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DECOMPOSITIONS OF MONTEL KÖTHE SEQUENCE SPACES[1]

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Abstract. *The following result has been recently proved by the author: Let E be a Fréchet Schwartz space with unconditional basis and with continuous norm; let F be any infinite dimensional subspace of E. Then we can write E as $G \oplus H$ where G and H do not have any subspace isomorphic to F. This theorem is extended here in two directions: (i) If E is a Montel Köthe sequence space (with certain additional assumptions which are satisfied by the examples described in the literature) and the subspace F is Montel non-Schwartz; (ii) If E is any Fréchet Schwartz space with unconditional basis (so the existence of continuous norm is dropped) and F is not isomorphic to* ω.

1 Introduction and notation

A Fréchet space E is primary if whenever $E = G \oplus H$ then either G or H is isomorphic to E. This property has been widely studied for Banach spaces. Concerning Fréchet (non Banach) spaces, it is known that the power space $X^{\mathbb{N}}$ is primary if X is the scalar field $I\!K$, ℓ_p $(1 \leq p \leq \infty)$, c_0, or $L_p([0,1])$ $(1 \leq p < \infty)$, see [14], [9], [1], [2]. Other primary Fréchet spaces are $\bigcap_{q>p} \ell_q$, $(1 \leq p < \infty)$, $\bigcap_{q<p} L_q$ $(1 < p \leq \infty)$ and the complementably universal elements of certain classes of Fréchet spaces, see [15], [4], [5]. However very little is known about primariness of Köthe sequence spaces though this is one of most important classes in the Theory of Fréchet spaces. The aim of our research is to give some insight into this subject. In fact, the following result (see [6, Corollary 1.6]) constitutes the starting point for this note.

Theorem 1 *Let E be a Fréchet Schwartz space with unconditional basis and with continuous norm and let F be an infinite dimensional subspace of E. Then we can write E as $G \oplus H$ such that G and H do not contain any subspace isomorphic to F. In particular E is not primary.*

We are interested in relaxing the hypothesis of this theorem in two directions. Firstly, we extend the theorem to certain Fréchet Montel spaces. In section 2 we obtain the result for certain Montel Köthe sequence spaces which include all the examples described so far in the literature. Secondly, we remove the hypothesis on the existence of a continuous norm. In section 3 the theorem is extended to Fréchet Schwartz spaces with unconditional basis, in this case we must clearly assume that F is not isomorphic to $\omega := I\!K^{\mathbb{N}}$. Our main tool is the linear topological invariant $\beta(\cdot,\cdot)$ due to Zahariuta (see [7], [8] and the survey paper [18]). We also introduce in section 3 a slight modification of $\beta(\cdot,\cdot)$ to be used for Fréchet spaces without a continuous norm.

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Fréchet spaces are usually denoted by $(E,(V_k))$ where (V_k) stands for a decreasing sequence of absolutely convex closed sets such that $(\frac{1}{k}V_k)$ is a 0-neigbourhood basis.

2 On certain Fréchet Montel spaces

Let I denote a countable index set. A matrix $A = (a_k(i))_{i\in I}$ is said to be a Köthe matrix on I if $0 < a_k(i) \le a_{k+1}(i)$, $k \in \mathbb{N}, i \in I$. We define the Köthe sequence space

$$\lambda_1(A) = \lambda_1(I,A) := \{(x_i) \in I\!K^I : \ \|(x_i)\|_k := \sum_{i=1}^{\infty} |x_i| a_k(i) < \infty, \ k \in \mathbb{N}\}.$$

Without loss of generality we assume that $a_1(i) = 1$ holds for every $i \in I$. Given a subset $J \subset I$ the sectional subspace of $\lambda_1(I,A)$ with respect to J is defined as

$$\lambda_1(J,A) \ := \{(x_i) \in \lambda_1(I,A) : \ x_i = 0 \ \forall \ i \notin J\}.$$

Note that $\lambda_1(I,A) \cong \lambda_1(J,A) \oplus \lambda_1(I\backslash J,A)$. For more information on Köthe sequence spaces we refer the reader to [13, IV.27]. The results of this section hold for Köthe spaces of order $p \in [1,\infty]$ but we keep $p = 1$ to simplify the notation and the statements.

We deal here with Köthe sequence spaces $\lambda_1(\mathbb{N}^2,A)$ that satisfy the following conditions:

$$\begin{aligned}
&\forall j \in \mathbb{N},\ \sup\{a_j(i,j) : \ i \in \mathbb{N}\} = \beta_j < \infty, && (C.1)\\
&\forall j,k \in \mathbb{N}, k \ge j,\ \lim_{i\to\infty} a_k(i,j)/a_{k+1}(i,j) = 0, && (C.2)\\
&\forall k \in \mathbb{N}, \varepsilon > 0, \exists m_0,\ \sup\{a_k(i,j)/a_{k+1}(i,j) : \ i \in \mathbb{N}, j \ge m_0\} < \varepsilon. && (C.3)
\end{aligned}$$

Köthe sequence spaces with conditions (C.1), (C.2) and (C.3) are Montel non-Schwartz. Indeed, (C.1) implies that $\lambda_1(\mathbb{N}^2,A)$ is not Schwartz (see the related notion of obliquely normalized basis by Bellenot [3, Definition 3.1]); by (C.2), the sectional subspace $\lambda_1(\mathbb{N}\times j,A)$ is Schwartz ($j \in \mathbb{N}$) which toghether with (C.3) imply that $\lambda_1(\mathbb{N}^2,A)$ is Montel. (See Floret's construction of Montel non-Schwartz spaces [10].) A Köthe sequence space is called a *canonical* Montel Köthe space if it satisfies conditions (C.1), (C.2) and (C.3). All the examples of Montel non-Schwartz Köthe sequence spaces found in the literature satisfy these conditions, see [12, I.31.5], [13, 27.21] and [16, pp. 217–221]. The purpose of this section is to prove the following result.

Theorem 2 *Let $\lambda_1(\mathbb{N}^2,A)$ be a canonical Montel Köthe sequence space and let F be any non-Schwartz subspace of $\lambda_1(\mathbb{N}^2,A)$. Then there is a subset $I \subset \mathbb{N}^2$ such that neither $\lambda_1(I,A)$ nor $\lambda_1(\mathbb{N}^2 \setminus I,A)$ contains a subspace isomorphic to F. In particular $\lambda_1(\mathbb{N}^2,A)$ is not primary.*

This theorem is proved after some preliminaries. Given U, V subsets of a linear space E, and denoting by $\mathcal{E}_V$ the family of finite dimensional subspaces of E spanned by elements of V, we define

$$\beta(V,U) := \sup\{dim\, L : \ L \in \mathcal{E}_V, L\cap U \subset V\}.$$

We refer the reader to [7] or [8] for more details and elementary properties of $\beta(\cdot,\cdot)$. Our first lemma can be obtained as Lemma 1.3 of [6], see also Lemma 7 below.

Lemma 3 *Let $(F,(U_k))$ be a Fréchet space isomorphic to a subspace of $(E,(V_k))$. For every $k \in \mathbb{N}$ there exist $\sigma(k) < \sigma(k+1) < \sigma(k+2) < \sigma(k+3)$, $k < \tau(k) < \tau(k+1) < \tau(k+2)$ and $M = M(k) > 0$ such that for every couple of scalars s and t we have*

$$\beta(U_{\sigma(k)} \cap tU_{\sigma(k+3)}, U_{\sigma(k)} \cup sU_{\sigma(k+2)}) \leq \beta(M(V_k \cap tV_{\tau(k+1)}), V_{\tau(k)} \cup sV_{\tau(k+2)}).$$

Lemma 4 *Let $\lambda_1(\mathbb{N}^2,A)$ be a canonical Montel non-Schwartz space. If U_k denotes the unit ball associated to the weight a_k, then for every natural numbers $p < q < r$ and every $s > 0$ there exists $t > 0$ such that*

$$\beta(U_p \cap tU_r, U_p \cup sU_q) = \infty.$$

Proof. By the definition of $\beta(\cdot,\cdot)$ it can be readily checked that $\beta(U_p \cap tU_r, U_p \cup sU_q)$ is bigger than or equal to the cardinal of the following set

$$T = \{(i,j) : \max\{a_p(i,j), \frac{1}{t}a_r(i,j)\} \leq \min\{a_p(i,j), \frac{1}{s}a_q(i,j)\}\}.$$

By conditions (C.1) and (C.3) we can find j_0 big enough to satisfy

$$\frac{1}{\beta_{j_0}}a_r(i,j_0) \leq \frac{1}{\beta_{j_0}}a_{j_0}(i,j_0) \leq 1 \leq a_p(i,j_0) \leq \frac{1}{s}a_q(i,j_0),\ \forall\, i \in \mathbb{N}.$$

Thus, with $t = \beta_{j_0}$, the set T contains $\{(i,j_0) :\ i \in \mathbb{N}\}$ and consequently $|T| = \infty$. □

Proof of Theorem 2 The proof is rather similar to the one of [6, Theorem 1.4] but some new ideas are necessary. By [3, Theorem 3.2.(II) or Corollary 3.5] F contains a subspace isomorphic to a canonical Montel Köthe space $\lambda_1(\mathbb{N}^2,B)$. It suffices to do the proof for $F = \lambda_1(\mathbb{N}^2,B)$. We denote by U_k (resp. V_k) the unit ball associated to the kth weight of $\lambda_1(\mathbb{N}^2,B)$ (resp. $\lambda_1(\mathbb{N}^2,A)$). Now, we select sequences of integers (m_n), (s_n) and (t_n), with $m_n < m_{n+1}$, as follows, first we set $m_1 = s_1 = t_1 = 1$, then for every $n \geq 2$ we choose successively m_n, s_n and t_n to satisfy the following properties

$$3(n-1)t_{n-1}a_p(i,j) \leq a_q(i,j), \forall\ p < q \leq n, j \geq m_n, \quad (1)$$
$$s_n := 3nm_n \sup\{\beta_j : j \leq m_n\}, \quad (2)$$
$$\beta(U_p \cap t_nU_r, U_p \cup s_nU_q) = \infty, \forall p < q < r \leq n+1. \quad (3)$$

The choice of m_n such that (1) holds is possible by condition (C.3). Then we set s_n according to (2). Finally, by Lemma 4 we select t_n to satisfy (3). The construction proceeds by induction. We put $I := \mathbb{N} \times \bigcup_{n\in\mathbb{N}}\{j :\ m_{2n} < j \leq m_{2n+1}\}$. Let us prove that F is not isomorphic to a subspace of $\lambda_1(I,A)$. The proof that F is not isomorphic to a subspace of $\lambda_1(\mathbb{N}^2 \setminus I,A)$ is analogous. The neighbourhoods $V_k \cap \lambda_1(I,A)$ are denoted again by V_k, $k \in \mathbb{N}$; this does not create any confusion.

If F is isomorphic to a subspace of $\lambda_1(I,A)$ then by Lemma 3, given $k \in \mathbb{N}$ we have

$$\beta(U_{\sigma(k)} \cap tU_{\sigma(k+3)}, U_{\sigma(k)} \cup sU_{\sigma(k+1)}) \leq \beta(M(V_k \cap tV_{\tau(k+2)}), V_{\tau(k)} \cup sV_{\tau(k+2)}) \quad (4)$$

for increasing indices $\sigma(\cdot)$'s, $\tau(\cdot)$'s, a constant M and every s,t. We prove that, taking $t = t_n$ and $s = s_n$ for n a big enough odd integer, the left hand side of (4) equals infinite while the right hand side is finite; this contradiction settles the proof. Indeed, let n be an odd integer with

$$n > \max\{2M, \sigma(k+3), \tau(k+2)\},$$

and set $t = t_n$, $s = s_n$. The left hand side of (4) is infinite by (3). We are done if we prove that the right hand side is finite. By (C.2) we can find i_n such that

$$a_{q+1}(i,j) \geq 3nt_n m_n a_q(i,j),\ \forall\, i \geq i_n, j \leq m_n, j \leq q. \tag{5}$$

Let us show that

$$\beta(M(V_k \cap t_n V_{\tau(k+2)}), V_{\tau(k)} \cup s_n V_{\tau(k+2)}) \leq i_n m_n. \tag{6}$$

Consider the following partition of I,

$$\begin{aligned} J_0 &= \{(i,j) \in I:\ i \leq i_n, j \leq m_n\}, \\ J_{1,r} &= \{(i,r) \in I:\ i > i_n\},\ \forall\, 1 \leq r \leq m_n \\ J_2 &= \{(i,j) \in I:\ j > m_{n+1}\}, \end{aligned}$$

and denote by P_0, $P_{1,r}$, P_2 the canonical projections from $\lambda_1(I,A)$ onto the sectional subspaces associated to J_0, $J_{1,r}$ and J_2 respectively. If the inequality in (6) fails then there exists $x \in \lambda_1(I,A)$ with $P_0(x) = 0$ and such that

$$[x] \cap (V_{\tau(k)} \cup s_n V_{\tau(k+2)}) \subset n(V_k \cap t_n V_{\tau(k+2)}). \tag{7}$$

Let $\alpha = \sup\{\gamma:\ \gamma x \in V_{\tau(k)} \cup s_n V_{\tau(k+2)}\}$. By (7), we have $\alpha x \in n(V_k \cap t_n V_{\tau(k+2)})$. We apply the projections and obtain

$$P_2(\alpha x) \in P_2(nt_n V_{\tau(k+2)}) \subset \frac{1}{3} V_{\tau(k)},$$

where the inclusion is a consequence of (1). Now fix $r \leq m_n$. If $\tau(k+2) \leq r$ then, by (2)

$$P_{1,r}(\alpha x) \in P_{1,r}(nV_k) \subset P_{1,r}(nV_1) \subset \frac{1}{3m_n} P_{1,r}(s_n V_r) \subset \frac{1}{3m_n} s_n V_{\tau(k+2)},$$

while, if $r < \tau(k+2)$, we obtain from (5)

$$P_{1,r}(\alpha x) \in P_{1,r}(nt_n V_{\tau(k+2)}) \subset \frac{1}{3m_n} V_{\tau(k+2)-1} \subset \frac{1}{3m_n} V_{\tau(k)}.$$

It follows that

$$\alpha x = \sum_{r=1}^{m_n} P_{1,r}(\alpha x) + P_2(\alpha x) \in \frac{2}{3}(V_{\tau(k)} \cup s_n V_{\tau(k+2)}),$$

a contradiction with the choice of α. This validates (6) and finishes the proof. □

Remark 1.4. Theorem 2 does not hold if the subspace F is Schwartz or if $\lambda_1(A)$ is any Montel (not necessarily canonical) Köthe sequence space. Indeed, given $B = (b_k)$ a Köthe matrix on $\mathbb{N}$ we define $C = (c_k)$ on $\mathbb{N}^2$ as follows:

$$c_k(i,j) = b_k(i),\ if\ k \leq i, \quad c_k(i,j) = b_i(i) b_{k-i}(j),\ if\ k > i.$$

Then for any subset $I \subset \mathbb{N}^2$ either $\lambda_1(I,C)$ or $\lambda_1(\mathbb{N}^2 \setminus I,C)$ contains a complemented copy of $\lambda_1(B)$ ([5, Proposition 2.(b)]). Consequently the assertion of Theorem 2 fails for the space $\lambda_1(\mathbb{N}^2,C)$ and the subspace $F = \lambda_1(B)$. Finally, observe that: (1) If $\lambda_1(B)$ is Schwartz then $\lambda_1(\mathbb{N}^2,C)$ is a canonical Montel non-Schwartz space; (2) If $\lambda_1(B)$ is Montel non-Schwartz then $\lambda_1(\mathbb{N}^2,C)$ is a Montel space which is not canonical. In the latter case we do not know if $\lambda_1(\mathbb{N}^2,C)$ is primary or not.

3 Fréchet Schwartz spaces with unconditional basis

Let I denote a countable index set and let $(E,(V_k))$ be a Fréchet space with an unconditional basis $(e_i)_{i\in I}$, i.e. every $x \in E$ can be written in an unique way as $x = \sum_{i\in I} x_i e_i$, $x_i \in \mathbb{K}$ and the series converges unconditionally. Given $J \subset I$, we denote by E_J the closed subspace spanned by $(e_i)_{i\in J}$, which is called the *sectional subspace* associated to J. There is a canonical projection $P_J : E \to E_J$, $\sum_{i\in I} x_i e_i \mapsto \sum_{i\in J} x_i e_i$; and we have $E \cong E_J \oplus E_{I\setminus J}$. We can assume that the neighbourhoods system has been chosen to satisfy $P_J(V_k) \subset V_k$, for every $k \in \mathbb{N}, J \subset I$, e.g. see [17, 1.18].

This section is devoted to prove the following theorem.

Theorem 5 *Let E be a Fréchet Schwartz space with an unconditional basis $(e_i)_{i\in I}$. Let F be an infinite dimensional subspace of E not isomorphic to ω. Then there exists $J \subset I$ such that E_J and $E_{J\setminus I}$ do not contain any subspace isomorphic to F. In particular E is not primary unless it is isomorphic to ω.*

We introduce a new invariant $\beta_0(\cdot,\cdot)$ to be able to handle spaces without a continuous norm. It is a modification of the invariant $\beta(\cdot,\cdot)$. Let E be a linear space; we denote by $[x]$ the line spanned by $x \in E$. Recall that $\mathcal{E}_V$ stands for the family of all finite dimensional subspaces spanned by elements of $V \subset E$. Given V, U subsets of E we define

$$\beta_0(V,U) = \sup\{dim\, L : \; L \in \mathcal{E}_V, \; L \cap U \subset V, \; [x] \not\subset U \; \forall\, x \in L\}.$$

The following elementary properties are straightforward.

Lemma 6 *Let E be a vector space and let U, $V \subset E$, then:*

(a) $\beta_0(A,B) \le \beta_0(V,U)$ *if $A \subset V$, $U \subset B$.*

(b) *If $T : E \to F$ is a linear injection then* $\beta_0(T(V),T(U)) = \beta_0(V,U)$.

(c) $\beta_0(\alpha V,U) = \beta_0(V,\alpha^{-1}U)$, $\forall\, \alpha > 0$.

(d) *If S is a subspace of E then* $\beta_0(V \cap S, U \cap S) \le \beta_0(V,U)$.

Lemma 7 *Let $(F,(U_k))$ be isomorphic to a subspace of the Fréchet space $(E,(V_k))$. Then for every $k \in \mathbb{N}$ there are increasing indices $\sigma(\cdot)$'s, $\tau(\cdot)$'s, and $M = M(k)$ such that*

$$\begin{aligned}&\beta_0(U_{\sigma(k)} \cap tU_{\sigma(k+2)}, \varepsilon U_k \cup U_{\sigma(k)} \cup sU_{\sigma(k+2)})\\ &\le \beta_0(M(V_{\tau(k)} \cap tV_{\tau(k+2)}), \varepsilon V_{\tau(k)} \cup V_{\tau(k+1)} \cup sV_{\tau(k+3)}),\end{aligned}$$

for every $s,t,\varepsilon > 0$.

Proof. The proof is similar to the one of [6, Lemma 1.3] so we give only the main details. We write $B < A$ to mean that $B \subset \lambda A$ for some $\lambda > 0$. If $T : F \to E$ is an isomorphism onto the image then, given $k \in \mathbb{N}$, we can find increasing indices $\sigma(\cdot)$'s and $\tau(\cdot)$'s to satisfy

$$\begin{aligned} T(U_k) \quad &> V_{\tau(k)} \cap T(F) > T(U_{\sigma(k)}) > V_{\tau(k+1)} \cap T(F) \\ &> T(U_{\sigma(k+1)}) > V_{\tau(k+2)} \cap T(F) > T(U_{\sigma(k+2)}) > V_{\tau(k+3)} \cap T(F). \end{aligned}$$

We fix $M = M(k)$ such that $B \subset M^{1/2}A$ for every couple $B < A$ in the chain above. The assertion follows from the elementary properties of $\beta_0(\cdot,\cdot)$ collected in Lemma 6. □

Lemma 8 *Let $(F,(U_k))$ be a nonnormable Fréchet space with a continuous norm. Assume that U_1 does not contain lines and that U_p and U_q do not induce equivalent topologies if $p \neq q$. Given integers $p < q < r$ and given $s > 1$ there exist t_0, $\varepsilon_0 > 0$ such that for every $t \geq t_0$ and $0 < \varepsilon \leq \varepsilon_0$ we have*

$$\beta_0(U_q \cap tU_r, \varepsilon U_p \cup U_q \cup sU_r) > 0.$$

Proof. We fix $x \in F$ such that $x \in U_q \backslash sU_r$. Then we choose ε_0 and t_0 such that $[x] \cap \varepsilon_0 U_p \subset U_q$ and $[x] \cap U_q \subset t_0 U_r$, respectively. Hence, if $t \geq t_0$ and $0 < \varepsilon \leq \varepsilon_0$ we have

$$[x] \cap (\varepsilon U_p \cup U_q \cup sU_r) = [x] \cap U_q \subset [x] \cap (U_q \cap tU_r),$$

consequently

$$\beta_0(U_p \cap tU_r, \varepsilon U_p \cup U_q \cap sU_r) \geq 1.$$

□

Proof of Theorem 5 Without loss of generality we can assume that $F = (F,(U_k))$ has a continuous norm and that U_p and U_q do not induce equivalent topologies if $p \neq q$. If E has a continuous norm this theorem is a particular case of [6, Corollary 1.6]. Assume that no continuous norm is defined on E. Two possibilities may occur. (i) There is $I_1 \subset I$ such that E_{I_1} has a continuous norm and $E_{I \backslash I_1}$ is isomorphic to ω; (ii) The set I can be written as a countable union of pairwise disjoint infinite sets, $I = \bigcup_{i \in \mathbb{N}} I_i$ in such a way that $E_i := E_{I_i}$ has a continuous norm and E is isomorphic to the product $\prod_{i \in \mathbb{N}} E_i$, see [11]. We give the details of the proof of the latter case; the proof of case (i) is obtained with small modifications indicated below.

We set some notation. The basic elements $\{e_j : j \in I_i\}$ are labelled as $\{e_{i,j} : j \in \mathbb{N}\}$. Hence $(e_{i,j})_j$ is an unconditional basis of E_i. A decreasing 0-neighbourhood basis of E_i is denoted by $(V_{k,i})_k$; to simplify the notation we assume that $V_{k,i} = E_i$, $\forall k < i$ and that $V_{i,i}$ does not contain lines; we also assume that $V_{q,i}$ is precompact in the topology induced by $V_{p,i}$, for every $i \in \mathbb{N}$ and all $i \leq p < q$. The 0-neighbourhoods (V_k) of E are defined by $V_k = \prod_{i \in \mathbb{N}} V_{k,i}$.

For every $i,n \in \mathbb{N}$ we denote by $Q_{n,i}$ (respectively $R_{n,i}$) the projection onto the sectional subspace spanned by $\{e_{i,j} : 1 \leq j \leq n\}$ (respectively $\{e_{i,j} : j > n\}$). Then we have

$$\forall\, i \in \mathbb{N},\ \forall p < q,\ \forall\, \delta > 0,\ \exists\, n_0 :\ V_q \cap R_{n,i}(E) \subset \delta V_p \cap R_{n,i}(E),\ \forall\, n \geq n_0. \tag{1}$$

This formula is obvious if $p < i$ and it follows by compactness arguments if $i \leq p$. As in Theorem 2 we construct sequences (m_n), (s_n), (t_n), (ε_n) as follows. Set $m_1 = s_1 = t_1 = \varepsilon_1 = 1$.

For $n \geq 2$ we choose successively m_n, s_n and the couple t_n, ε_n to satisfy

$$(n-1)t_{n-1}V_q \cap R_{m_n,i}(E) \subset \tfrac{1}{2^i 4}V_p \cap R_{m_n,i}(E), \quad \forall\, i \leq n-1,\ \forall\, p < q \leq n, \quad (2)$$
$$nV_i \cap Q_{m_n,i}(E) \subset \tfrac{1}{2^i 4}s_n V_n \cap Q_{m_n,i}(E), \quad \forall\, i \leq n-1, \quad (3)$$
$$\beta_0(U_p \cap t_n U_r, \varepsilon_n U_p \cup U_q \cup s_n U_r) > 0, \quad \forall\, p < q < r \leq n+1. \quad (4)$$

The existence of m_n is ensured by (1). The choice of s_n can be done since V_i induces a norm on the finite dimensional subspace $Q_{m_n,i}(E)$. Finally, t_n and ε_n are selected by Lemma 7. (Note that to prove the case (i), i.e. when $E \cong E_{I_1} \oplus \omega$, the choice of m_n is done only for $i = 1$. This is the only difference between both proofs.) We set $J = \mathbb{N} \times \bigcup_{n\in\mathbb{N}} \{j : m_{2n} < j \leq m_{2n+1}\}$, and show that F is not isomorphic to a subspace of E_J. The proof that F is not isomorphic to a subspace of $E_{\mathbb{N}^2 \setminus J}$ is analogous. The neighbourhoods $V_k \cap E_J$ are denoted again by V_k, $k \in \mathbb{N}$; this does not create any confusion and simplifies the notation. To get a contradiction, if F is isomorphic to a subspace of E_J, then given any $k \in \mathbb{N}$, by Lemma 7 there are increasing indices $\sigma(\cdot)$'s, $\tau(\cdot)$'s and M such that

$$\begin{aligned}&\beta_0(U_{\sigma(k)} \cap tU_{\sigma(k+2)}, \varepsilon U_k \cup U_{\sigma(k)} \cup sU_{\sigma(k+2)})\\ &\leq \beta_0(M(V_{\tau(k)} \cap tV_{\tau(k+2)}), \varepsilon V_{\tau(k)} \cup V_{\tau(k+1)} \cup sV_{\tau(k+3)}),\end{aligned} \quad (5)$$

for every $s, t, \varepsilon > 0$. We fix an odd integer n with

$$n > \max\{M, \sigma(k+2), \tau(k+3)\},$$

and set $t = t_n$, $s = s_n$, $\varepsilon = \varepsilon_n$. Then the left hand side of (5) is bigger than zero by (4). We are done if we check that the right hand side equals zero. Assume on the contrary that there is $x \in E_J$ such that

$$\begin{aligned}&[x] \cap (\varepsilon_n V_{\tau(k)} \cup V_{\tau(k+1)} \cup s_n V_{\tau(k+3)}) \subset n(V_{\tau(k)} \cup t_n V_{\tau(k+2)}),\\ &[x] \not\subset \varepsilon_n V_{\tau(k)} \cup V_{\tau(k+1)} \cup s_n V_{\tau(k+3)}.\end{aligned} \quad (6)$$

By the latter condition we can define

$$\alpha = \sup\{\gamma :\ \gamma x \in \varepsilon_n V_{\tau(k)} \cup V_{\tau(k+1)} \cup s_n V_{\tau(k+3)}\} < \infty.$$

By the first inclusion of (6), we have $\alpha x \in n(V_{\tau(k)} \cap t_n V_{\tau(k+2)})$. Let us consider the following partition of J,

$$J_1 = \{(i,j) \in J :\ i \leq \tau(k),\ j \leq m_n\},$$
$$J_2 = \{(i,j) \in J :\ i \leq \tau(k),\ j > m_{n+1}\},$$
$$J_3 = \{(i,j) \in J : i > \tau(k)\},$$

and denote by P_1, P_2 and P_3, respectively, the projections onto the associated sectional subspaces. For every $i \leq \tau(k)$, we have by (3)

$$\begin{aligned}P_1(\alpha x) \cap Q_{m_n,i}(E) &\in nV_{\tau(k)} \cap Q_{m_n,i}(E) \subset nV_i \cap Q_{m_n,i}(E)\\ \subset \tfrac{1}{2^i 4}s_n V_n \cap Q_{m_n,i}(E) &\subset \tfrac{1}{2^i 4}s_n V_{\tau(k+3)},\end{aligned}$$

whence

$$P_1(\alpha x) \in \frac{1}{4}s_n V_{\tau(k+3)}.$$

By (2), for every $i \leq \tau(k)$ it follows

$$\begin{aligned} P_2(\alpha x) \cap R_{m_{n+1,i}}(E) &\in nt_n V_{\tau(k+2)} \cap R_{m_{n+1,i}}(E) \\ \in \tfrac{1}{2^i 4} V_{\tau(k+1)} \cap R_{m_{n+1},i}(E) \subset \tfrac{1}{2^i 4} V_{\tau(k+1)}, \end{aligned}$$

thus

$$P_2(\alpha x) \in \frac{1}{4} V_{\tau(k+1)}.$$

Finally $[P_3(\alpha x)] \subset V_{\tau(k)}$, in particular

$$P_3(\alpha x) \in \frac{1}{4} \varepsilon_n V_{\tau(k)}.$$

Altoghether we deduce

$$\alpha x = (P_1 + P_2 + P_3)(\alpha x) \in \frac{3}{4} (\varepsilon_n V_{\tau(k)} \cup V_{\tau(k+1)} \cup s_n V_{\tau(k+3)}),$$

in contradiction with the definition of α. □

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