}$ with $1/n < \epsilon$. It suffices to show that $B(X) \subset V^n[e]$, with e the neutral element of B(X) and V the $\mathcal{L} \vee \mathcal{R}$-vicinity defined by $V := \{(f, g) : g$ ● fl(U, ε) $\cap N(U, \epsilon) f\}$. This can be done in exactly the same way as in the proof of 2.6. For any $f \in B(X)$ define $f_0(t) := e$ ($t \in [0, 1[$), for $i = 1, \ldots, n$ let

$$f_i(t) := \begin{cases} f(t) & : \ 0 \leq t \leq i/n \\ g(t) & : \ \text{otherwise} \end{cases}$$

Then $f_i \in B(X)$, and f_{i-1} and f_i agree on $[0, 1[\setminus[(i-1)/n, i/n[$, whence we obtain $\lambda((f_{i-1} f_i^{-1})^{-1}(U)) \geq 1 - 1/n > 1 - \epsilon$ and $\lambda((f_i^{-1} f_{i-1})^{-1}(U)) > 1 - \epsilon$. But this implies $(f_{i-1}, f_i) \in V$, and thus $f = f_n \in V^n[e]$.
X embeds in B(X) via the map ϕ sending $x \in X$ to the constant function $t \mapsto x$. ϕ is a topologica1 embedding because of the equation

$$N(U, \epsilon) \cap \phi(X) = \phi(U)$$

□

Recall that a topologica1 group X is called an SIN-group (resp. an ASIN-group) if $\mathcal{L}$ and $\mathcal{R}$ agree (resp. agree on some neighborhood of e), cf. [25].

Remarks 5.7. (1) Remarks 2.1 concerning metrizability, connectedness and separability of X are easily transferred to the case of topologica1 groups.
(2) lt is easy to check that the left uniformity of the topologica1 group $B(X)$ is equal to the uniformity obtained by embedding $(X, \mathcal{L})$ as in 2.6; and similarly for $\mathcal{R}$ and CV $\mathcal{R}$. For C $\wedge \mathcal{R}$ this is also true, although the proof is somewhat more complicated and is omitted.
(3) It is easy to check that $\phi(X)$ is a normal subgroup of B(X) iff X is abelian and that B(X) is an SIN-group iff X is SIN.

As a last property of the group constructed in 5.6, we note

Proposition 5.8. *If the topological group X does not carry the coarsest topology, $B(X)$ is not $\mathcal{L} \wedge \mathcal{R}$-precompact.*

Proof. Let $U = U^{-1} \in \mathcal{U}_e$ with $U^4 \neq X$. Let $F \subset B(X)$ finite. We will show that $N(U, \frac{1}{4})FN(U, \frac{1}{4}) \neq B(X)$. There exist $0 = t_0 < t_1 < < t_n = 1$ such that each $f \in F$ is constant on $[t_{i-1}, t_i[$ for $1 \leq i \leq n$. Choosen $\in X \setminus U^4$ and put $s_i := \frac{1}{2}(t_{i-1} + t_i)$ for $1 \leq i \leq n$. Then $U^2 \cap UaU = \emptyset$, whence, for each $f \in F$, $e \notin Uf(t)U$ or $a \notin Uf(t)U$ for $0 \leq t < 1$. Defining $g \in B(X)$ by $g(r) := e$ for $t \in \bigcup_{i=1}^n [t_{i-1}, s_i[$ and $g(t) := a$ for $t \in \bigcup_{i=1}^n [s_i, t_i[$, it follows that

$$(*) \qquad \lambda(\{t : g(t) \in Uf(t)U\}) \leq \frac{1}{2}.$$

We finish by showing that $g \notin N(U, \frac{1}{4})FN(U, \frac{1}{4})$. Otherwise $g = h_1 f h_2$ with $h_1, h_2 \in N(U, \frac{1}{4})$ and $f \in F$. Then $\lambda(h_i^{-1}(U)) > \frac{3}{4}$ for $i = 1, 2$, whence $\lambda(h_1^{-1}(U) \cap h_2^{-1}(U)) > \frac{1}{2}$. But for $t \in h_1^{-1}(U) \cap h_2^{-1}(U)$ one has $g(t) \in Uf(t)U$. This contradicts $(*)$. □

In contrast to C $\vee \mathcal{R}$-precompactness, $\mathcal{L} \vee \mathcal{R}$-boundedness of a group does not imply the SIN-property, as the following examples show.

Examples 5.9. on $\mathcal{L}$ V $\mathcal{R}$-boundedness and SIN.
(a) The topologica1 group B(X) from 5.6, although $\mathcal{L} \vee \mathcal{R}$-bounded, is not SIN for any non-SIN-group X, since SIN is inherited by subgroups.
(b) Another C V $\mathcal{R}$-bounded group that is not SIN is the group X of all orientation-preserving self-homeomorphisms of the compact interval $[0, 1]$, X provided with the topology of uniform convergence. By Atkin [1], (6.8) (b), X is bounded and non-SIN. We show now that it is $\mathcal{L}$ V $\mathcal{R}$-bounded. The sets $U_\epsilon := \{x \in X : \max\{|x(t) - t| : 0 \leq t \leq 1\} \leq \epsilon\}$ with $\epsilon > 0$ form a basis of $\mathcal{U}_1(\mathcal{X})$, and the sets $V_\epsilon := \{(x, y) \in X \times X : y \in xU_\epsilon \cap U_\epsilon x\}$ with $\epsilon > 0$ form

a basis of $\mathcal{L} \vee \mathcal{R}$. The $\mathcal{L} \vee \mathcal{R}$-boundedness results from $(V_{1/n})^n[e] = X$ for all $n \in \mathbb{N}'$, the latter following obviously from

$$(*) \qquad \forall \epsilon, \delta > 0 : V_\delta[U_\epsilon] 3 \ U_{\epsilon+\delta},$$

$V_\epsilon[e] = U_\epsilon$, and $U_1 = X$. To prove $(*)$, let $y \in U_{\epsilon+\delta}$. It is easy to see that an $x \in U_\epsilon$ is defined by $x(t) := y(t)$ if $|y(t) - t| \leq \epsilon$, .x(f) $:= t + \epsilon$ if $y(t) \geq t + \epsilon$, and $x(t) := t - \epsilon$ if $y(t) \leq t - \epsilon$, and that $\max\{|y(t) - x(t)| : 0 \leq t \leq 1\} \leq 6$, hence $y \circ x^{-1} \in U_\delta$. Since the graphs of x^{-1} and y^{-1} arise from those of x and y by reflection about the diagonal of $X \times X$, also $\max\{ |y^{-1}(t) - x^{-1}(t)| : 0 \leq t \leq 1\} \leq 6$, which means that $x^{-1} \circ y \in U_\delta$. So $y \in xU_\delta$ il $U_\delta x \subset V_\delta[U_\epsilon]$. This proves $(*)$. □

In this example, X is not a Banach Lie group, since every bounded Banach Lie group is SIN by [1], (6.8) (b); for instance the unitary group of l^2 is such a group by loc. cit.. (3.5).

6. INFRABOUNDEDNESS

In this section we introduce yet another rather weak notion of boundedness. It is not derived from a uniform structure. We then compare the different notions.

Definition 6.1. *Let* **X De** *a topological group. We call a subset $A \subset$* **X infrabounded** *(resp.* strongly **infrabounded)** *if for all $U \in \mathcal{U}_e$ there exist $n \in \mathbb{N}$ and finite $F \subset$* **X** *(resp. $F \subset A$) with $A \subset$* **(FU)>'.** *A subgroup* **Y** *of X* **respects infraboundedness** *if a subset of* **Y** *is infrabounded in Y if (and only if) it is infrabounded in X (confer 2.1).*

We do not know whether there is always a uniform structure on X such that the bounded sets with respect to the uniformity are exactly the infrabounded subsets. Plainly, Y < X is strongly infrabounded. if it is infrabounded in itself. We shall not discuss strong infraboundedness systematically, it will appeal- in 6.2. 6.4, 6.14, 6.15, 6.16, 6.19 and 6.31.

Proposition 6.2. *Let X be a topological group in which every infrabounded subset is strongly infrabounded. Then for all $A \subset X$: A is infrabounded iff every countable uniformly $\mathcal{L} \wedge \mathcal{R}$-discrete subset of A is infrabounded.*

Proof. The necessity is obvious. For every $A \subset X$ which is not infrabounded there exist $V \in \mathcal{U}_e$ and an inductively constructed set $B = \{a_0, a_1, \}$ satisfying $a_0 = e$ and

$$(*) \qquad a_n \in A \quad \setminus \quad (\{a_0, , a_{n-1}\}V)^{2n} \quad (n \geq 1)$$

We have in particular $a_n \notin V^n\{a_0, , a_{n-1}\}V^n$ (note $a_0 = e$), hence the $\mathcal{L} \wedge \mathcal{R}$-discreteness of the sequence. From the construction and the assumption on X it is clear that the set $B \setminus \{a_0\}$ is a countable subset of A which is not infrabounded. □

The following proposition states the obvious implications between the various notions of boundedness

Proposition 6.3. *Foro topological group X and $A \subset X$ the following implications hold : A CV $\mathcal{R}$-bounded $\Rightarrow$ A bibounded $\Rightarrow$ A $\mathcal{L}$-bounded (or $\mathcal{R}$-bounded) $\Rightarrow$ A LA $\mathcal{R}$-bounded $\Rightarrow$ A strongly infrabounded $\Rightarrow$ A infrabounded.*

If X is an SIN-group (i.e. has a neighborhood base of conjugation-invariant subsets) then every infrabounded set is $\mathcal{L} \vee \mathcal{R}$-bounded, i.e. all notions of boundedness are identical. If X is locally compact, all notions are equivalent to relative compactness.

Proof. The chain of implications is obvious from the definitions and from the characterization in 5.1. If X is an SIN-group, all the uniformities coincide (see [25], Proposition 2.17) and the same holds for the related definitions of boundedness. Furthermore, if $U \in \mathcal{U}_e$ is a conjugation-invariant neighborhood, then for every infrabounded A we have $A \subset (FU)^n = F^n U^n$, with $n \in \mathbb{N}$ and F finite, which implies the $\mathcal{L}$-boundedness of A. But then A is also $\mathcal{L} \vee \mathcal{R}$-bounded. If $U \in \mathcal{U}_e$ is compact and $A \subset X$ is infrabounded, then $A \subset (FU)^n$ for some finite F, and the product is compact. □

As was shown in Example 5.3, the second and third implications in 6.3 generally cannot be reversed; whereas we know nothing about the first implication. We show now that a (strongly) infrabounded group need not be $\mathcal{L} \wedge \mathcal{R}$-bounded. The example has been communicated to us by V.V. Uspenskij; we present it in a slightly sharpened form.

Example 6.4. There is a topological group X with a $\mathcal{U}_e$-base consisting of subgroups, which has the following two properties :

$$(i) \qquad \forall V \in \mathcal{U}_e \exists f \in X : X = VfVfV$$

$$(ii) \qquad \exists V \in \mathcal{U}_e \forall F \subset X \text{ finite } : X \neq FVFVF$$

By (i), X is (strongly) infrabounded, by (ii) it is not $\mathcal{L} \wedge \mathcal{R}$-precompact and hence not $\mathcal{L} \wedge \mathcal{R}$-bounded.

The construction of X: Let (U, d) be the Urysohn universal metric space of diameter 1, as constructed in [26], VI. U satisfies:

(a) Every finite metric space Y of diameter ≤ 1 has an isometric embedding into U (cf. [26], VI).

(b) Every isometry $U_1 \to U_2$ between two finite subsets U_1 and U_2 of U extends to an isometry of U onto itself (cf. [26], VII).

Let X be the group of isometries of U onto itself, endowed with the topology of pointwise convergence with respect to the discrete topology on U. Then X has a $\mathcal{U}_e$-base consisting of subgroups and satisfies (i) and (ii), as will be shown with the aid of the following two lemmas:

Lemma 1: Let $A := \{a_1, \dots, a_n\} \subset U$ and $V := \{g \in X : \forall 1 \leq i \leq n : g(a_i) = a_i\}$. Then for every $f \in X$:

$$VfV = \{g \in G : \forall 1 \leq i, j \leq n : d(g(a_i), a_j) = d(f(a_i), a_j)\}.$$

Proof of Lemma 1: Concerning "$\subset$": If $g = v_1 f v_2$, with $v_i \in V$, then $d(g(a_i), a_j) = d(v_1 f v_2(a_i), a_j) = d(f v_2(a_i), v_1^{-1}(a_j)) = d(f(a_i), a_j)$, with the last equality due to the fact that v_1, v_2 leave the a_i fixed.

Concerning "$\supset$": Let $g \in X$ be such that for all i, j : $d(g(a_i), a_j) = d(f(a_i), a_j)$. Then there is an isometry between $A \cup g(A)$ and $A \cup f(A)$, defined by $a_i \mapsto a_i$ and $g(a_i) \mapsto f(a_i)$ $(1 \leq i \leq n)$. This is well-defined because if $g(a_i) = a_j$, then $d(f(a_i), a_j) = d(g(a_i), a_j) = 0$, implying $f(a_i) = a_j$; therefore $a_j \mapsto a_j$ and $g(a_i) \mapsto f(a_i)$ are consistent. Let $h \in X$ be

any extension of this isometry, by (b), h exists. Then we have $h \in V$ and $f^{-1}hg \in V$: $f^{-1}hg(a_i) = f^{-1}f(a_i) = a_i$. This yields $g = h^{-1}ff^{-1}hg \in VfV$.
Lemma 2: Let $A \subset U$ be finite and $\frac{1}{2}\mathrm{diam}A \leq c \leq 1$. Then there is $g \in$ X with $\forall a, b \in A$: $d(g(a), b) = c$.
Proof of Lemma 2: Define a metric don $Y := A \times \{0, 1\}$, by setting

$$\tilde{d}((a,i),(b,j)) := \begin{cases} d(a,b) & : \quad i = j \\ c & : \quad i \neq j \end{cases}$$

By (a) there exists an isometric embedding ϕ : Y $\rightarrow$ X. Fori = 0, 1, let $B_i := \phi(A \times \{i\})$. Then B_i is isometric to A, with $d(b_1, b_2) = c$, for $b_i \in$ B,. Let g_1, $g_2 \in$ X with $g,(A) = B_i$, they exist by (b). Then $g = g_2^{-1}g_1$ isasdesired: $d(g(a), b) = d(g_2^{-1}g_1(a), b) = d(g_1(a), g_2(b)) = c$, since $g_1(a) \in B_1$ and $g_2(b) \in B_2$.
Proof of (i): We can assume $V = \{g \in X : \forall 1 \leq i \leq n : g(a_i) = a_i\}$, for some $A = \{a_1, . \ , a_n\}$. Pickf $\in$ X with $d(f(a_i), a_j) = 1$ for all $1 \leq i,j \leq n$, as provided by Lemma 2. Applying Lemma 1, we compute $P :=$ *V'as*

$$(+) \qquad P = \{g \in x : \forall 1 \leq i,j \leq n : d(g(a_i), a_j) = l\}.$$

Since $V^2 = V$, we have t" show P^2 = X. Let $g \in$ X. Applying Lemma 2 t" $A \cup g(A)$, we obtain an isometry $h \in$ X satisfying $d(h(a_i), a_j) = d(h(a_i), g(a_j)) = 1$, for all $1 \leq i,j \leq$ n. Let $p \in$ X be an extension of the isometry $h(A) \cup g(A) \rightarrow h(A) \cup A$, defined by $h(a_i) \mapsto a_i$ resp. $g(a_i) \mapsto h(a_i)$. By definition "fp: $p^{-1}(a_i) = h(a_i) = pg(a_i)$, whence $d(p^{-1}(a_i), a_j) = 1 = d(pg(a_i), a_j)$. Then $(+)$ implies $p^{-1}, pg \in P$, whence $g = p^{-1}pg \in P$'.
Proof of (ii): Let a • U and $V := \{g \in X : g(a) = a\}$. Then we have $V \in \mathcal{U}_e$. For any finite $F \subset X$ and any $g \in$ X we have the following equivalences:

$$\begin{aligned} g \in FVFVF &\Leftrightarrow \exists f_1, f_2 \in F : f_1^{-1}gf_2^{-1} \in VFV \\ &\Leftrightarrow \exists f_1, f_2, f_3 \in F : d(f_1^{-1}gf_2^{-1}(a), a) = d(f_3(a), a) \\ &\Leftrightarrow \exists f_1, f_2, f_3 \in F : d(gf_2^{-1}(a), f_1(a)) = d(f_3(a), a). \end{aligned}$$

Applying Lemma 2 t" the finite set $\{f(a) : f \in F\} \cup \{f^{-1}(a) : f \in F\}$, we obtain that $d(gf_2^{-1}(a), f_1(a))$ takes any value between $1/2$ and 1, as g runs through X. On the other hand, $d(f_3(a), a)$ *can* take only finitely many values, whence $X \neq FVFVF$. □

Conceming the fifth implication in 6.3 and the property of respecting infraboundedness for open subgroups, and also for the succeeding Remark 6.6 we present

Example 6.5. of a topologica1 group X with an infrabounded open subgroup G which is not strongly infrabounded. Because of the openness of G this means that G is not infrabounded in itself. In particular, G is an open subgroup which does not respect infraboundedness. Moreover, X contains an open normal subgroup which does not respect infraboundedness.

Let Y be an infinite group and S the subgroup of $Y^{\mathbb{Z}}$ consisting of all $(y_n)_{n \in \mathbb{Z}}$ for which $\{n \in \mathbb{Z} : y_n \neq e_Y\}$ is finite. The normal subgroups

$$H_k := \{ (y_n)_{n \in \mathbb{Z}} \in S : y_n = e_Y \text{ for } n < k\} \qquad (k \in \mathbb{Z})$$

of S form a basis of $\mathcal{U}_e(S)$ for a Hausdorff SIN-group topology $\mathcal{S}$ on S, with which we equip S. For each $m \in \mathbb{Z}$, $\sigma_m : S \to S$, $(y_n)_{n\in\mathbb{Z}} \mapsto (y_{n+m})_{n\in\mathbb{Z}}$, is a topological automorphism of S (note that $\sigma_m(H_k) = H_{k-m}$); and $m \mapsto$ LT,,, defines a homomorphism $\sigma : \mathbb{Z} \to Aut(S)$. We form the topological semidirect product $X := S \times_\sigma \mathbb{Z}$, with $\mathbb{Z}$ carrying the discrete topology. The subgroups $G_k := H_k \times \{0\}$ $(k \in \mathbb{Z})$ of X form a basis of neighborhoods of $(e_S, 0)$ in X; and for $m, k \in \mathbb{Z}$,

$$(e_S, m)G_{k+m}(e_S, m)^{-1} = \sigma_m(H_{k+m}) \times \{0\} = G_k$$

Therefore each G_k is infrabounded.

But G_k is not strongly infrabounded: For every $l > k$ and finite $F \subset G_k$, one has $(FG_l)^n = F^n G_l$ (since H_l / S), which is a proper subset of G_k since H_l has infinite index in H_k. With a view to 6.24 (ii) below we note also that the open normal subgroup $S \times \{0\}$ contains no open subgroup G that respects boundedness. Otherwise, for $k \in \mathbb{Z}$ with $G_k \subset$ G, G_k would be infrabounded in G and hence in $S \times \{0\}$ which leads to a contradiction similar to the preceding one. □

Remark 6.6. V. Uspenskij has communicated to us in a letter of 26 April, 1992 the sul-prising result and its interesting proof that a topological group X is precompact if for every $U \in \mathcal{U}_e$ there is a finite $F \subset$ X such that X $= FUF$. By contrast, the open subgroups G_k <X of Example 6.5 are not even strongly infrabounded although for every $k \in \mathbb{Z}$ and $U \in \mathcal{U}_e$ there is $x \in$ X such that $G_k \subset xUx^{-1}$.

We now give a table of the basic properties of the various notions of boundedness:

Proposition 6.7. *Let X be a topological group. Let $A, B \subset X$ be both bounded in one of the previously defined senses, and let $x \in X$. The following table gives an overview which of the various sets constructed from A, B and x are bounded in the same sense: A "+" means, that the set is always bounded, "-" indicates that there is a counterexample, the "?" indicates that the question is unsettled.*

	$\mathcal{C}$	$\mathcal{R}$	*bibounded*	$\mathcal{L} \wedge \mathcal{R}$	$\mathcal{L} \vee \mathcal{R}$	*infra*
$C \subset A$	+	+	+	+	+	+
$A \cup B$	+	+	+	+	+	+
$\overline{A}$	+	+	+	+	+	+
$xA \cup Ai$	+	+	+	+	+	+
AB	+	+	+		?	+
A^{-1}			+	+	+	+

Substituting "bounded" by "precompact" in the first five columns yields the same table, except for the "?" which turns into a "+". (Note that a "+" in this case means that the set is precompact, not just bounded.)

Proof. The first two rows are obvious. For the uniformities the third row is due to 1.3 (1), using the fact that all uniformities induce the group topology. For infraboundedness, the proof is easy: If A is infrabounded and $U \in \mathcal{U}_e$, then $A \subset (FU)^n$, for some $n \in \mathbb{N}$ and $F \subset X$ finite. Thus $\overline{A} \subset UA \subset ((F \cup \{e\})U)^{n+1}$.

Now for the fourth row: translations are uniform automorphisms ofthe canonical uniformities (see [25], 2.24), which implies a "+" for these cases as well as for biboundedness. For infraboundedness it is a corollary of the fifth row.
For the fifth row let $A, B \subset X$ be $\mathcal{L}$-bounded. Let $U \in \mathcal{U}_e$. Since B is $\mathcal{L}$-bounded, there exist finite $F \subset B$ and $n \in \mathbb{N}$ with $B \subset FU^n$. Pick $W \in \mathcal{U}_e$ with $Wf \subset fU$, for all f in the finite set F. By $\mathcal{L}$-boundedness of A we have finite $H \subset A$ and $m \in \mathbb{N}$ satisfying $A \subset HW^m$. Hence $AB \subset HW^mFU^n \subset HFU^{m+n}$, with finite HF. The proof for $\mathcal{R}$-bounded A, B is analogous, implying the bibounded case. The counterexample for $\mathcal{L} \wedge \mathcal{R}$-boundedness can not be given in this paper, as it would require too much space. The infrabounded case is obvious.
For the counterexamples in the last row, confer Example 5.3. For the bibounded case the "+" is clear. The inversion is a uniform automorphism of the uniform spaces $(X, \mathcal{L} \wedge \mathcal{R})$ and $(X, \mathcal{L} \vee \mathcal{R})$, which implies a "+" in the corresponding places. The proof for the case of infraboundedness is again easy: If $A \subset (FU)^n$, then $A^{-1} \subset ((F^{-1} \cup \{e\})U^{-1})^{n+1}$.
If we substitute "bounded" by "precompact", most of the above argument can easily be adapted; only the fifth row requires a different treatment. For $\mathcal{L} \wedge \mathcal{R}$, the same counterexample mentioned previously also works for this case. As to $\mathcal{L} \vee \mathcal{R}$, apply 1.7 to the cases $\mathcal{R}$ and $\mathcal{L}$. For the remaining cases, it suffices to deal with the left uniformity. Let $U \in \mathcal{U}_e$, choose $V \in \mathcal{U}_e$ with $V^2 \subset U$. Since B is $\mathcal{L}$-precompact, there exists finite $F \subset B$ with $B \subset FV$. Pick $W \in \mathcal{U}_e$ with $Wf \subset fV$, for all f in F. There exists $H \subset A$ finite satisfying $A \subset HW$. Hence $AB \subset HW^mFV^n \subset HFV' \subset HFU$, with finite HF. □

For the proof of the partial result 6.9 on products of $\mathcal{L} \vee \mathcal{R}$- (resp. $\mathcal{L} \wedge \mathcal{R}$-) bounded sets we need

Lemma 6.8. *If X is an ASIN-group and $A \subset X$ is infrabounded then*

$$(*) \qquad \forall U \in \mathcal{U}_e \exists V \in \mathcal{U}_e \forall x \in A : Vx \subset xU$$

Proof. By [25], 10.17, since X is an ASIN-group,

$$(**) \qquad \exists U' \in \mathcal{U}_e \forall V \in \mathcal{U}_e \exists W \in \mathcal{U}_e \forall x \in U' : Wx \subset xV.$$

With $U' \in \mathcal{U}_e$ from $(**)$ we have $A \subset (FU')^n$ with finite $F \subset X$ and $n \in \mathbb{N}$. For $x \in A$ write $x = y_1u_1 \ldots y_nu_n$ with $y_i \in F$ and $u_i \in U'$. Now let $U \in \mathcal{U}_e$. Putting $V_1 := U$, $(**)$ yields a $W_1 \in \mathcal{U}_e$ sucht that $W_1u \subset uV_1$ for all $u \in U'$. Hence

$$xU = y_1u_1 \ldots y_nu_nV_1 \supset y_1u_1 \ldots y_nW_1u_n.$$

There is a $V_2 \in \mathcal{U}_e$ such that $y^{-1}V_2y \subset W_1$ for all $y \in F$. Thus $xU \supset y_1u_1 \ldots y_{n-1}u_{n-1} V_2y_nu_n$. Proceeding inductively, the result follows. □

Proposition 6.9. *Let X be an ASIN-group and $A, B \subset X$. (i) If A,B are $\mathcal{L} \vee \mathcal{R}$- (resp. $\mathcal{L} \wedge \mathcal{R}$-) bounded in themselves, then so is AB. (ii) If A, B are $\mathcal{L} \wedge \mathcal{R}$-bounded and A or B is $\mathcal{L} \wedge \mathcal{R}$-bounded in itself, then AB is $\mathcal{L} \wedge \mathcal{R}$-bounded.*

Proof. (i): In both cases, A, A^{-1}, B, B^{-1} are infrabounded. Therefore, by 6.8, given $U \in \mathcal{U}_e$ there is $V \in \mathcal{U}_e$, $V \subset U$, such that for all $x \in A \cup B$ one has $Vx \subset xU$ and $xV \subset Ux$. If A, B

are $CV\mathcal{R}$-bounded in themselves there are finite sets $E\subset A$ and $F\subset B$ and $n\in\mathbb{N}'$ such that, for each $a\in A$ and $b\in B$, there are $a_0, a_1, .., a_n \in A$ and $b_0, b_1, , b_n \in B$ with $a_0 \in E$, $a_n = a$, $b_0 \in F$, $b_{,,} = b$, and $a_{i+1} \in a_iV \cap Va_i$, $b_{i+}, \in b_iV \cap Vb_i$ for $0 \leq i < n$. Consequently, $a_0b_0 \in EF$, $a_nb_n = ab$, $a_ib_i \bullet AB$, and $a_{i+1}b_{i+1} \in a_iVb_iV \cap Va_iVb_i \subset a_ib_iUV \cap VUa_ib_i \subset a_ib_iU^2 \cap U^2a_ib_i$ for $0 \leq i<n$. This shows thut AB is $\mathcal{L}\vee\mathcal{R}$-bounded in itself. If A, B are $\mathcal{L}\wedge$ R-bounded in themselves one argues similarly, considering $a_0, a_1, , a_n \in A$ and $b_0, b_1, , b_{,,} \in B$ with $a_{i+1} \in Va_iV$ and $b_{i+1} \in Vb_iV$.- The proof of (ii) is similar.

In regard to the hypothesis of $\mathcal{L}\wedge\mathcal{R}$-boundedness in itself. confer Proposition 6.18. Every open subgroup of a topologica1 group X respects $\mathcal{V}$-boundedness for each $V \in \{C, \mathcal{R}, \mathcal{L}\wedge\mathcal{R}, \mathcal{L}$ V R}, because of 5.1. The following result on infraboundedness-respecting open subgroups will help to establish cases in which infraboundedness coincida with other kinds of boundedness.

Theorem 6.10. *Let X be a topological group and $G < X$. Consider the following conditions.*
(i) G has finite index.
(ii) There are $R, S \subset X$, R finite, such that $X = GSR$ and such that

$$(*) \qquad \forall U \in \mathcal{U}_e \exists V \in \mathcal{U}_e \forall s \in S : sVs^{-1} \subset U$$

(iii) $\forall U \in \mathcal{U}_e \exists V \in \mathcal{U}_e \forall x \in X \exists g \in G : xVx^{-1} \subset gUg^{-1}$.
(iv) $\forall U \in \mathcal{U}_e \exists V \in \mathcal{U}_e \forall x \in X \exists$ *finite* $E \subset G \exists$ *finite* $F \subset X \exists m \in \mathbb{N} : xV \subset (EU)^m F$.
Then $(i) \Rightarrow (ii) \Rightarrow (iii) \Rightarrow (iv)$, and if G is open, then (iv) implies that G respects infraboundedness.

Proof. The proof of the implications between (i) through (iv) is simple (for $(ii) \Rightarrow (iii)$ use $\bigcap_{r\in R} r^{-1}V_I$ wh V open from $(*)$). Now let (iv) hold, let $A \subset$ G be infrabounded and $U \in \mathcal{U}_1$, $U \subset$ G. To prove $A \subset$ (EU)" for some finite $E \subset$ G and $n \in \mathbb{N}$, choose $V \in \mathcal{U}_1$ according to (iv). One has

$$(1) \qquad A \subset (FV)^m \text{ with some finite } F \subset X \text{ and } m \in \mathbb{N}.$$

An easy induction, using (iv), shows

$$\forall i \in \mathbb{N} \exists \text{ finite } E_i \subset G \exists \text{ finite } F_i \subset X \exists p_i \in \mathbb{N} : (FV)^i \subset (E_iU)^{p_i} F_i$$

So, by (1), $A \subset (E_mU)^{p_m}F_m \cap G$. Since A, E_m, U are contained in G we obtain, with $E := E_m \cup (F_m \cap G)$ and $n := p_m + 1$, that indeed $A \subset (EU)^n$. □

Corollary 6.11. *Let X be a topological group and G an open subgroup of finite index. If G is infrabounded, then it is infrabounded in itself.*

Corollary 6.12. *Let X be an infrabounded topological group. Then every open subgroup G of X that contains an open normal subgroup N is infrabounded in itself.*

Proof. With some finite $F \subset X$ and $n \in \mathbb{N}$ one has X $= (FN)^n = F^nN$. Hence G has finite index and 6.10 (i) shows that G respects infraboundedness. Since G is also infrabounded, the result follows. □

Corollary 6.13. *An open subgroup G of a topological group X respects infraboundedness if the group GZ, where Z denotes the center of X, has finite index.*

Proof. This is immediate from 6. 10 (ii). □

We doubt whether every open subgroup of an ASIN-group respects infraboundedness, which is plainly true for SIN-groups.

We now look for circumstances under which some of the notions of boundedness coincide. Some results hold for all subsets of the groups of a certain class, in other results we put conditions on the subsets. Our first result also concerns ASIN-groups.

Proposition 6.14. *Ler X be a topological group and let $A \subset X$ be strongly infrabounded. A is $\mathcal{L}$-bounded under any of the following three conditions.*

(i) $\mathcal{U}_e$ has a basis $\mathcal{B}$ such that $\forall U \in \mathcal{B} \forall x \in A : Ux \subset xU$.

(ii) A is a semigroup and

$$(*) \qquad \forall U \in \mathcal{U}_e \exists V \in \mathcal{U}_e \forall x \in A : \; Vx \subset xU \qquad \textit{(confer } 6.8).$$

(iii) A is $\mathcal{L} \wedge \mathcal{R}$-bounded and $(*)$ holds.

Condition (ii) implies (i). An analogous "right version" is obtained by passing to the opposite topological group.

Proof. If (i) holds, note that (FU)” $\subset F^n U^n$ for all $F \subset A$ and $n \in \mathbb{N}$.
If (ii) holds, the set $W := \bigcup_{x \in A} x^{-1} Vx$ with any $V \in \mathcal{U}_e$ satisfies $x^{-1} Wx \subset W$ for all $x \in A$, which yields (i).
If (iii) holds, for given $U \in \mathcal{U}_e$ pick $V \subset U$ according to $(*)$. By assumption one has $A \subset V^n F V^n$ with finite $F \subset A$ and $n \in \mathbb{N}$, whence $A \subset FU$’” by $(*)$. □

Three corollaries of 6.14, case (ii) (and 6.8) are

Proposition 6.15. *Let X be an ASIN-group. A subsemigroup of X is bibounded if it is strongly infrabounded.*

Proposition 6.16. *Let X be an ASIN-group and G a strongly infrabounded subgroup. Then G is bounded and $\{U \in \mathcal{U}_e(X) : \; \forall x \in G : \; xUx^{-1} = U\}$ is a basis of $\mathcal{U}_e(X)$. In particular, every infrabounded ASIN-group is a bounded SIN-group.*

Proposition 6.17. *Let X be an ASIN-group. If $A \subset X$ is $\mathcal{L} \wedge \mathcal{R}$-bounded, it is bibounded.*

Proof. A is $\mathcal{L}$-bounded by 6.14, and this may be applied to A^{-1}. □

Using the last part of 6.14 we can show also the following result on $\mathcal{L} \vee \mathcal{R}$-boundedness.

Proposition 6.18. *Let X be an ASIN-group and let $A \subset X$ be $\mathcal{L} \wedge \mathcal{R}$-bounded in itself. Then A is $\mathcal{L} \vee \mathcal{R}$-bounded in itself.*

Proof. Let $U \in \mathcal{U}_e$. The last part of 6.14 applied to A and A^{-1} shows

$$\exists V \in \mathcal{U}_e : V \subset U \text{ and } \forall x \in A : Vx \subset xU \text{ and } xV \subset Ux$$

By assumption, there are a finite set $F \subset A$ and $n \in \mathbb{N}'$ such that for every $a \in A$ there are $b \in F$ and $x_0, x_1, , x_n \in A$ such that $x_0 = b$, $x_n = a$ and $x_i \in Vx_{i-}$, V for $1 \leq i \leq n$, whence $x_i \in x_{i-1} U^2 \cap U^2 x_{i-1}$ by the choice of V. This proves the assertion. □

A similar result is

Proposition 6.19. *Let X be a topological group and A an open, strongly infrabounded submonoid of X. Suppose that X is an ASIN-group or that*

$$(*) \qquad \mathcal{B} := \{U \in \mathcal{U}_e(X) : \forall x \in A : xUx^{-1} = U\} \text{ is a basis of } \mathcal{U}_e(X)$$

Then A is an SIN-subgroup of X and bounded in itself.

For the proof we need

Lemma 6.20. *Let X be a topological group. Every open $\mathcal{L}$- (or $\mathcal{R}$-) bounded monoid $A \subset X$ is a subgroup of X.*

Proof. It is enough to consider the case that A is $\mathcal{L}$-bounded. To show $A^{-1} \subset A$, let $U = U^{-1} \in \mathcal{U}_e(X)$, $U \subset A$. By assumption, $A \subset FU^n$ with finite $F \subset X$ and $n \in \mathbb{N}$. For given $x \in A$ there are $k, l \in \mathbb{N}$, $k < l$, and $b \in F$ such that $x^k, x^l \in bU^n$. Hence $x^{-l+k} \in U^{2n} \subset A$ and $x^{-1} = x^{l-k-1}U^{2n} \subset A$ since $l - k - 1 \in \mathbb{N}$. □

Proof of 6.19: Since A and A^{-1} are infrabounded, the last part of 6.14 shows that $(*)$ is satisfied if X is ASIN. Suppose now that $(*)$ holds and let $U \in \mathcal{B}$, $U \subset A$. By assumption. $A \subset$ (FU)" with finite $F \subset A$ and $n \in \mathbb{N}$, whence $A = F^nU^n$. Hence A is $\mathcal{L}$-bounded in itself. and by 6.20 a subgroup of X, which is SIN because of $(*)$. □

6.21 Example/Remark In 6.20, the condition that A be $\mathcal{L}$- or $\mathcal{R}$-bounded cannot be weakened to $LA\,\mathcal{R}$-boundedness, as the following counterexample shows. Let T be an infinite set and X the group of all bijections $x : T \to T$ and let $X_1 := \{x \in X : \{t \in T : x(t) \neq t\} \text{ is finite}\}$. Equip X with the group topology $\mathcal{T}$ for wbich $\{V_F : F \subset T \text{ finite}\}$ with $V_F := \{x \in X_1 : x(t) = t \text{ for } t \in F\}$ is a basis of $\mathcal{U}_e(X, \mathcal{T})$. Then $(X, \mathcal{L} \wedge \mathcal{R})$ is precompact by the argument from [25], 9.14 in which only the neighborhoods U_F are to be replaced by the sets V_F. Let $A \subset X$ be the open monoid generated by X_1 and an element $a \in X$ of infinite order. Then $A = X_1\{a^n : n \in \mathbb{N}\}$ and A is $LA\,\mathcal{R}$-precompact, but A is not a subgroup, since $a^{-1} \notin A$. □

Remark 6.22. Plainly, in a topologica1 group X. e has a neighborhood that is bounded (resp. precompact) for $\mathcal{L}$ or $\mathcal{R}$ iff e has a bibounded (resp. $\mathcal{L} \vee \mathcal{R}$-precompact) neighborhood, and this is equivalent with X to be uniformly locally bounded for $\mathcal{L}$ and $\mathcal{R}$ (resp. uniformly locally $\mathcal{L}$ V $\mathcal{R}$-precompact; see 1.19). "Uniformly locally bounded" may be replaced here by B-conservative, by the result about AB in 6.7. This yields a corollary to 3.13, $(a) \Leftrightarrow (c) \Leftrightarrow (d)$, generalizing [15], Theorem 3 to arbitrary topological groups, for $\mathcal{L}$ and $\mathcal{R}$. Similarly, in ASIN-groups, if e has a neighborhood U which is $\mathcal{L} \wedge \mathcal{R}$-bounded in itself, then the vicinity $U_{\mathcal{L} \wedge \mathcal{R}}$ is $\mathcal{L} \wedge \mathcal{R}$-B-conserving (by 6.9). and another corollary to 3.13, for $\mathcal{L} \wedge \mathcal{R}$-boundedness, results.

We do not know if X is uniformly locally $CV\mathcal{R}$- (resp. $CA\,\mathcal{R}$-) bounded if e has an $\mathcal{L}$ V $\mathcal{R}$- (resp. $\mathcal{L} \wedge \mathcal{R}$-) bounded neighborhood.

Proposition 6.23. *Ler X be a topological group and let e have an $\mathcal{L}$-bounded (resp. $\mathcal{L}$-precompact) neighborhood (cf. 6.22). Then every infrabounded subset is bibounded (resp. $\mathcal{L} \vee \mathcal{R}$- precompact).*

Proof. If $A \subset$ X is infrabounded and $U \in \mathcal{U}_e$ is bibounded (resp. precompact) then $A \subset$ *(FU)"* with $F \subset$ X finite and $n \in \mathbb{N}$. $(FU)^n$ is a finite product of bibounded (resp. $\mathcal{L} \vee \mathcal{R}$-precompact) sets and thus itself bibounded (resp. $\mathcal{L} \vee \mathcal{R}$-precompact), by 6.7, whence A is bibounded (resp. C V $\mathcal{R}$-precompact).

The following proposition was show by Atkin in a weaker form (cf. [1], 3.2 (b)).

Proposition 6.24. *Let X be a topological group and let $A \subset X$ meet only finitely many left (resp. right) H-cosets of X, for any open subgroup H of X. (i) A is $\mathcal{L} \wedge \mathcal{R}$-bounded iff it is $\mathcal{L}$-bounded (resp. $\mathcal{R}$-bounded). (ii) If every open subgroup of X contains an open subgroup which respects infraboundedness and if A is infrabounded then A is $\mathcal{L}$-bounded (resp. $\mathcal{R}$-bounded).*

Proof. For (i) let A be $\mathcal{L} \wedge \mathcal{R}$-bounded and $U \in \mathcal{U}_e$ symmetric. (U) is an open subgroup, hence $A \subset$ F(U) for a finite $F \subset A$. For any $x \in F$, $x^{-1}A \cap \langle U \rangle$ is $\mathcal{L} \wedge \mathcal{R}$-bounded, so that $x^{-1}A \cap \langle U \rangle \subset U^k E U^k$ for some finite $E \subset x^{-1}A$ **n** $\langle U \rangle \subset \langle U \rangle$ and $k \in \mathbb{N}$. U is symmetric and E is finite, hence $E \subset U^l$ for some l, and $x^{-1}A \cap (U) \subset U^{2k+l}$. Choosing k and /big enough for all $x \in F$, we obtain $A = \bigcup_{x \in F} x(x^{-1}A \cap \langle U \rangle) \subset FU$’“+‘, as desired. A modification of this argument yields (ii): *(U)* contains an open subgroup G which respects infraboundedness, $A \subset FG$ with finite $F \subset A$, *etc.* □

Corollary 6.25. Let X be a topological group und A an C $\wedge \mathcal{R}$-bounded subset meeting only a finite number of $\mathcal{L}$-pseudocomponents of X. Then A is bibounded.

Proof. If A meets finitely many $\mathcal{L}$-pseudocomponents, it meets only finitely many left (resp. right) H-cosets, for any open subgroup H. Hence the statement follows from 6.24(i). □

On account of Theorem 6. 10 and of 6.24 (ii) we have

Theorem 6.26. *Let X be a topological group. Suppose that every open subgroup has finite index. Then a subset of X is infrabounded iff it is bibounded.*

Corollary 6.27. A topological group is bounded iff it is infrabounded and every open subgroup has finite index.

Proof. The necessity was stated in Remark 5.2 (1) and Proposition 6.3, and the sufficiency is immediate from the theorem. □

Note that every open normal subgroup of an infrabounded topological group has finite index.

Corollary 6.28. Every infrabounded topological group with open $\mathcal{L}$-pseudocomponent P of e is bounded.

Proof. *P* **has finite** index by the last remark. So 6.27 yields the result. □

Corollary 6.25 applies in particular to groups with only one $\mathcal{L}$-pseudocomponent. However, in this context a stronger statement is easily obtained:

Proposition 6.29. *Let X be a topological group whose $\mathcal{L}$-pseudocomponent P of e has finite index, X = FP with finite $F \subset X$. If $A \subset X$ is infrabounded then for every $U \in \mathcal{U}_e$ there exists $n \in \mathbb{N}$ such that $A \subset FU^n \cap U^n F \cap F^{-1}U^n \cap U^n F^{-1}$. Note the special case X = P.*

Proof. As P is closed, P is open by the assumption. Let $U = U^{-1} \in \mathcal{U}_e$, $U \subset P$ without restriction. Then $\langle U \rangle = P$, so $A \subset F(U)$, and the proof of 6.24 yields $A \subset FU$’ with some $i \in \mathbb{N}$. Since also A^{-1} is infrabounded, one deduces $A^{-1} \subset FU^j$ for some $j \in \mathbb{N}$. Since

$X = PF = F^{-1}P$, the full assertion follows. □

A further corollary to 6.24 is

Proposition 6.30. *Let X be a topological group with open $\mathcal{L}$-pseudocomponent P of e. Then a subset $A \subset X$ is bibounded if A is $\mathcal{L} \wedge \mathcal{R}$-bounded and also if P respects infraboundedness and A is infrabounded.*

Proof. P being an open normal subgroup there are $F \subset X$ finite and $n \in \mathbb{N}$ such that $A \subset P^nFP^n = FP = PF$, respectively $A \subset (FP)^n = F^nP = PF^n$. Therefore 6.24 applies and gives the result. □

Concerning the hypothesis on P see end of Example 6.31. Regarding infraboundedness of open connected $\mathcal{L}$-pseudocomponents and for a discussion of the hypotheses of 6.30 we present the following example similar to 6.5 which however does not yield the "contrast" at the rnd of 6.6.

Example 6.31. of a topological group X whose $\mathcal{L}$-pseudocomponent P of e is open, connected, and infrabounded, but not strongly infrabounded. In particular, the open normal subgroup P does not respect infraboundedness, and in the second part of 6.30 (with $A = P$) the hypothesis that P respect infraboundedness is not superfluous.

Let $(B(\mathbb{R}), \mathcal{S}_0)$ be the bounded TVS constructed in 4.1, let $\mathcal{S}_1$ be the vector space topology induced on $B(\mathbb{R})$ by the product $\mathbb{R}^{[0,1[}$, and equip $B(\mathbb{R})$ with the Hausdorff vector space topology $\mathcal{S} := \mathcal{S}_0 \vee \mathcal{S}_1$. The torus group $T := \mathbb{R}/\mathbb{Z}$ acts on $[0,1[$ naturally by "translation modulo 1", $(t, s) \mapsto t(s)$ $(t \in T, s \in [0,1[)$. Plainly, this yields for every $t \in T$ a topological automorphism σ_t of $(B(\mathbb{R}), \mathcal{S})$, $\sigma_t(f)(s) = f(t(s))$, and a homomorphism $\sigma : T \to Aut(B(\mathbb{R}), \mathcal{S})$ giving rise to the topological semidirect product $X := (B(\mathbb{R}), \mathcal{S}) \times_\sigma T$ with T carrying the discrete topology. The sets

$$W_\epsilon := \{f \in B(\mathbb{R}) : \lambda(f^{-1}(]-\epsilon, \epsilon[)) > 1 - \epsilon\} \qquad (\epsilon > 0)$$

form a basis of $\mathcal{U}_0(B(\mathbb{R}), \mathcal{S}_0)$, the sets

$$V_\epsilon(E) := W_\epsilon \cap \{f \in B(\mathbb{R}) : |f(s)| \leq \epsilon \text{ for } s \in E\}$$

with $\epsilon > 0$ and finite $E \subset [0,1[$ form a basis of $\mathcal{U}_0(B(\mathbb{R}), \mathcal{S})$, and the sets $U_\epsilon(E) := V_\epsilon(E) \times \{0\}$ form a basis of $\mathcal{U}_e(X)$. Clearly the $\mathcal{L}$-pseudocomponent P of e in X is equal to the open connected normal subgroup $B(\mathbb{R}) \times \{0\}$ of X. To prove that P is infrabounded consider the basic neighborhood $U_\epsilon(E)$. There is $n \in \mathbb{N}$ such that $W_\epsilon^n = B(\mathbb{R})$ since $(B(\mathbb{R}), \mathcal{S}_0)$ is connected. Choose $t \in T$ such that $t(E) \cap E = \emptyset$. Then

$$(*) \qquad U_\epsilon(t(E)) = \sigma_t(V_\epsilon(E)) \times \{0\} = (0,t)U_\epsilon(E)(0,t)^{-1}$$

Using a function $h \in B(\mathbb{R})$ with $h([0,1[) \subset [0,1]$, $h(E) \subset \{1\}$ and $h(t(E)) \subset \{0\}$, every $(f, 0) \in W_\epsilon \times \{0\}$ can be written as

$$(f, 0) = (hf + (f - hf), 0) \in (V_\epsilon(t(E)) + V_\epsilon(E)) \times \{0\} \subset (0,t)U_\epsilon(E)(0,t)^{-1}U_\epsilon(E),$$

by $(*)$. Hence $((0,t)U_\epsilon(E)(0,t)^{-1}U_\epsilon(E))^n \supset W_\epsilon^n \times \{0\} = B(\mathbb{R}) \times \{0\}$ which proves that P is infrabounded. But P is not strongly infrabounded. Otherwise P would be infrabounded in itself, and hence bibounded in itself (since abelian), hence $(B(\mathbb{R}), \mathcal{S}_1)$ would be bibounded, a contradiction.- X is not infrabounded since otherwise X and ***P*** would be bibounded by 6.28: In this example T may be replaced by any dense subgroup of $\mathbb{R} / \mathbb{Z}$, in particular by a cyclic dense subgroup.

Questions 6.32. Are $\mathbb{Z}$ and SL(2, $\mathbb{C}$) bounded, non-precompact for some group topology ? SL(2, $\mathbb{C}$) admits no $\mathcal{L}$-precompact Hausdorff group topology, see [20], Corollary 9.12.
(ii) Is there an abelian group X with two group topologia S and $\mathcal{T}$ such that S is bibounded, $\mathcal{T}$ is precompact, and $\mathcal{T} \vee$ S is not bibounded ?

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