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WEIGHTED SPACES OF HOLOMORPHIC FUNCTIONS AND SEQUENCE SPACES[1]

JOSÉ BONET, DIETMAR VOGT

Abstract. *Our aim in this note is twofold. Firstly we show that, given any Köthe echelon space of order one, a weighted inductive limit of Banach spaces of holomorphic functions on the disc can be constructed such that the strong dual of the sequence space is isomorphic to a complemented subspace of the projective hull associated with the weighted inductive limit. It is also proved that, under some mild assumptions, a weighted inductive limit of spaces of holomorphic functions is a (DFS)-space (and hence the projective description holds) if and only if the associated weights satisfy the condition (S) of Bierstedt, Meise and Summers.*

1 Introduction.

Let G be an open domain in $\mathbb{C}$ and let $V = (v_n)_n$ be a decreasing sequence of strictly positive, continuous weights on G. Let $A = (a_n)_n$ be a Köthe matrix on $\mathbb{N}$, i.e. an increasing sequence of strictly positive weights on $\mathbb{N}$. As in [2] we consider the following sets of weights associated respectively with V and A. $\overline{V}$ is the set of all the upper semicontinuous functions $\overline{v} : G \to [0, \infty[$ such that $\overline{v}/v_n$ is bounded on G for all $n \in \mathbb{N}$. Analogously, $\overline{\Lambda}$ is the set of all the strictly positive sequences $\overline{\lambda} = (\overline{\lambda}(j))_j$ such that $a_n\overline{\lambda}$ is bounded on $\mathbb{N}$ for all $n \in \mathbb{N}$.

The spaces

$$H\overline{V}(G) := \{f \in H(G) \mid \overline{v}f \text{ is bounded in } G \ \forall \overline{v} \in \overline{V}\}$$

and

$$K_\infty = K_\infty(\overline{\Lambda}) := \{x = (x_j)_j \mid (\overline{\lambda}(j)x_j)_j \text{ is bounded } \forall \overline{\lambda} \in \overline{\Lambda}\}$$

are endowed with the locally convex topologies defined by the family of seminorms

$$p_{\overline{v}}(f) := \sup_{z \in G} \overline{v}(z)|f(z)| \ , \ p_{\overline{\lambda}}(x) := \sup_j \overline{\lambda}(j)|x_j|$$

respectively. The space K_∞ is in fact isomorphic to the strong dual of the Köthe echelon space of order one $\lambda_1(A)$ associated with the matrix A [3, 10]. The space $H\overline{V}(G)$ is the projective hull associated with the weighted inductive limit $VH(G) := \mathrm{ind}_n Hv_n(G)$, defined as the countable inductive limit of the sequence of Banach spaces of holomorphic functions

$$Hv_n(G) := \{f \in H(G) \mid \sup_{z \in G} v_n(z)|f(z)| < \infty\},$$

$n \in \mathbb{N}$. These inductive limits appear often in applications to complex analysis or linear partial differential equations. We refer to [1, 2] for further details. The problem of projective

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description of weighted inductive limits of spaces of holomorphic functions [2] asks whether $VH(G)$ and $H\overline{V}(G)$ coincide topologically. It is known [2] that these two spaces coincide algebraically and they even have the same bounded sets. They also coincide topologically if the sequence of weights V satisfies the following condition (S): for every n there is $m > n$ such that the function v_m/v_n vanishes at infinity on G. See [2]. The problem of projective description was answered recently in the negative in [7]. A more natural example for spaces of entire functions was later given in [6].

The main part of this article has two sections. In section 2 we assume that G is bounded and we show that, under some mild restrictions on the boundary of G, given any Köthe echelon space of order one $\lambda_1(A)$, there is a decreasing sequence of strictly positive continuous weights $V = (v_n)_n$ on G such that the strong dual of $\lambda_1(A)$ is isomorphic to a complemented subspace of the projective hull $H\overline{V}(G)$ of the weighted inductive limit $VH(G)$. Selecting appropriate Köthe matrices A for which the echelon space is not distinguished (cf. [9, 31.7], or [5, 13] for a general characterization), we obtain examples of weighted inductive limits $VH(G)$ such that its topology cannot be described by the weighted sup-seminorms defined by the associated system of weights $\overline{V}$. This extends the first counterexamples of [7], and permits to obtain examples in which the set G is the unit open disc D and the weights can be selected with some additional properties. Another approach to get counterexamples was taken by the authors in [8]. In that article a necessary condition is given for the projective description to hold for $VH(G)$.

In section 3 we prove that, under some natural assumptions on the sequence of weights $V = (v_n)_n$, the weighted inductive limit $VH(G)$ satisfies that the linking maps are compact (which is sufficient for the projective description to hold) if and only if the sequence of associated weights $\tilde{V} = (\tilde{v}_n)_n$ satisfies the condition (S) defined above. Given a strictly positive, continuous weight v on G, the associated weight is defined by $\tilde{v}(z) := 1/\sup\{|f(z)| \ ; \ f \in H(G), |f| \leq 1/v \text{ on } G\}$. Clearly $1/\tilde{v}(z)$ coincides with the norm of the evaluation on z as an element of the dual $Hv(G)'$ of $Hv(G)$. Associated weights and its relation to weighted inductive limits have been extensively studied in [1]. The main result of section 3 improves and clarifies certain results in sections 2 and 3 of [1].

Our notation for functional analysis and locally convex spaces is standard. We refer the reader to [10]. Unexplained notation for weighted inductive limits can be seen in [2].

2 Complemented subspaces isomorphic to sequence spaces.

For a compact subset K of $\mathbb{C}$ which is the closure of its interior, we denote by $A(K)$ the Banach algebra of all holomorphic functions on the interior of K which have a continuous extension to the whole set K.

In this section we make the following **assumptions**:

(1) G is a bounded open domain in $\mathbb{C}$ with closure K and boundary ∂G.

(2) There is a discrete sequence $(z_j)_j$ in ∂G of different peak points of $A(K)$ (cf [12]) converging to an element z_∞ of ∂G.

(3) $V = (v_n)_n$ is a decreasing sequence of strictly positive continuous weights on G which are bounded by 1 on G, and such that each v_n has a strictly positive continuous extension w_n to $G \cup \{z_j \mid j \in \mathbb{N}\}$.

(4) The Köthe matrix $A = (a_n)_n$ is defined by $a_n(j) = w_n(z_j)^{-1}$ for every $n, j \in \mathbb{N}$.

We prove the following main result.

Theorem 1 *The space K_∞ is isomorphic to a complemented subspace of $H\overline{V}(G)$.*

Proof. By continuity we select, for each j, a closed neighbourhood U_j of z_j in K such that $U_j \cap U_i = \emptyset$ if $j \neq i$, $\lim_j \operatorname{diam} U_j = 0$ and

$$\frac{1}{2} \leq \frac{v_n(z)}{w_n(z_j)} \leq 2 \ \ \forall 1 \leq n \leq j \ \ \forall z \in U_j \cap G. \qquad (1)$$

There is $\overline{\lambda}_0 \in \overline{\Lambda}$ such that $0 < \overline{\lambda}_0(j) \leq a_1(j)^{-1}$ for all j (cf. [3]). We select a sequence $(\varepsilon_j)_j$ such that

$$0 < \varepsilon_j < \overline{\lambda_0}(j) 2^{-j-1} \ \ \forall j \in \mathbb{N} \quad \text{and} \quad \sum_{j \in \mathbb{N}} \varepsilon_j =: \varepsilon < \infty \qquad (2)$$

Since z_j is a peak point in $A(K)$, there is $g_j \in A(K)$ such that $g_j(z_j) = 1$ and $|g_j(z)| < 1$ for all $z \in K$, $z \neq z_j$. If we denote by C_j the closure of $K \setminus U_j$, then g_j is a continuous function on the compact set C_j, and the sequence of powers $(|g_j|^k)_k$ is a decreasing sequence of continuous functions on C_j, such that $\lim_k g_j^k(z) = 0$ for all $z \in C_j$. By the theorem of Dini, for each j there is $k(j)$ such that $|g_j^{k(j)}(z)| \leq \varepsilon_j$ for all $z \in C_j$. We put, for each j, $e_j := g_j^{k(j)}$. Then, for every j,

$$e_j \in A(K), \ \ e_j(z_j) = 1 \ \text{ and } \ |e_j(z)| \leq \varepsilon_j \ \forall z \in K \setminus U_j. \quad (3)$$

Now, since $\lim_{z \to x_j} e_j(z) = e_j(z_j) = 1$, for each j, we find an open subset V_j of $U_j \cap G$ such that

$$\left| \frac{1}{\mu(V_j)} \int_{V_j} e_j(z) d\mu(z) - 1 \right| \leq \varepsilon_j. \quad (4)$$

Here μ denotes the Lebesgue measure. After this preparation we establish some preliminary facts about the weights which will be needed in the proof below.

(i) $\forall \overline{\lambda} \in \overline{\Lambda} \ \exists \overline{\mu} \in \overline{\Lambda} \ : \ \overline{\lambda} \leq \overline{\mu}$ and $\varepsilon_j \leq 2^{-j-1} \overline{\mu}(j) \ \ \forall j$.

(ii) $\forall \overline{v} \in \overline{V} \ \exists \overline{\lambda} \in \overline{\Lambda} \ \forall j \ : \ \sup_{z \in U_j \cap G} \overline{v}(z) \leq \overline{\lambda}(j)$.

(iii) $\forall \overline{\lambda} \in \overline{\Lambda} \ \exists \overline{v} \in \overline{V} \ \forall j \ : \ \overline{\lambda}(j) \leq \inf_{z \in V_j} \overline{v}(z)$.

Proof of (i): Given $\overline{\lambda} \in \overline{\Lambda}$, take $\overline{\mu} := \max(\overline{\lambda}, \overline{\lambda_0})$.

Proof of (ii): Given $\overline{v} \in \overline{V}$, for each n there is $\alpha_n > 0$ such that $\overline{v} \leq \alpha_n v_n$. We define, for each j, $\overline{\lambda}(j) := \sup_{z \in U_j \cap G} \overline{v}(z)$. To show $\overline{\lambda} \in \overline{\Lambda}$, we fix n, and, for $j \geq n$ we apply (1) to get

$$\overline{\lambda}(j) a_n(j) = \frac{\overline{\lambda}(j)}{w_n(z_j)} \leq \alpha_n \frac{\sup_{z \in U_j \cap G} v_n(z)}{w_n(z_j)} \leq 2\alpha_n.$$

Consequently

$$\sup_j \overline{\lambda}(j) a_n(j) \leq \max(2\alpha_n, \overline{\lambda}(1) a_n(1), ..., \overline{\lambda}(n-1) a_n(n-1)).$$

Proof of (iii): Given $\overline{\lambda} \in \overline{\Lambda}$, for each n, there is $\beta_n > 0$ such that $\overline{\lambda} a_n \leq \beta_n$. We define $\overline{v}(z) := \overline{\lambda}(j)$ if $z \in U_j \cap G$ for some j, and $\overline{v}(z) := 0$ otherwise for $z \in G$. Clearly $\overline{v}$ is upper semicontinuous and non-negative. To check $\overline{v} \in \overline{V}$, we fix n. If $z \in U_j \cap G$ for some $j \geq n$, we apply (1) to get

$$\frac{\overline{v}(z)}{v_n(z)} = \frac{\overline{\lambda}(j)}{v_n(z)} \leq \beta_n \frac{w_n(z_j)}{v_n(z)} \leq 2\beta_n.$$

Therefore, if we set $\gamma_j := \inf(v_n(z) : z \in U_j \cap G)$, we have, for all $z \in G$,

$$\frac{\overline{v}(z)}{v_n(z)} \leq \max(2\beta_n, \frac{\overline{\lambda}(1)}{\gamma_1}, ..., \frac{\overline{\lambda}(n-1)}{\gamma_{n-1}}).$$

Once this is proved, we define the maps

$$\psi : K_\infty \longrightarrow H\overline{V}(G), \ \psi(x) := \sum_j x_j e_j,$$

and

$$\Phi : H\overline{V}(G) \longrightarrow K_\infty, \ \Phi(f) := (\frac{1}{\mu(V_j)} \int_{V_j} f(z) d\mu(z))_j.$$

Both are well-defined, continuous and linear maps. We first prove it for ψ: We fix $x \in K_\infty$. There is $C > 0$ such that $|x_j| \leq C\overline{\lambda}_0(j)^{-1}$ for all j (with $\overline{\lambda}_0$ as in (2)). For every $z_0 \in K$, $z_0 \neq z_\infty$, there are j_0 and a neighbourhood W of z_0 which does not intersect U_j for $j \geq j_0$. If z belongs to this neighbourhood W, we have

$$\sum_{j \geq j_0} |x_j e_j(z)| \leq C \sum_{j \geq j_0} \overline{\lambda}_0(j)^{-1} \varepsilon_j \leq C \sum_{j \geq j_0} 2^{-j-1},$$

and the series defining $\psi(x)$ converges (absolutely and) uniformly on W. This implies $\psi(x) \in H(G)$ and it is continuous on $K \setminus \{z_\infty\}$ for all $x \in K_\infty$. In order to check $\psi(x) \in H\overline{V}(G)$ and the continuity of ψ, given $\overline{v} \in \overline{V}$, we select $\overline{\lambda} \in \overline{\Lambda}$ as in (ii) and then we take $\overline{\mu} \geq \overline{\lambda}$ as in (i). Let $x \in K_\infty$ satisfy $\sup_j \overline{\mu}(j)|x_j| \leq 1$. There is $C > 0$ such that $\overline{v}(z) \leq C$ for all $z \in G$. For $z \in G$, there are two possible cases.

Case 1: z does not belong to any U_j, then, by (3),

$$\overline{v}(z) |\sum_j x_j e_j(z)| \leq \overline{v}(z) \sum_j |x_j e_j(z)| \leq C.$$

Case 2: $z \in U_{j_0}$ for some j_0, then

$$\overline{v}(z)|\sum_j x_j e_j(z)| \leq \overline{v}(z)|x_{j_0} e_{j_0}(z)| + \overline{v}(z) \sum_{j \neq j_0} |x_j e_j(z)| \leq$$

$$\leq C + |x_{j_0}| \sup_{z \in U_{j_0} \cap G} \overline{v}(z)|e_{j_0}(z)| \leq C + |x_{j_0}| \overline{\lambda}(j_0) \leq C + 1.$$

From where the conclusion follows.

We prove now that Φ is also well defined and continuous. Given $f \in H\overline{V}(G)$, we show $\Phi(f) \in K_\infty$ (and the estimates yield also the continuity). For $\overline{\lambda} \in \overline{\Lambda}$, we select $\overline{v} \in \overline{V}$ as in (iii). For each j we have

$$\overline{\lambda}(j) \frac{1}{\mu(V_j)} | \int_{V_j} f(z) d\mu(z)| \leq \frac{1}{\mu(V_j)} \int_{V_j} \overline{v}(z)|f(z)| d\mu(z) \leq$$

$$\leq \sup_{z \in G} \overline{v}(z)|f(z)|.$$

from where the conclusion follows.

It is now easy to see that, if $x = (x_j)_j \in K_\infty$, then, by the estimates already shown for the continuity of ψ, $(\Phi\psi - id)(x) =: (y_j)_j$ satisfies

$$y_j = (\frac{1}{\mu(V_j)} \int_{V_j} e_j(z) d\mu(z) - 1) x_j + \sum_{k \neq j} \frac{x_k}{\mu(V_j)} \int_{V_j} e_k(z) d\mu(z).$$

We can now apply (3) and (4) to conclude

$$|y_j| \leq |\frac{1}{\mu(V_j)} \int_{V_j} e_j(z) d\mu(z) - 1||x_j| + \sum_{k \neq j} \frac{|x_k|}{\mu(V_j)} | \int_{V_j} e_k(z) d\mu(z)| \leq$$

$$\leq \varepsilon_j |x_j| + \sum_{k \neq j} |x_k| \sup_{z \in V_j} |e_k(z)| \leq \sum_k \varepsilon_k |x_k|.$$

And the latter series converges. Indeed, since $x \in K_\infty$, given $\overline{\lambda}_0$ as in (2), we have

$$\sum_k \varepsilon_k |x_k| \leq \sum_k 2^{-k-1} |x_k| \overline{\lambda}_0(k) \leq \frac{1}{2} p_{\overline{\lambda}_0}(x).$$

We consider now the following subset of $\overline{\Lambda}$.

$$\Gamma := \{\overline{\lambda} \in \overline{\Lambda} \mid \overline{\lambda}_0 \leq \overline{\lambda} \leq 1 \text{ on } \mathbb{N}\}.$$

For every $\overline{\lambda} \in \overline{\Lambda}$, we define $\overline{\mu} := \min(a_1^{-1}, \max(\overline{\lambda}, \overline{\lambda}_0))$, with $\overline{\lambda}_0$ as in (2). Hence, since $a_1^{-1} \leq 1$ we have $\overline{\mu} \in \Gamma$ and there is $C \geq 1$ such that $\overline{\lambda} \leq C\overline{\mu}$. Accordingly the set of multiples of the elements of Γ permit to define a fundamental system of seminorms on K_∞.

In order to apply the von Neumann series (cf. [7, Lemma 2]), to conclude that K_∞ is isomorphic to a complemented subspace of $H\overline{V}(G)$, we only have to show that, for all $\overline{\lambda} \in \Gamma$,

$$p_{\overline{\lambda}}((\Phi\psi - id)(x)) \leq \frac{1}{2} p_{\overline{\lambda}}(x) \quad \forall x \in K_\infty.$$

To see this we fix $\overline{\lambda} \in \Gamma$. If we write, as above $(y_j)_j := (\Phi\psi - id)(x)$, and use the estimates already established, we have for $x \in K_\infty$

$$p_{\overline{\lambda}}(y) \leq \sup_j |y_j| \leq \frac{1}{2} p_{\overline{\lambda}_0}(x) \leq \frac{1}{2} p_{\overline{\lambda}}(x).$$

□

This result permits to get easily examples of weighted inductive limits $VH(D)$ of holomorphic functions defined on the unit disc D such that the projective hull $H\overline{V}(D)$ of $VH(D)$ contains the strong dual of a non-distinguished Köthe echelon space, and hence it is not bornological. Accordingly, in this case, these two spaces do not coincide topologically.

Before we construct the example we first observe that every point in the boundary T of D is a peak point of the disc algebra $A := A(K)$ for $K = \overline{D}$. We put $z_j := e^{(1/j)i}$ and we take any non-distinguished Köthe echelon space $\lambda_1(A)$ of order 1 [9, 10], such that the Köthe matrix satisfies $1 \leq a_1$ (which is not a restriction). Clearly in defining $VH(G)$ we may assume the weights of the form

$$v_n(z) = e^{-\varphi_n(z)}$$

where $\varphi_n(z)$ is continuous and subharmonic. Otherwise we might replace $v_n(z)$ by $e^{-\varphi_n(z)}$ with $\varphi_n(z) = \sup\{\log|f(z)| : \sup_{z\in D} v_n(z)|f(z)| \leq 1\}$. See also section 3. Here those $\varphi_n(z)$ which are harmonic are of particular interest. If $\psi_n(z)$ is the harmonic conjugate, then

$$u_n(z) = e^{-(\varphi_n(z)+i\psi_n(z))}$$

is holomorphic, $|u_n(z)| = v_n(z)$ and we have the case of weights which are the modulus of a holomorphic function. We define, for each n, a function α_n on T in the following way: α_n takes the value $\log(a_n(j))$ in a small arc $I_{n,j}$ around z_j for each j and it is 0 otherwise. The length of the arcs $I_{n,j}$ is selected so that the function α_n belongs to L^1 for all n. We define φ_n as the function on D given by the Poisson integral $P[\alpha_n]$ [11], hence it is a harmonic function. We set $v_n := e^{-\varphi_n}$ for each n. The sequence of positive weights $V = (v_n)$ satisfies all our general assumptions and by our theorem the strong dual of $\lambda_1(A)$ is isomorphic to a complemented subspace of $H\overline{V}(D)$.

With the same argument given above we obtain the following consequence.

Corollary 2 *Given a Köthe matrix $A = (a_n)_n$ with $a_1 \geq 1$, there is a decreasing sequence $V = (v_n)_n$ of strictly positive continuous weights on the unit disc $D \subset \mathbb{C}$ such that the strong dual of the Köthe echelon space $\lambda_1(A)$ is isomorphic to a complemented subspace of the projective hull $H\overline{V}(D)$ of the weighted inductive limit $VH(D)$. Moreover the weights v_n can be selected such that v_n coincides with the modulus of a holomorphic function on D.*

3 Characterization of a property of weighted inductive limits.

In this section we assume G is an arbitrary connected open subset of $\mathbb{C}$ and $V = (v_n)_n$ is a decreasing sequence of strictly positive, continuous weights v_n on G such that $\tilde{v}_n(z)$ is finite for every $n \in \mathbb{N}$ and $z \in \mathbb{C}$, hence $\tilde{v}_n$ is a weight on G for each $n \in \mathbb{N}$. At several points we consider the following condition which should be compared with [1, 1.13].

(A) For every $n \in \mathbb{N}$ and $z \in G$, $1/\tilde{v}_n(z)$ coincides with

$$\sup\{|g(z)|\ ;\ g \in H(G), |g| \leq 1/v \text{ on } G \text{ and } vg \text{ vanishes at infinity on } G\}.$$

A function g is said to vanish at infinity on G if for every $a > 0$ there is a compact subset K of G such that $|g(z)| < a$ for every $z \in G \setminus K$. Condition (A) means exactly that, for each $n \in \mathbb{N}$ and each $z \in G$, $1/\tilde{v}_n(z)$ coincides with the norm of the evaluation δ_z at the point z in the dual of the Banach space $H(v_n)_0(G)$ of all those holomorphic functions $f \in H(G)$ such that $v_n|f|$ vanishes at infinity on G endowed with the norm induced by $Hv_n(G)$. The condition (A) is satisfied if G is a balanced open subset of $\mathbb{C}$ and each v_n is a decreasing, radial weight on G such that $H(v_1)_0(G)$ contains the polynomials. In this case the polynomials are dense in each $H(v_n)_0(G)$ and we have $H(v_n)_0(G) = H(\tilde{v}_n)_0(G)$ algebraically for each n. We refer to [4] for more details.

An inductive limit $E = \text{ind}_n E_n$ of Banach spaces is a (DFS)-space if for every n there is $m > n$ such that the linking map $E_n \to E_m$ is compact. By a result of Baernstein, if $VH(G)$ is a (DFS)-space, then the projective description holds and $VH(G) = H\overline{V}(G)$ holds topologically, [2]. Moreover if a sequence $V = (v_n)_n$ of weights satisfies condition (S), it is easy to see that for each n there is $m > n$ such that $Hv_n(G)$ is continuously included in $H(v_m)_0(G)$ with compact inclusion. In particular, $VH(G)$ coincides with $V_0H(G) := \text{ind}_n H(v_n)_0(G)$ and both spaces are (DFS)-spaces.

Our main result in this section improves [1, 2.1.(a) and 3.5].

Theorem 3 *Let $V = (v_n)_n$ be a decreasing sequence of strictly positive continuous weights on a connected open subset G of $\mathbb{C}$ such that $\tilde{v}_n$ is a weight on G for each $n \in \mathbb{N}$. Let $\tilde{V} = (\tilde{v}_n)_n$ be the sequence of associated weights. Assume that $V_0H(G) = \tilde{V}_0H(G)$ holds algebraically.*

(a) If $H(v_1)_0(G)$ separates points of G, then the following conditions are equivalent:

(1) $VH(G)$ is a (DFS)-space,

(2) the sequence $\tilde{V} = (\tilde{v}_n)_n$ satisfies condition (S), i.e. for each n there is $m > n$ such that $\tilde{v}_m/\tilde{v}_n$ vanishes at infinity on G.

(b) If moreover the condition (A) is satisfied, then (1) and (2) in part (a) are also equivalent to

(3) $V_0H(G)$ is a (DFS)-space.

Proof. (a) We assume first that (2) is satisfied. Since $VH(G) = \tilde{V}H(G)$ holds algebraically, hence topologically, (1) follows from our remarks above.

We prove now that (1) implies (2). We fix $n \in \mathbb{N}$ and assume there is $k > n$ such that the inclusion $j : Hv_n(G) \to Hv_k(G)$ is compact. Since $V_0H(G) = \tilde{V}_0H(G)$, we can apply Baire category theorem to get $m > k$ with $H(v_k)_0(G) \subset H(\tilde{v}_m)_0(G)$. This implies that the $\sigma(H(v_k)_0(G)', H(v_k)_0(G))$ limit of $\tilde{v}_m(z)\delta_z$ as z approaches the boundary of G is 0. Since $H(v_k)_0(G)$ separates points of G, we can apply the compactness of the transpose of j to conclude the $Hv_n(G)'$-limit of $\tilde{v}_m(z)\delta_z$ as z approaches the boundary is also 0. This means $\tilde{v}_m/\tilde{v}_n$ vanishes at infinity on G, as desired.

(b) The proof that (2) implies (3) follows also from our remarks before the proof since $V_0H(G) = \tilde{V}H_0(G)$ by assumption. The converse can be shown by an adaptation of the argument given above since the norm of δ_z in $H(v_m)_0(G)'$ coincides with $1/\tilde{v}_m(z)$ for all $z \in G$ by condition (A). □

Lemma 4 *Let G be a connected open subset of $\mathbb{C}$ and let v be a strictly positive continuous weight on G. Let $z_1 \neq z_2$ be two points in G. Then either the space $Hv_0(G)$ has dimension less or equal 1 or there is $f \in Hv_0(G)$ such that $f(z_1) \neq f(z_2)$. In particular, if $\dim(Hv_0(G)) \geq 2$, the space $Hv_0(G)$ separates points of G.*

Proof. The kernel of $\delta_{z_1} \in Hv_0(G)'$ is either 0 (and the whole space must have dimension less or equal 1) or there is $f \in Hv_0(G)$ such that $f \neq 0$ but $f(z_1) = 0$. If $f(z_2) \neq 0$, the proof is complete. Assume $f(z_2) = 0$. This implies both z_1 and z_2 are isolated zeros of f. Let k be the order of the zero z_1 for f. The function $g(z) = (z - z_1)^{-k} f(z)$ belongs to $Hv_0(G)$ and satisfies $g(z_1) \neq 0$ but $g(z_2) = 0$. □

If G is a bounded open subset of $\mathbb{C}$ such that $Hv_0(G) \neq 0$, then $\varphi f \in Hv_0(G)$ for all $\varphi \in H^\infty$ and $f \in Hv_0(G)$. By Lemma 4, in this case, $Hv_0(G)$ separates points of G. In case $G = \mathbb{C}$ and $v(z) = |z|$, $Hv_0(G)$ contains only the constants, has dimension 1 and does not separate points of $G = \mathbb{C}$.

Corollary 5 *Let $V = (v_n)_n$ be a decreasing sequence of strictly positive continuous weights on a bounded connected open subset G of $\mathbb{C}$ such that $v_n = \tilde{v}_n$ for each n. If $H(v_1)_0(G) \neq 0$ and $V_0H(G) = \tilde{V}_0H(G)$, then the following conditions are equivalent:*

(1) $VH(G)$ is a (DFS)-space,

(2) the sequence V satisfies condition (S).

Corollary 6 *Let $V = (v_n)_n$ be a decreasing sequence of radial, strictly positive, continuous weights on a balanced open subset G of $\mathbb{C}$ such that $H(v_1)_0(G)$ contains the polynomials. Let $\tilde{V} = (\tilde{v}_n)_n$ be the sequence of associated weights. The following conditions are equivalent:*

(1) $VH(G)$ is a (DFS)-space,

(2) $V_0H(G)$ is a (DFS)-space,

(3) the sequence $\tilde{V}$ satisfies condition (S).

To close this section we give an example, different from the ones in [1], to show that in general the conditions (1) and (2) in theorem 3 are not equivalent. Let G_1 be the disc in $\mathbb{C}$ of center 0 and radius 2. For each n let u_n be the radial weight on G_1 which is identically 1 on the disc of radius 1 and coincides with $(2-r)^n$ for $r \in [1,2[$. Clearly $U = (u_n)_n$ satisfies condition (S) on G_1, hence $UH(G_1)$ is a (DFS)-space, and $u_n = \tilde{u}_n$ for each n. We set $G := G_1 \setminus \{0\}$ and, for each n, we denote by v_n the restriction of u_n to G. By the theorem of removable singularities, $Hu_n(G_1)$ can be canonically identified, using restrictions, with $Hv_n(G)$. Accordingly $VH(G)$ is isomorphic to $UH(G_1)$, hence it is a (DFS)-space. Clearly $V = (v_n)_n$ does not satisfy condition (S) at the point 0 of the boundary of G. Observe that in this example, if $f \in H(G_1)$ belongs to $H(v_n)_0(G)$ for some n and $v_n|f| \leq 1$ on G, then $f(0) = 0$ and, by Schwarz lemma, $|f(z)| \leq |z|$ for each $|z| < 1$. Accordingly if $|z| < 1$, we have $\tilde{v}_n(z) = 1$ but $1/\sup\{|f(z)| \ ; \ f \in H(v_n)_0(G), v_n|f| \leq 1 \text{ on } G\} = 1/|z|$. This implies $g(z) = z \in H(v_1)_0(G) \setminus H(\tilde{v}_n)_0(G)$ for all $n \in \mathbb{N}$. Moreover condition (A) is not satisfied.

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J. Bonet
Departmento de Matemática Aplicada
E.T.S. Arquitectura
Universidad Politécnica de Valencia
E-46071 Valencia
SPAIN
E-mail address: jbonet@pleiades.upv.es

D. Dietmar Vogt
FB. Mathematatik
Bergische Universität
Gaussstrasse 20
D-42097 Wuppertal
GERMANY
E-mail address: dvogt@uni-wuppertal.de