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TRANSFER ARGUMENTS FOR SPACES OF OPERATORS AND TENSOR PRODUCTS

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Abstract. *We prove two abstract theorems which allow to transfer stability results of topological properties for spaces $L_b(E,F)$ of operators between two locally convex spaces (with the topology of uniform convergence on all bounded sets) into stability results for injective tensor products, and vice versa. Various examples demonstrate the usefulness of these arguments.*

1 Introduction

As usual we denote by $L_b(E,F)$ the space of bounded and linear operators between two locally convex spaces E and F endowed with the topology of uniform convergence on all bounded sets. Recall that the injective tensor product $E'_b \otimes_\varepsilon F$ of the strong dual of E with F can be identified with the topological subspace $\mathcal{F}(E,F)$ of all finite rank operators in $L_b(E,F)$.

Shortly after Taskinen's counterexamples [19] in 1986 to Grothendieck's problème des topologies (When can every bounded subset of the projective tensor product of two Fréchet spaces be localized?) and the closely related (DF)-space problem (If E is a Fréchet space and F is a (DF)-space, when is $L_b(E,F)$ a (DF)-space?) it became very much clear that questions of this type up to some point are finite dimensional in nature. For finite dimensional Banach spaces $L_b(E,F)$ and $E'_b \otimes_\varepsilon F$ coincide isometrically, hence at least "philosophically" the topological structures of $L_b(E,F)$ and $E'_b \otimes_\varepsilon F$ for many infinite dimensional spaces E, F are similar in many respects - and this is reflected by a long list of papers published after 1986 (see the references of this paper and [9]).

Using ultrapower and desintegration techniques we give two abstract devices which allow to transfer results on topological properties of $E'_b \otimes_\varepsilon F$ into results on topological properties of $L_b(E,F)$, and vice versa. We prove that under very mild conditions, $L_b(E,F)$ can be considered as a complemented subspace of an ultrapower of $E'_b \otimes_\varepsilon F$. Hence for any topological property on locally convex spaces which is stable under the formation of ultrapowers and complemented subspaces, every stability result of this property for injective tensor products can also be formulated for spaces of operators. By the so-called desintegration technique we prove conversely: If E is a locally convex space and X is a Banach space with the bounded approximation property, then any topological property which is inherited by quotients and projective tensor products with normed spaces, transfers from $L_b(E,X)$ to $E'_b \otimes_\varepsilon X$.

In the last section we try to illustrate the usefulness of our transfer arguments by various examples.

We shall use standard notation and notions from the theory of locally convex spaces, Banach spaces and tensor products (see e.g. [15], [8]). If E is a locally convex space, $\mathcal{U}_0(E)$

and $\mathcal{B}(E)$ denote the system of (absolutely convex) 0-neighbourhoods and bounded subsets in E, respectively, and $cs(E)$ is the set of all continuous seminorms.

We also recall some notations and definitions of ultrapowers of locally convex spaces (see [13]). Let A be any set, I be an index set, and $\mathcal{D}$ an ultrafilter on I. Then the set-theoretic ultrapower $A^I/\mathcal{D}$ is the cartesian power A^I, factored by the equivalence relation: $(a_i) \sim (b_i)$ if $\{i \,:\, a_i = b_i\} \in \mathcal{D}$. The equivalence class containing the family (a_i) is denoted by $(a_i)/\mathcal{D}$. Now, given a locally convex space E, we define

$$\text{fin}(E^I/\mathcal{D}) := \{(x_i)/\mathcal{D} \in E^I/\mathcal{D} \,:\, \lim_{\mathcal{D}} p(x_i) < \infty \ \text{ for all } \ p \in cs(E)\},$$

$$\mu(E^I/\mathcal{D}) := \{(x_i)/\mathcal{D} \in E^I/\mathcal{D} \,:\, \lim_{\mathcal{D}} p(x_i) = 0 \ \text{ for all } \ p \in cs(E)\}.$$

Denote the quotient map from $\text{fin}(E^I/\mathcal{D})$ onto $\text{fin}(E^I/\mathcal{D})/\mu(E^I/\mathcal{D})$ by q, and put $(x_i)_{\mathcal{D}} := q((x_i)/\mathcal{D})$.

The *full ultrapower* $(E)_{\mathcal{D}}$ is the space $(E)_{\mathcal{D}} := \text{fin}(E^I/\mathcal{D})/\mu(E^I/\mathcal{D})$, the topology of which is generated by the family of seminorms $p((x_i)_{\mathcal{D}}) := \lim_{\mathcal{D}} p(x_i)$, where $(x_i)_{\mathcal{D}} \in (E)_{\mathcal{D}}$ and p ranges over $cs(E)$.

We now consider the following distinguished subset of $\text{fin}(E^I/\mathcal{D})$:

$$\text{bd}(E^I/\mathcal{D}) := \{(x_i)/\mathcal{D} \in E^I/\mathcal{D} \,:\, \exists I_0 \in \mathcal{D} \,:\, \ (x_i)_{i \in I_0} \ \text{ bounded in } \ E\}.$$

The *bounded ultrapower* $[E]_{\mathcal{D}}$ is the subspace of $(E)_{\mathcal{D}}$ generated by $\text{bd}(E^I/\mathcal{D})$.

Some of the topological properties we are interested in are only inherited by ultrapowers taken with respect to good ultrafilters (by a good ultrafilter on an index set we will always mean an ultrafilter which is countably incomplete and $\aleph^+$-good with respect to a suitable cardinality $\aleph$). For all information on these notions and their relevance in the context of locally convex spaces see ([5], [13] and [12]).

In what follows we will use the space C_2 of Johnson assuming $C_2' = C_2$. This amounts to choosing a sequence $(F_k)_{k\in\mathbb{N}}$ of finite-dimensional Banach spaces which is dense in the set of all finite-dimensional Banach spaces endowed with the Banach-Mazur distance and letting C_2 be the l_2-direct sum of $\oplus_k F_k \times \oplus_k F_k'$.

2 Ultrapower techniques

Our purpose in this section is to transfer properties from $E_b' \otimes_\varepsilon F$ to $L_b(E,F)$. To do this we will take bounded ultrapowers of the first space and obtain the second space as a quotient or even complemented subspace of a bounded ultrapower of the first one.

Definition 1 *A pair (E,F) of locally convex spaces is said to have the equicontinuous approximation property (shortly, EAP) if, for each $T \in L(E,F)$, there is an equicontinuous net $\{T_i\}_{i\in I}$ in $\mathcal{F}(E,F)$ such that $T(x) = \lim_i T_i(x)$, for all $x \in E$.*

Note that, in particular, (E,F) has the EAP if either E or F has the BAP.

The following technical lemma will be needed:

Lemma 2 *Let (E,F) be a pair of locally convex spaces with EAP and $\aleph = |S|$ a cardinality. Consider the index set*

$$I := \{(M,U,A) \ / \ M \subset E \text{ finite}, \ U \in \mathcal{U}_0(F), \ A \subset S \text{ finite}\}$$

together with its natural order

$$(M,U,A) \leq (M',U',A') \text{ if } M \subset M', \ U' \subset U \text{ and } A \subset A'.$$

Then there is a countably incomplete and $\aleph^+$-good ultrafilter $\mathcal{D}$ on I refining the order filter such that for all $T \in L(E,F)$ there is an equicontinuous net $\{T_i\}_{i\in I}$ in $\mathcal{F}(E,F)$ satisfying $T(x) = \lim_{\mathcal{D}} T_i(x)$, for all $x \in E$.

Proof. Without loss of generality we assume that S is such that $\aleph \geq \max\{|\mathcal{U}|,|E|\}$, for some 0-basis $\mathcal{U}$ in F. Given $T \in L(E,F)$, let $\mathcal{W}$ be an equicontinuous subset of $\mathcal{F}(E,F)$ such that T belongs to the closure of $\mathcal{W}$ with respect to the pointwise convergence topology. For each $i = (M,U,A) \in I$ we select (and fix) $T_i \in \mathcal{W}$ such that

$$(T - T_i)(M) \subset U.$$

Obviously $\lim_i T_i(x) = T(x)$, for each $x \in E$. Moreover, if we consider the order filter basis

$$\mathcal{A} := \{I_i \ / \ i \in I\}, \quad I_i := \{j \in I \ / \ j \geq i\},$$

then we easily observe that

$$|I_i| = \aleph \quad \text{for all } i \in I.$$

Apply now [5, 6.1.4 and the remarks before 6.1.8] to get an ultrafilter $\mathcal{D}$ which refines the order filter and is $\aleph^+$-good. On the other hand, the order filter in I is countably incomplete, hence so is $\mathcal{D}$.

From now on I and $\mathcal{D}$ will be fixed as in the preceding lemma, where $\aleph$ is any cardinal not smaller than the cardinal of some 0-basis in $E'_b \otimes_\varepsilon F$.

Theorem 3 *Let F be a locally convex space which is a complemented subspace of $(F'_b)'_e$.*

(a) If E is quasibarrelled and the pair (E,F) has the EAP, then there is a continuous surjection

$$\phi : [E'_b \otimes_\varepsilon F]_{\mathcal{D}} \longrightarrow L_b(E,F).$$

Moreover, if E or F has the BAP, then there is a monomorphism

$$\psi : L_b(E,F) \longrightarrow [E'_b \otimes_\varepsilon F]_{\mathcal{D}}$$

such that $\phi \circ \psi = \mathrm{id}_{L(E,F)}$; that is, $L_b(E,F)$ is a complemented subspace of $[E'_b \otimes_\varepsilon F]_{\mathcal{D}}$.

(b) If X is an arbitrary Banach space, then $L_b(X,F)$ is a complemented subspace of $[C_2 \otimes_\varepsilon F]_{\mathcal{D}}$.

Proof. (a) Identify $\mathcal{F}(E,F)$, as a topological subspace of $L_b(E,F)$, with $E'_b \otimes_\varepsilon F$, and define

$$\begin{array}{rccc} \tilde{\phi}: & [E'_b \otimes_\varepsilon F]_{\mathcal{D}} & \longrightarrow & L_b(E,(F'_b)'_e) \\ & (T_i)_{\mathcal{D}} & \mapsto & T(x) := \lim_{\mathcal{D}} T_i(x) \end{array}$$

This limit is a well-defined element in F'', since $\{T_i(x)\}_{i\in I}$ is bounded in F, for every $x \in E$, and every bounded subset of F is $\sigma(F'',F')$-relatively compact. T is obviously linear and for each $U \in \mathcal{U}_0(F)$, there is $V \in \mathcal{U}_0(E)$ such that $T(V) \subset U^{oo}$, since the quasibarrelledness of E implies the equicontinuity of $\{T_i\}_{i\in I}$ and U^{oo} is $\sigma(F'',F')$-closed. This yields $T \in L(E,(F'_b)'_e)$.

The continuity of $\tilde{\phi}$ follows from the fact that, if $B \in \mathcal{B}(E)$ and $U \in \mathcal{U}_0(F)$, then the set

$$\tilde{U} := (W(B,U))_{\mathcal{D}} \subset [E'_b \otimes_\varepsilon F]_{\mathcal{D}}$$

is a 0-neighbourhood in $[E'_b \otimes_\varepsilon F]_{\mathcal{D}}$ for which $\tilde{\phi}(\tilde{U}) \subset W(B,U^{oo})$.

Now define $\phi := P \circ \tilde{\phi}$, where $P : L_b(E,(F'_b)'_e) \to L_b(E,F)$ is the canonical projection. ϕ is continuous and, if $T \in L(E,F)$, there is an equicontinuous net $\{T_i\}_{i\in I}$ in $E' \otimes F$ such that $T(x) = \lim_{\mathcal{D}} T_i(x)$, for all $x \in E$. Then $\phi((T_i)_{\mathcal{D}}) = T$ and we get the surjectivity of ϕ.

If E (resp., F) has the BAP, there is an equicontinuous net $\{P_i\}_{i\in I}$ in $\mathcal{F}(E)$ such that $x = \lim_{\mathcal{D}} P_i(x)$, for all $x \in E$ (resp., $\{Q_i\}_{i\in I}$ in $\mathcal{F}(F)$ such that $y = \lim_{\mathcal{D}} Q_i(y)$, for all $y \in F$).

For $T \in L(E,F)$ define $\psi(T) := (T \circ P_i)_{\mathcal{D}}$ (resp., $\psi(T) := (Q_i \circ T)_{\mathcal{D}}$). Then $\psi(T)$ is a well-defined element in $[E'_b \otimes F]_{\mathcal{D}}$, and

$$\psi : L_b(E,F) \longrightarrow [E'_b \otimes_\varepsilon F]_{\mathcal{D}}$$

is linear, continuous and satisfies $\phi \circ \psi = \mathrm{id}_{L(E,F)}$. From the last equality we conclude that ψ is a monomorphism.

(b) Let $C_2 = l_2((X_n)_n)$ be the Johnson space. For each $M \subset X$ finite, choose (and fix) $k(M) \in \mathbb{N}$ and an isomorphism $T_M : \mathrm{span}(M) \to X_{k(M)}$ such that $\max\{\| T_M \|, \| T_M^{-1} \|\} < 1 + |M|^{-1}$. Let us define

$$\psi : L_b(X,F) \longrightarrow [C_2 \otimes_\varepsilon F]_{\mathcal{D}}$$

by $\psi(T) := (T_i)_{\mathcal{D}}$, where $T_i := T \circ T_M^{-1} \circ P_{k(M)}$ whenever $i = (M,U,S)$ and $P_{k(M)} : C_2 \to X_{k(M)}$ is the projection onto the $k(M)$-th coordinate. ψ is well-defined, linear and continuous. Now let

$$\begin{array}{rccc} \tilde{\phi}: & [C_2 \otimes_\varepsilon F]_{\mathcal{D}} & \longrightarrow & L_b(X,(F'_b)'_e) \\ & (T_i)_{\mathcal{D}} & \mapsto & T(x) := \lim_{\mathcal{D}} T_i \circ j_{k(M)} \circ T_M(x) \end{array}$$

where $j_{k(M)} : X_{k(M)} \to C_2$ is the canonical injection, $i = (M,U,S)$ and the limit is taken for those $i = (M,U,S)$ such that $x \in M$. $\tilde{\phi}$ is well-defined, linear and continuous. To conclude, define $\phi := P \circ \tilde{\phi}$, where $P : L_b(X,(F'_b)'_e) \to L_b(X,F)$ is the canonical projection, and observe that $\phi \circ \psi = \mathrm{id}_{L(X,F)}$.

The following result is an immediate consequence of Theorem 3 (in case (a) we use the open mapping theorem for Fréchet spaces).

Corollary 4 *Let E and F be l.c.s. such that E is quasibarrelled and F is a complemented subspace of $(F'_b)'_e$. Consider one of the following two cases* *(a) Let P be a property in metrizable l.c.s. which is stable under the formation of quotients and good ultrapowers, and assume that (E,F) has the EAP.*

(b) Let P be a property in locally convex spaces which is stable under complemented subspaces and good ultrapowers, and assume that E or F has the BAP.

Then if $E_b' \otimes_\varepsilon F$ has P, the space $L_b(E,F)$ also has property P.

Moreover, if P is a property as in (b) which is satisfied by $C_2 \otimes_\varepsilon F$, then $L_b(X,F)$ has property P for all Banach spaces X.

Remark. The reason why we have to assume that F is complemented in $(F_b')_e'$ is not only technical: Indeed, in [17, Proposition 2.6] the second author showed that there are Fréchet spaces E and a Banach $\mathcal{L}_\infty$-space Z not complemented in its bidual, such that $L_b(E,Z)$ is not quasinormable. While the first author proved in [6, 4.5.5] that $G \otimes_\varepsilon Z$ is quasinormable whenever Z is a Banach $\mathcal{L}_\infty$-space and G is quasinormable (which is the case for $G = E_b'$ if E is Fréchet).

3 Desintegration techniques

Based on the so-called desintegration technique (see e.g. [3], [8]) we now provide a general device which allows us to transfer topological and geometrical properties from spaces $L_b(E,F)$ of operators to their underlying injective tensor products $E_b' \otimes_\varepsilon F$.

Theorem 5 *Let P be a property in locally convex spaces which is stable under quotients and projective tensor products with normed spaces.*

(1) If X is a Banach space with the BAP and E is a locally convex space such that $L_b(E,X)$ has property P, then $E_b' \otimes_\varepsilon X$ has P.

(2) If X' (resp., X) has the BAP and $L_b(X,F)$ (resp., $L_b(X',F)$) has property P, then $X' \otimes_\varepsilon F$ (resp., $X \otimes_\varepsilon F$) has P.

(3) If $L_b(C_2,E)$ has P, then $X \otimes_\varepsilon E$ has P for all normed spaces X.

(4) If $L_b(E,C_2)$ has P, then $E_b' \otimes_\varepsilon X$ has P for all normed spaces X.

Proof. We start with the proof of (1) (the proof of (2) follows similar lines): Consider for an arbitrary Banach space Y the tensor contraction

$$\begin{array}{rccc} C: & (Y \otimes_\varepsilon X) \otimes_\pi (X' \otimes_\varepsilon X) & \to & Y \otimes_\varepsilon X \\ & (y \otimes x_1) \otimes (x_1' \otimes x) & \mapsto & x_1'(x_1)(y \otimes x) \end{array}$$

which is well-defined, linear and continuous; we show that it is even surjective and open: Take some $z = \sum y_k \otimes x_k \in Y \otimes X$ and look at is associated operator

$$T_z \in \mathcal{F}(Y',X), \ \ T_z(y') := \sum y'(y_k)x_k.$$

Since X has the λ-BAP for some $\lambda \geq 1$, there is some $S \in \mathcal{F}(X,X)$ such that $\| S \| \leq 2\lambda$ and

$$S \circ T_z = T_z$$

(see e.g. [8, Corollary 16.9]). Clearly, S defines some $z_S \in X' \otimes X$ for which $C(z \otimes z_S) = z$ and

$$\pi(z \otimes z_S) = \varepsilon(z)\varepsilon(z_S) \leq 2\lambda\varepsilon(z).$$

Now an easy check of the proof of [8, Theorem 3.5] shows that for each locally convex space Y, in particular for $Y = E'_b$, the above tensor contraction is a topological surjection. To conclude consider the following continuous and linear mapping

$$\begin{array}{rrcl} \Psi: & L_b(E,X) \otimes_\pi (X' \otimes_\varepsilon X) & \rightarrow & E'_b \otimes_\varepsilon X \\ & T \otimes (x' \otimes x) & \mapsto & T'x' \otimes x \end{array}$$

Since the following diagramm commutes, Ψ is even a topological surjection:

$$\begin{array}{ccc} L_b(E,X) \otimes_\pi (X' \otimes_\varepsilon X) & \longrightarrow & E'_b \otimes_\varepsilon X \\ \uparrow & \nearrow & \\ (E'_b \otimes_\varepsilon X) \otimes_\pi (X' \otimes_\varepsilon X) & & \end{array}$$

Clearly, this gives the claim. Proof of (3) ((4) is again similar): By [8, 29.7 Corollary and 35.3 Theorem] we know that the tensor contraction

$$(X \otimes_\varepsilon C_2) \otimes_\pi (C_2 \otimes_\varepsilon E) \longrightarrow X \otimes_\varepsilon E$$

is a topological surjection, and hence as above

$$\begin{array}{rcl} (X \otimes_\varepsilon C_2) \otimes_\pi L(C_2, E) & \rightarrow & X \otimes_\varepsilon E \\ (x \otimes y) \otimes T & \mapsto & x \otimes Ty \end{array}$$

is a topological surjection.

The following corollary summarizes the spirit of this note.

Corollary 6 *Let P a property in locally convex spaces which is stable under quotients, projective tensor products with normed spaces and good ultrapowers. Then for each quasibarrelled locally convex space E complemented in $(E'_b)'_e$ and for each Banach space X complemented in X'' the following hold:*

(1) Let X have the BAP. Then $L_b(E,X)$ has P if and only if $E'_b \otimes_\varepsilon X$ has P.
(2) Let X' have the BAP. Then $L_b(X,E)$ has P if and only if $X' \otimes_\varepsilon E$ has P.
(3) $L_b(X,E)$ has P for all X if and only if $X \otimes_\varepsilon E$ has P for all X.
(4) $L_b(E,X)$ has P for all X if and only if $E'_b \otimes_\varepsilon X$ has P for all X.

4 Examples

In this section we want to illustrate the usefulness of the previous results and provide particular examples - without trying to be as complete as possible.

We first recall some topological invariants from the theory of locally convex spaces which, like properties as "(DF)", "barrelled" or "bornological", are in general not stable under the formation of spaces of linear operators and injective tensor products: A locally convex space E is *quasinormable* if

$$\forall U \in \mathcal{U}_0(E) \quad \exists V \in \mathcal{U}_0(E) \quad \forall \varepsilon > 0 \quad \exists B \in \mathcal{B}(E): \quad V \subset \varepsilon U + B,$$

and E is said to satisfy the *strict Mackey convergence condition*, (SMCC), if

$$\forall B \in \mathcal{B}(E) \quad \exists C \in \mathcal{B}(E) \quad \forall \varepsilon > 0 \quad \exists U \in \mathcal{U}_0(E): \quad U \cap B \subset \varepsilon C.$$

In connection with his study of ultrapowers of locally convex spaces, Heinrich [13] introduces the *density condition*, (DC), as follows: E satisfies (DC) if given any function $\lambda : \mathcal{U}_0(E) \to]0, +\infty[$ and and arbitrary element $V \in \mathcal{U}_0(E)$, there always exist a finite subset $\mathcal{F}$ of $\mathcal{U}_0(E)$ and an element $B \in \mathcal{B}(E)$ such that

$$\bigcap_{U \in \mathcal{F}} \lambda(U)U \subset B + V.$$

This property was intensively studied by Bierstedt and Bonet [1] in the context of Fréchet spaces.

The space E is said to satisfy the *dual density condition*, (DDC), (resp., the *strong dual density condition*, (SDDC)) (see [2]) if given any function $\lambda : \mathcal{B}(E) \to]0, +\infty[$ and an arbitrary element $A \in \mathcal{B}(E)$, there always exist a finite subset $\mathcal{F}$ of $\mathcal{B}(E)$ and $U \in \mathcal{U}_0(E)$ such that

$$A \cap U \subset \overline{\Gamma\left(\bigcup_{B \in \mathcal{F}} \lambda(B)B\right)}, \quad (\text{resp., } \; A \cap U \subset \Gamma\left(\bigcup_{B \in \mathcal{F}} \lambda(B)B\right)).$$

In the context of Fréchet spaces, Vogt introduced the classes (Ω_φ) and (DN_φ) for the splitting theory of exact sequences of Fréchet spaces. See e.g. [21].

Corollary 7 *Let E and F be l.c.s. such that E is quasibarrelled and F is a complemented subspace of $(F_b')_e'$.*

(a) If (E,F) has the EAP, E is a (DF)-space, F is a Fréchet space and $E_b' \otimes_\varepsilon F$ is quasinormable (resp., satisfies condition (Ω_φ)), then $L_b(E,F)$ is also quasinormable (resp., satisfies (Ω_φ)).

(b) If E or F has the BAP and $E_b' \otimes_\varepsilon F$ is quasinormable (resp., has the (DC), is a (DF)-space, is a (DF)-space with the (DDC), is a (DF)-space with the (SMCC), is a metrizable space with condition (DN_φ)), then $L_b(E,F)$ has the same property (in the fourth case $L_b(E,F)$ is a (DF)-space which has even the (SDDC) instead of only the (DDC), which in particular implies that it is bornological).

(c) If X is a Banach space then all the properties considered in (b) are transferred from $C_2 \otimes_\varepsilon F$ to $L_b(X,F)$.

Proof. (a) Quasinormability and (Ω_φ) are properties which are stable under the formation of quotients and good ultrapowers (see [12, Remarks after Corollary 3] and [16, 3.3.6]). Apply then (a) in Corollary 4.

(b) and (c): Quasinormability and the rest of properties in (b) are all stable by taking good ultrapowers (see [12] and [16, 3.3.6]). In the case when $E_b' \otimes_\varepsilon F$ is a (DF)-space with the (DDC), we have that $[E_b' \otimes_\varepsilon F]_\mathcal{D}$ is a (DF)-space with the (SDDC) [12, Proposition 2]. We again conclude by Corollary 4.

Recall that a locally convex space G is in space$(\mathcal{L}_r)$ if for each continuous seminorm p on G, there is another continuous seminorm $q \geq p$ such that the canonical map from $\hat{E}_q$ into $\hat{E}_p''$

factorizes through some $L_r(\mu)$. Banach spaces in space($\mathcal{L}_r$) for $r = 1$ (resp., $r = \infty$) are the $\mathcal{L}_1$-spaces (resp., $\mathcal{L}_\infty$-spaces) in the sense of Lindenstrauss and Pełczyński; for $1 < r < \infty$ they coincide with the class of all $\mathcal{L}_r$-spaces in the sense of Lindestrauss and Pełczyński together with all Banach spaces isomorphic to a Hilbert space.

Corollary 8 *Let E be a Fréchet space and F a (DF)-space complemented in $(F_b')_e'$. Then in each of the following cases $L_b(E,F)$ is a (DF)-space:*
(1) E is a Banach space and F is in space($\mathcal{L}_1$).
(2) E is a Banach $\mathcal{L}_p$-space ($1 \leq p \leq 2$) and F is in space($\mathcal{L}_\infty$).

Proof. Both results follow from Corollary 4 and their counterpart for ε-tensor products: Statement (1) is a consequence of [7, Proposition 2], and (2) of [7, Proposition 3] and [3].

For related results see for example [10] ($L_b(l_1,F)$ is (DF) for every locally complete (DF)-space F) and Taskinen [20].

Following [9] we say that a locally convex space E has type 2 (resp., cotype 2) whenever for a basis $\mathcal{U}$ of 0-neighbourhoods all E_U have this property. We speak of uniform type 2 (resp., uniform cotype 2) if all E_U have type 2 (resp., cotype 2) with a uniform constant not depending on U. Examples will be given below.

Corollary 9 *(1) Let E be a Fréchet space of uniform type 2 with the density condition. Then $E_b' \otimes_\varepsilon X$ is a bornological (DF)-space for each Banach space of cotype 2.*
(2) Let E and F be Fréchet spaces of uniform type 2 with the density condition. Then $E_b' \otimes_\varepsilon F_b'$ is a bornological (DF)-space.

The following lemma, which is based on Pisier's factorization theorem (see e.g. [8] or [18]) will show that it suffices to check that $E_b' \otimes_\varepsilon l_2$ for each E as above is a bornological (DF)-space

Lemma 10 *Let P be a property in locally convex spaces which is stable under quotients and projective tensor products. Then $E_1 \otimes_\varepsilon E_2$ has property P provided the E_k's have cotype 2 and their injective tensor products with l_2 have P.*

Proof. We know from [8, Theorem 35.3 and (1) in Proposition 29.3] that

$$\begin{array}{rrcl} \Phi: & (E_1 \otimes_\varepsilon l_2) \otimes_\pi (l_2 \otimes_\varepsilon E_2) & \to & E_1 \otimes_{\omega_2} E_2 \\ & (x \otimes \xi) \otimes (\zeta \otimes y) & \mapsto & <\xi,\zeta> x \otimes y \end{array}$$

is a topological surjection - here ω_2 is the tensor norm associated to the Banach operator ideal $\mathcal{L}_2$ of all operators factoring through a Hilbert space (see [8, 17.12]). By Pisier's factorization theorem (see e.g. [8, 31.4])

$$X \otimes_\varepsilon Y = X \otimes_{\omega_2} Y$$

(isomorphically) for all cotype 2 spaces X and Y; hence, since the E_k's have cotype 2, we easily obtain that

$$E_1 \otimes_\varepsilon E_2 = E_1 \otimes_{\omega_2} E_2$$

(topologically) which completes the proof.

Proof. [Proof of the corollary] We deduce these results from their analogues for spaces of operators. Indeed, as proved in [9, Prop. of sect. 4 and Ex. 5 of sect. 5] under the above assumptions $L_b(E,X)$ as well as $L_b(E,F_b')$ are bornological (DF)-spaces. Consider the property "bornological (DF)-space", which is stable under quotients and projective tensor products (see e.g. [15, 11.3.8]). By Theorem 5 (1) we know that $E_b' \otimes_\varepsilon l_2$ is a bornological (DF)-space. In view of the lemma it remains to check that E_b' has uniform cotype 2. By [9, Proposition of section 1] it is possible to choose a fundamental system $\mathcal{B}$ of absolutely convex, bounded sets in E such that E_B has type 2 for all $B \in \mathcal{B}$. Then the polars B^o, $B \in \mathcal{B}$, form a basis of 0-neighbourhoods of E_b'. Since each $(E_b')_{B^o}$ can be considered as an isometric subspace of $(E_B)_b'$ and duals of type 2 spaces have cotype 2, we get what we wanted.

We can give some concrete examples of bornological injective tensor products. For $1 < p \leq 2$, let

$$l_{p-} := \operatorname{ind}_{q<p} l_q$$

$$S_{p-} := \operatorname{ind}_{q<p} S_q$$

where S_q stands for the Schatten q-class. And, for $1 \leq p < 2$, consider

$$L_{p+}[0,1] := \operatorname{ind}_{q>p} L_q[0,1].$$

These are strong duals of Fréchet spaces of uniform type 2 (see [11] for the Schatten classes) and, moreover, by Hölder's inequality they satisfy the (SMCC) (we refer to [14] for l_{p-} and to [4] for $L_{p+}[0,1]$). Hence they even are strong duals of Fréchet spaces with the density condition.

Examples.

(1) For p as above and for each Banach space X of cotype 2 the injective tensor products $l_{p-} \otimes_\varepsilon X$, $S_{p-} \otimes_\varepsilon X$ and $L_{p+}[0,1] \otimes_\varepsilon X$ are bornological (DF)-spaces.

(2) For E, F equal to any of the spaces l_{p-}, S_{p-} or $L_{p+}[0,1]$ and p as above, we have that the tensor product $E \otimes_\varepsilon F$ is a bornological (DF)-space.

Counterexample.

For all $2 < p < \infty$ there is a Banach space X of cotype 2 such that $l_{p-} \otimes_\varepsilon X$ is not (DF). To see this, by [9, Example 3 in section 4], consider a Banach space X of cotype 2 such that $L_b(l_{p'+},X)$ is not (DF). Assume that $l_{p-} \otimes_\varepsilon X$ was a (DF)-space. Then $L_b(l_{p'+},X)$, as a complemented subspace of some ultrapower $[l_{p-} \otimes_\varepsilon X]_{\mathcal{D}}$, would be a (DF)-space, which is a contradiction.

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