Note di Matematica Vol. 17, 189-195 (1997)

ON THE "THREE-SPACE PROBLEM" FOR SPACES OF POLYNOMIALS[1]

JOSÉ M. ANSEMIL, FERNANDO BLASCO, SOCORRO PONTE

Abstract. *The problem of when equality between any two of the usual topologies on spaces of homogeneous continuous polynomials on a real or complex locally convex space is a "three-space property" is considered. For all possible cases a positive result or a counterexample is given.*

1 Introduction

The study of when a property is a "three-space property" ($3SP$) has drawn the attention of several authors. For a recent survey on the particular case of Banach spaces see [9] and for the locally convex space case see [22]. We recall that a property is a $3SP$ for a space E if E has the property when F and E/F have it for a subspace F of E. We will say that a property is a $3SP$ for closed (resp. complemented) subspaces if E has the property when for a closed (resp. complemented) subspace F of E, F and E/F have it.

For a given real or complex locally convex space E and a natural number n, denote by $\mathcal{P}(^nE)$ the space of all n-homogeneous continuous polynomials on E (recall that the elements in $\mathcal{P}(^nE)$ are the restrictions to the diagonal of the n-linear continuous mappings from $E \times \cdots \times E$ into the scalar field). On $\mathcal{P}(^nE)$ we consider the natural topologies τ_o, τ_b, β and τ_ω (see the definitions below) and we study when the equality $\tau = \sigma$ for two of those topologies is a $3SP$. We give an answer for all possible choices of τ and σ. The topologies on $\mathcal{P}(^nE)$ we consider are defined as follows:

-τ_o is the topology of uniform convergence on all convex balanced compact subsets of E. When E is quasi-complete, τ_o is the compact open topology.

-τ_b is the topology of uniform convergence on bounded subsets of E. When $n = 1$, τ_b is the strong topology on the dual E' of E.

-β is the strong topology on $\mathcal{P}(^nE)$ as the dual space of $\hat{\otimes}^n_{s,\pi}E$, the completion of the n-fold projective symmetric tensor product of E by itself n times (see [23], [13], [14]). For $n = 1$, β and τ_b agree.

-τ_ω is the Nachbin ported topology defined as the inductive limit of the normed spaces $\mathcal{P}(^nE_V)$, when V ranges over the family of all open, convex and balanced neighborhoods of 0 in E (note that the natural normed topology on $\mathcal{P}(^nE_V)$ agrees with τ_b).

It is easy to see that $\tau_o \leq \tau_b \leq \beta \leq \tau_\omega$ on $\mathcal{P}(^nE)$. Sometimes these topologies agree, but in general do not (see [12]). It is even possible that $\tau_o < \tau_b < \beta < \tau_\omega$ on $\mathcal{P}(^nE)$ for a Fréchet

---

[1] 1991 *Mathematics Subject Classification*. Primary 46A03, 46G20. Secondary 46E10.

Key words and phrases: three-space problem, spaces of polynomials.

First and third authors partially supported by UCM grant PR295/95-6140. Second author partially supported by DGES grant PB96-0651-C03-02.

Grothendieck: A pair (E, F) of locally convex spaces has property (BB) if every bounded subset B in the completed projective tensor product $E\hat{\otimes}_\pi F$ is contained in the closed convex balanced hull of $C \otimes D$, being C bounded in E and D bounded in F.

For every bounded subset C in E and for every polynomial $P \in \mathcal{P}(^nE)$,

$$| P |_C = \sup\{| P(x) | : x \in C\} = \sup\{| Q(x \otimes \cdots \otimes x) | : x \in C\} = | Q |_{\bar{\Gamma}(\otimes^n C)},$$

where Q denotes the linearization of P through the tensor product $\hat{\otimes}^n_{s,\pi} E$ and $\bar{\Gamma}(\otimes^n C)$ denotes the closed convex balanced hull of $\otimes^n C$. So, for a locally convex space E we have $\tau_b = \beta$ on $\mathcal{P}(^nE)$ if and only if E has the $(BB)_{n,s}$ property.

Counterexample 3 *Let E_1 be the cartesian product $\mathbb{C}^{\mathbb{N}}$ and E_2 the direct sum $\mathbb{C}^{(\mathbb{N})}$. Then $\tau_o = \tau_\omega$ on $\mathcal{P}(^nE_i)$ for $i = 1, 2$, but $\tau_o \neq \beta$ (and hence $\tau_o \neq \tau_w$) on $\mathcal{P}(^n(E_1 \times E_2))$.*

Proof.

The space $E_1 = \mathbb{C}^{\mathbb{N}}$ is in fact the first known example of an infinite dimensional Fréchet space such that $\tau_o = \tau_\omega$ on $\mathcal{P}(^nE_1)$ [4]. On the other hand, $E_2 = \mathbb{C}^{(\mathbb{N})}$ is the strong dual of the Fréchet-Montel space E_1 and for such spaces τ_o agrees with τ_ω [11]. Both are classical results in infinite dimensional holomorphy which can be obtained easily using the $(BB)_{n,s}$ property: E_1 and E_2 are Montel spaces and both have the $(BB)_{n,s}$ property which follows straightforward from [15, 1.3 Prop. 7 and Cor. 2], then $\tau_o = \tau_b = \beta$ in both cases. $(\mathcal{P}(^nE_1), \beta)$ as the strong dual of the Fréchet Montel space $\hat{\otimes}^n_{s,\pi}\mathbb{C}^{\mathbb{N}}$ is bornological and $(\mathcal{P}(^nE_2), \beta)$ as the strong dual of the DFM space $\hat{\otimes}^n_{s,\pi}\mathbb{C}^{(\mathbb{N})}$ is also bornological. Since τ_ω is the bornological topology associated with τ_o, and hence with β, for metrizable or DFM spaces [12, Ex. 1.37 and Ex. 1.38], we have that $\tau_o = \tau_\omega$ on $\mathcal{P}(^nE_i)$, $i = 1, 2$.

To see that $\tau_o \neq \beta$ on $\mathcal{P}(^n(E_1 \times E_2))$ we are going to see that $E_1 \times E_2$ has not the $(BB)_{n,s}$ property. It is known that this particular couple (E_1, E_2) has not the (BB) property [15, 1.1.2] and when a couple of spaces (E_1, E_2), each of them with the $(BB)_{2,s}$ property, as this is the case, does not have the (BB) property, then $E_1 \times E_2$ does not have the $(BB)_{2,s}$ property. This follows from the following formula (see [2]) which holds for every pair of locally convex spaces E_1, E_2:

$$\hat{\otimes}^2_{s,\pi}(E_1 \times E_2) \cong (E_1 \hat{\otimes}_{s,\pi} E_1) \times (E_1 \hat{\otimes}_\pi E_2) \times (E_2 \hat{\otimes}_{s,\pi} E_2).$$

Since the $(BB)_{n,s}$ property implies the $(BB)_{2,s}$ property [7] this gives that E does not have the $(BB)_{n,s}$ property. Another proof of this can be found in [12, Ex. 1.39]. □

Remark 4 In the context of Fréchet spaces we don't know if it is possible that any of the equalities $\tau_o = \beta$ or $\tau_o = \tau_\omega$ on $\mathcal{P}(^nE)$, for $n \geq 2$, are $3SP$ (for $n = 1$ the answer is yes and follows from Theorem 2). For Fréchet spaces, β and τ_ω are different in general but we have the following theorem.

Theorem 5 *If E is a Fréchet space, the equality $\tau_o = \beta$ on $\mathcal{P}(^nE)$ is a $3SP$ for closed subspaces if and only if the equality $\tau_o = \tau_\omega$ is.*

Proof.

Assume $\tau_o = \beta$ on $\mathcal{P}(^nE)$ is a $3SP$. Let F be a closed subspace of E and assume that $\tau_o = \tau_\omega$ on $\mathcal{P}(^nF)$ and on $\mathcal{P}(^n(E/F))$. Then by hypothesis $\tau_o = \beta$ on $\mathcal{P}(^nE)$. But, as we have seen, since E is a Fréchet space, the above equality implies that E is Montel and for Fréchet-Montel spaces E with the $(BB)_{n,s}$ property, $\hat{\otimes}^n_{s,\pi}E$ is also a Fréchet-Montel space, hence its strong dual, which is $\mathcal{P}(^nE)$ endowed with β, is bornological. Then $\beta = \tau_\omega$ on $\mathcal{P}(^nE)$.

If we now assume that $\tau_o = \beta$ on $\mathcal{P}(^nF)$ and on $\mathcal{P}(^n(E/F))$, then F and E/F are Montel spaces with the $(BB)_{n,s}$ property. Hence the spaces $\hat{\otimes}^n_{s,\pi}F$ and $\hat{\otimes}^n_{s,\pi}(E/F)$ are Fréchet-Montel spaces and their strong duals $(\mathcal{P}(^nF),\beta)$ and $(\mathcal{P}(^n(E/F)),\beta)$ are bornological. So $\tau_o = \tau_\omega$ on these spaces. The hypothesis gives $\tau_o = \tau_\omega$ on $\mathcal{P}(^nE)$ and then $\tau_o = \beta$ on $\mathcal{P}(^nE)$. □

4 The 3SP for $\tau_b = \beta$ and $\tau_b = \tau_\omega$ on $\mathcal{P}(^nE)$

As we mentioned in section 3, for a locally convex space E the equality $\tau_b = \beta$ on $\mathcal{P}(^nE)$ is equivalent to E having the $(BB)_{n,s}$ property. Property $(BB)_{n,s}$ is inherited by passing to complemented subspaces [7]. Now we are going to see that this property can be lost by passing to cartesian products. This implies that $(BB)_{n,s}$ is not a $3SP$ even for complemented subspaces or, equivalently, that $\tau_b = \beta$ on $\mathcal{P}(^nE)$ is not a $3SP$ for complemented subspaces.

Counterexample 6 *Consider as E_1 the Fréchet space given by the projective limit $\ell^{p+} := \bigcap_{q>p}\ell^q = proj_n \ell^{(p+\frac{1}{n})}$, with $2 \le p < \infty$ (see [20]), and let $E_2 = C_2$ be the ℓ^2-sum of a sequence of finite dimensional Banach spaces dense in the set of all finite dimensional spaces with respect to the Banach-Mazur distance [18]. Then $\tau_b = \beta$ on $\mathcal{P}(^2E_1)$ and $\tau_b = \beta$ on $\mathcal{P}(^2E_2)$, but $\tau_b \neq \beta$ on $\mathcal{P}(^2E)$ where $E = E_1 \times E_2$. So $\tau_b = \beta$ on $\mathcal{P}(^2E)$ is not a 3SP even for complemented subspaces.*

Proof.

E_1 has the property $(BB)_{2,s}$: It is proved in [10, Ex. 5] that (E_1,E_1) has the (BB) property and hence E_1 has the $(BB)_{2,s}$ property. On the other hand, since E_2 is a Banach space, (E_2,E_2) has the (BB) property [15, 1.3. Prop. 5] and hence E_2 has the $(BB)_{2,s}$ property. Let us see how $E_1 \times E_2$ does not have the property $(BB)_{2,s}$. It is proved in [21, Count. 2] that this particular couple (E_1,E_2) does not have the (BB) property, then the argument at the end of counterexample 3 gives that $E_1 \times E_2$ does not have the $(BB)_{2,s}$ property. □

Remark 7 For the particular spaces we have considered in the above counterexample it happens that the strong topology on the corresponding spaces of polynomials agrees with the Nachbin ported topology τ_ω. Note that E_1 has the Heinrich density condition (DC) [16], a consequence of the fact that E_1 is quasinormable which follows from [20]. This implies that $\hat{\otimes}^2_{s,\pi}E_1$ has the (DC) [5, 1.7] and then it is distinguished; so its strong dual is bornological and then $\beta = \tau_\omega$ on $\mathcal{P}(^nE_1)$ ($\hat{\otimes}^2_{s,\pi}E_1$ is Fréchet). On the other hand since E_2 is a Banach space $\beta = \tau_\omega$ and both agree with the natural normed topology on $\mathcal{P}(^nE_2)$. Then the above counterexample also shows that the equality $\tau_b = \tau_\omega$ on $\mathcal{P}(^nE)$ is not a $3SP$ (even for complemented subspaces).

5 The $3SP$ for $\beta = \tau_\omega$ on $\mathcal{P}(^nE)$

The next counterexample shows that $\beta = \tau_\omega$ on $\mathcal{P}(^nE)$ is not a $3SP$ even for complemented subspaces.

Counterexample 8 *Let E_1 be the Banach space ℓ^1 and let E_2 be the Köthe sequence space $\lambda_p(A)$, where p is any real number greater than n and A is the Grothendieck-Köthe matrix [19, Ch. 31] (or any Köthe matrix which does not satisfy condition (D), see [5, Th. 2.6]). Then $\beta = \tau_\omega$ on $\mathcal{P}(^nE_1)$ and on $\mathcal{P}(^nE_2)$ but $\beta \neq \tau_\omega$ on $\mathcal{P}(^n(E_1 \times E_2))$.*

Proof.
Since E_1 is Banach, $\beta = \tau_\omega$ on $\mathcal{P}(^nE_1)$. On the other hand, since $p > n$, the Fréchet space $\hat{\otimes}^n_{s,\pi}\lambda_p(A)$ is distinguished [8] and then $(\mathcal{P}(^n\lambda_p(A)), \beta)$ is bornological, so $\beta = \tau_\omega$ on $\mathcal{P}(^nE_2))$. Let us see how $\beta \neq \tau_\omega$ on $\mathcal{P}(^n(E_1 \times E_2))$. The space $\hat{\otimes}^2_{s,\pi}(E_1 \times E_2)$ is complemented in $\hat{\otimes}^n_{s,\pi}(E_1 \times E_2)$ [7] and the formula at the end of counterexample 3 gives that $\ell^1\hat{\otimes}_\pi\lambda_p(A)$ is complemented on this space. Since $\ell^1\hat{\otimes}_\pi\lambda_p(A)$ is not distinguished (it is distinguished if and only if $\lambda_p(A)$ has the density condition [6, 2.4] and because of our assumption on A, $\lambda_p(A)$ has not this condition), we have that $\hat{\otimes}^n_{s,\pi}(E_1 \times E_2)$ is not distinguished which gives that $(\mathcal{P}(^n(E_1 \times E_2)), \beta)$ is not bornological and then $\beta \neq \tau_\omega$ on $\mathcal{P}(^nE_1 \times E_2)$. □

Remark 9 If we denote by $\mathcal{H}_b(E)$ the space of all entire functions of bounded type on a complex locally convex space E endowed with its natural topology τ_b of uniform convergence on bounded subsets, then the arguments in the proofs of Counterexample 1 and Theorem 2 show that the equality $\tau_0 = \tau_b$ on $\mathcal{H}_b(E)$ is not a $3SP$ in general, but it is a $3SP$ for Fréchet spaces with respect to closed subspaces. The argument in Counterexample 3 also shows that the equality $\tau_0 = \tau_\omega$ on $\mathcal{H}(E)$, the space of all entire functions on a complex locally convex space E, is not a $3SP$. For the definition of τ_ω on spaces of holomorphic functions see [12, Def. 2.33]. Note that $(\mathcal{P}(^nE), \tau_o)$ (resp. $(\mathcal{P}(^nE), \tau_\omega)$) are complemented subspaces of $(\mathcal{H}(E), \tau_o)$ (resp. $(\mathcal{H}(E), \tau_\omega)$) [12, 2.40 and 2.41]. According [3] $\tau_0 = \tau_\omega$ on $\mathcal{H}(E)$ for Fréchet spaces E is a $3SP$ for closed subspaces if and only it it is a $3SP$ for closed subspaces on $\mathcal{P}^n(E)$ for all n.

References

[1] J. M. Ansemil, F. Blasco, S. Ponte, *About quasinormability and topologies on spaces of polynomials*, J. Math. Ann. Appl. **213** (1997), 534-539.

[2] J. M. Ansemil, K. Floret, *The symmetric tensor product of a direct sum of locally convex spaces*, Studia Math. **129** (1998), 285-295.

[3] J. M. Ansemil, S. Ponte, *The compact open and the Nachbin ported topologies on spaces of holomorphic functions*, Arch. Math. **51** (1988), 65-70.

[4] J. A. Barroso, *Topologias nos Espaços de Aplicaçoes holomorfas entre espaços localmente convexos*, Anais Acad. Bras. de Ciencias **43** (1971), 527-546.

[5] K. D. Bierstedt, J. Bonet, *Density conditions in Fréchet and DF-spaces*, Rev. Mat. Univ. Complut. Madrid **2** (1989), 59-75.

[6] K. D. Bierstedt, J. Bonet, *Stefan Heinrich's Density Condition for Fréchet Spaces and the Characterization of the Distinguished Köthe Echelon Spaces*, Math. Nach. **135** (1988), 149-180.

[7] F. Blasco, *Complementation in spaces of symmetric tensor products and polynomials*, Studia Math. **123** (1997), 165-173.

[8] F. Blasco, *Polynomials on Köthe echelon spaces*, Arch. Math. (Basel) **70** (1998), 147-152.

[9] J. F. Castillo, M. González, *Three-space Problems in Banach Spaces Theory,* Lecture Notes in Math., vol. 1667, Springer, Berlin, 1997.

[10] A. Defant, A. Peris, *Maurey's extension theorem and Grothendieck's problème des topologies*, J. London Math. Soc. (2), 58, 3 (1998) 679-696.

[11] S. Dineen, *Holomorphic functions on strong duals of Fréchet-Montel spaces*, Infinite Dimensional Holomorphy and Applications (M.C. Matos, ed.), North-Holland Math. Studies, vol. 12, North-Holland, Amsterdam, 1977, pp. 147-166.

[12] S. Dineen, *Complex Analysis in Locally Convex Spaces*, North-Holland Math. Studies, vol. 57, North-Holland, Amsterdam, 1981.

[13] S. Dineen, *Complex Analysis on Infinite Dimensional Spaces*, Springer Monograph in Mathematics, 1999.

[14] K. Floret, *Natural norms on symmetric tensor products of normed spaces*. These proceedings.

[15] A. Grothendieck, *Produits tensoriels topologiques et espaces nucléaires*, Mem. Amer. Math. Soc. **16** (1995).

[16] S. Heinrich, *Ultrapowers of locally convex spaces and applications, I,* Math. Nachr. **118** (1984), 285-315; *II*, Math. Nachr. **121** (1985), 211-229.

[17] H. Jarchow, *Locally Convex Spaces*, B. G. Teubner, Stuttgart, 1981.

[18] W. B. Johnson, *Factoring compact operators*, Israel J. Math. **9** (1971), 337-345.

[19] G. Köthe *Topological Vector Spaces I*, Springer-Verlag, Berlin, 1969.

[20] G. Metafune, W. B. Moscatelli, *On the space* $\ell^{p+} = \bigcap_{q>p} \ell^q$, Math. Nachr. **147** (1990), 7-12.

[21] A. Peris, *Quasinormable spaces and the problem of topologies of Grothendieck*, Ann. Acad. Sci. Fenn., Series A, I. Math. **19** (1994), 167-203.

[22] W. Roelcke, S. Dierolf *On the three-space-problem for topological vector spaces*, Collec. Math. **32** (1981), 3-25.

[23] R. Ryan, *Applications of topological tensor products to infinite dimensional holomorphy*. Ph. D. Thesis, Trinity College Dublin, 1980.

[24] J. Taskinen, *Counterexamples to "Problème des Topologies" of Grothendieck*, Ann. Acad. Sci. Fenn, Series A, I. Math. Dissertationes, **63** (1986).

J.M. Ansemil
Departamento de Análisis Matemático
Facultad de Matemáticas
Universidad Complutense de Madrid
28040-Madrid
SPAIN
E-mail address: JM_Ansemil@mat.ucm.es

F. Blasco
Departamento de Matemáticas Aplicadas a Los Recursos Naturales
E.T. Superior de Ingenieros de Montes
Universidad Politécnica de Madrid
28040-Madrid
SPAIN
E-mail address: fblasco@montes.upm.es

S. Ponte
Departamento de Análisis Matemático
Facultad de Matemáticas
Universidad Complutense de Madrid
28040-Madrid
SPAIN
E-mail address: Socorro_Ponte@mat.ucm.es