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ALMOST RIEMANNIAN SPACES WITH SMALL EXCESS

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Abstract. *Let M be a space in the class of almost Riemannian spaces of positive injectivity radius with curvature uniformly bounded below, diameter uniformly bounded above by D, and the D-excess of M is almost zero. We classify M in terms of C^1-fibration.*

An Alexandrov space is called geodesically complete provided every geodesic can be extended to a geodesic defined on all of R. A geodesically complete Alexandrov space is called an almost Riemannian space. T. Yamaguchi studied the fibration theorem for certain class of Riemannian manifolds [5]. The result was extended by F. Wilhelm to the category of almost Riemannian spaces [3].

Theorem 1. *For compact metric spaces, X and Y, let $d_{GH}(X,Y)$ denote the Gromov-Hausdorff distance between X and Y. If X is a compact, almost Riemannian space with curvature $\geq k$ and positive injectivity radius, then there is an $\epsilon(X)>0$ so that if M is an almost Riemannian space with positive injectivity radius which satisfies $curv(M) \geq k$ and $d_{GH}(M,X)<\epsilon$, then there is a C^1-fibration, $f: M \to X$, which has the following properties:*
1. f is a submersion, and given any $\eta>0$, there is a $\delta>0$ so that if $d_{GH}(M,X)<\delta$, then

$$|\|df(\zeta)\| - 1| < \eta$$

for every unit vector $\zeta \in TM$ which is perpendicular to a fiber of f.
2. Given any $\eta>0$, there is a $\delta>0$ so that if $d_{GH}(M,X)<\delta$, then for every $x \in X$, $f^{-1}(x)$ is connected and has diameter less than η with respect to the extrinsic metric from M.

It is well known from the classical Bott-Samelson Theorem [1] that a nonsimply connected Blaschke Riemannian m-manifold is diffeomorphic to RP^m. The aim of our paper here is to examine this result in the more general setting of almost Riemannian spaces.

We first review the notions of excess function. Given two points p and q in a compact metric spaces X, the (p,q) - excess function $e_{p,q}: X \to R$ is given by $e_{p,q}(x) = d(p,x) + d(x,q) - d(p,q)$. Given $p \in X$ and $\triangle > 0$, the $\triangle$-excess of X at p is

$$e_p^{\triangle} = \max_{x \in B_{\triangle}(p)} \min_{q \in S_{\triangle}(p)} e_{p,q}(x)$$

where $B_{\triangle}(p) = \{x \in X | d(x,p) \leq \triangle\}$ and $S_{\triangle}(p) = \{x \in X | d(x,p) = \triangle\}$. The $\triangle$-excess of X is $e^{\triangle}(X) = \max_{p \in X} e_p^{\triangle}$.

If the $\triangle$-excess of a Riemannian manifold M is 0, then the injectivity radius of M is greater than or equal to $\triangle$. However, if $e^{\triangle}(M)$ is close to zero does not imply that the injectivity radius of M is almost greater than or equal to $\triangle$ [3].

Nevertheless, the condition of small $\triangle$-excess is a natural extension of the injectivity radius bound. The condition that injectivity radius is greater than or equal to $\triangle$ implies that for any

$p \in M$, the metric $\triangle$-sphere $S_\triangle(p)$ around p satisfies $S_\triangle(p) \neq \emptyset$ and for any $x \in B_\triangle(p)$ there exists a minimal geodesic γ from p to $S_\triangle(p)$ which contains x. The condition of $e^\triangle(M)$ being small implies that for any p and $x \in B_\triangle(p)$, there exists minimal geodesic γ from p to $S_\triangle(p)$ with small distance between x and γ.

Our main theorem can be stated as follows:

Theorem 2. *Given $k \in R$, $D>0$ and $n \in N$. Let $\mathcal{M}_k^D(n)$ denote the class of all almost Riemannian spaces with positive injectivity radius with curvature $\geq k$, diameter D and dimension $\leq n$. There exists $\epsilon(k,D,n)>0$ such that if $M \in \mathcal{M}_k^D(n)$ and $e^{D-\epsilon}(M)<\epsilon$, then there is a C^1-fibration $f : M \to N$ where N is*
1. simply connected, or
2. homotopic to RP^m where $2 \leq m \leq n$, or
3. 1-dimensional space.

Proof. Take a sequence $\{\epsilon_i>0\}$ such that $\lim_{i\to\infty} \epsilon_i = 0$. Take a sequence $\{M_i\} \subset \mathcal{M}_k^D(n)$ for which $e^{D-\epsilon_i}(M_i)<\epsilon_i$. It follows from Gromov's precompactness theorem that such sequence has a subsequence which converges to a compact length space N that has curvature $\geq k$. By Proposition 3(ii) in [2] (continuity of $\triangle$-excess with respect to the Gromov-Hausorff topology), $e^D(N) = 0$. It is easy to see that N is geodesically complete and hence almost Riemannian [3]. An almost Riemannian space with D-excess equal to 0 has injectivity radius $\geq D$ (cf. Proposition 3(iii) of [2]), hence D. Suppose N is simply connected, we are done by applying Theorem 1. Suppose not, we claim that N is homeomorphic to RP^m.

In fact, we know that $\Pi_1(N)$, the fundamental group of N, is finitely generated because N is compact. Let $\pi : \tilde{N} \to N$ be the universal covering projection and $\{\omega_i\}_{i=1}^k$ be the set of all generators of $\Pi_1(N)$. Fix a point p in N and $p_0 \in \pi^{-1}(p)$. Let $p_i := \omega_i(p_0)$. The Dirichlet region $U_0 \subset \tilde{N}$ is defined as

$$U_0 := \{x \in \tilde{N} | d(p_0,x)<d(p_i,x) \text{ for all } i = 1,\ldots,k\}.$$

Let p_i be chosen such that $d(p_0,p_i) = d(p_0,\pi^{-1}(p)\backslash\{p_0\})$. If $r \in \overline{U_0} \bigcap \overline{U_i}$ satisfies

$$d(p_0,r) = d(p_i,r)<d(p_j,r) \text{ for all } j \neq i,0$$

then there is a small positive number h such that every point $q \in B(r,h) \bigcap \overline{U_0} \bigcap \overline{U_i}$ is the midpoint of a unique minimizing geodesic joining p_0 to p_i of length 2.

Furthermore, for every $i,j \neq 0$, one has

$$\overline{U}_0 \bigcap \overline{U}_i \bigcap \overline{U}_j = \emptyset.$$

Suppose there exists a point $q \in \overline{U}_0 \bigcap \overline{U}_i \bigcap \overline{U}_j$, then the minimizing geodesics joining p_0 to p_i and p_0 to p_j would bifurcate at the point q.

Let Σ_p' be the space of the set of geodesics with origin p endowed with a metric in which the distance is the angle between the geodesics at p. Its metric completion is called the space of direction at p and is denoted Σ_p [4]. Let $\tilde{\Sigma}_{p_0} := \{\xi \in \Sigma_{p_0} | \exp_{p_0} 2\xi = p_i\}$. Then the above argument implies that $\tilde{\Sigma}_{p_0}$ is open and closed in Σ_{p_0}. However, Σ_{p_0} is connected if dim $N = m \geq 2$, and hence $\Sigma_{p_0} = \tilde{\Sigma}_{p_0}$. This means that $\tilde{N}$ has only two fundamental domains and that $\Pi_1(N) = \mathbf{Z}_2$. Since every geodesic emanating from p has length 2, it follows that $\Sigma'_p = \Sigma_p$. From this we have that the cut locus to p_0 is a single point p_i and that Σ_p is isometric to the standard unit $m-1$ sphere, i.e. p is non-singular.

If $C_p \subset N$ is the cut locus to p, then

$$C(p) = \pi(\overline{U}_0 \bigcap \overline{U}_i) = S^{m-1}(1)\,/\,i,$$

where i is some free, topological involution. Thus N is homotopy equivalent to $\mathbf{RP}^m$.

The Theorem is now followed by applying Theorem 1.

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