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ON THE p-STRESS ENERGY TENSOR AND ITS APPLICATIONS

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Abstract. *We define the p-stress energy tensor and obtain a monotonicity inequality and Liouville-type theorem for p-harmonic maps.*

1 Introduction

Let $f : (M,g) \to (N,h)$ be a smooth map between Riemannian manifolds with metrics g and h respectively. Then its energy density $e(f) : M \to R$ is defined by

$$e(f) := \frac{1}{2}|df|^2,$$

where $|\cdot|$ denotes the Hilbert-Schmidt norm of the differential df of f, which is the differential 1-form with values in the induced vector bundle $f^{-1}TN(TN =$ the tangent bundle of $N)$ over M. A.H. Taub suggested that the stress energy tensor, which is defined by

$$S_f := e(f)g - f^*h$$

should be useful in the theory of harmonic maps(see [1]), where f^*h denotes a pull-back 2-tensor field by f, which is symmetric and semi-positive. Indeed, recent developments have confirmed Taub's prediction (cf.[1,4,5]).

In this note, we define the *p-stress energy tensor* of f and show that it is closely related to the theory of p-harmonic maps.

2 Preliminaries

Let (M,g) and (N,h) be Riemannian manifolds of dimension m and n respectively. For a smooth map $f : (M,g) \to (N,h)$ and each $p \in [2,\infty)$, the *p-energy density* $e_p(f)$ and the *p-energy* $E_p(f)$ of f are respectively defined by

$$e_p(f) := \frac{1}{p}|df|^p,$$

$$E_p(f) := \int_M e_p(f)\,dv_g,$$

where dv_g is the volume element of M. The p-energy $E_p(f)$ may be infinite, but when M is compact, it has to be finite. We call a symmetric 2-tensor $\tilde{S}_f$

$$\tilde{S}_f := e_p(f)g - |df|^{p-2}f^*h$$

the *p-stress energy tensor*, which is a natural generalization of the stress energy tensor S_f. A smooth map f is said to be *p-harmonic* if it is a critical point of p-energy functional, that is,

$$\left.\frac{dE_p(f_t)}{dt}\right|_{t=0} = 0$$

for any one-parameter family of maps $f_t : M \to N$ with $f_0 = f$. Note that 2-harmonic maps are harmonic maps by definition. We denote by ∇ and ${}^N\nabla$ the Levi-Civita connections of M and N respectively. Let $\tilde{\nabla}$ denotes the induced connection on the induced vector bundle $f^{-1}TN$ from ${}^N\nabla$ and f. For a local orthonormal frame field $\{e_i\}_{i=1}^m$ on M, we define the *p-tension field* $\tau_p(f)$ of f by

$$\begin{aligned} \tau_p(f) &:= \sum_{i=1}^m [\tilde{\nabla}_{e_i}(|df|^{p-2} df(e_i)) - |df|^{p-2} df(\nabla_{e_i} e_i)] \\ &= \sum_{i=1}^m [\bar{\nabla}_{e_i}(|df|^{p-2} df)](e_i), \end{aligned} \quad 2.1$$

where $\bar{\nabla}$ is the induced connection on the vector bundle $T^*M \otimes f^{-1}TN$. In the case of $p = 2$, $\tau_p(f)$ is nothing but the tension field $\tau(f)$. The first variation formula (cf.[3]) for a smooth map $f : M \to N$ is given by

$$\left.\frac{dE_p(f_t)}{dt}\right|_{t=0} = -\int_M h(V, \tau_p(f))\, dv_g,$$

where $V := \frac{df_t}{dt}|_{t=0}$ may be viewed as a vector field in N along f, that is, $V \in \Gamma(f^{-1}TN)$(=the set of smooth cross-sections of $f^{-1}TN$). Therefore a smooth map $f : M \to N$ is a p-harmonic map if and only if the p-tension field $\tau_p(f) = 0$. In the sequel we use the same notation ∇ to denote different connections on different bundles and use summation conventions, namely summing up repeated indices over the range of indices unless otherwise stated.

3 Some properties of the p-stress energy tensor

In this section, we calculate the divergence of $\tilde{S}_f$ and obtain two important integral formulas, which are useful to study monotonicity inequalities and Liouville-type theorems for p-harmonic maps. First of all, we have

Theorem 1 *The p-stress energy tensor $\tilde{S}_f$ of any smooth map $f : (M,g) \to (N,h)$ has divergence*

$$div\tilde{S}_f = - < \tau_p(f), df >, \quad 3.1$$

where $div\tilde{S}_f$ denotes the divergence of $\tilde{S}_f$.

Proof. Choose a local orthonormal frame field $\{e_i\}_{i=1}^m$ near an arbitrary point $x \in M$ with $\nabla_{e_i} e_j|_x = 0$. For any $X \in T_xM$ and at the point x

$$\begin{aligned}
(div\tilde{S}_f)(X) = & \ (\nabla_{e_i}\tilde{S}_f)(e_i, X) \\
= & \ \nabla_{e_i}(\tilde{S}_f)(e_i, X) - \tilde{S}_f(e_i, \nabla_{e_i} X) \\
= & \ \nabla_{e_i}\left[\tfrac{1}{p}|df|^p g(e_i, X) - |df|^{p-2} h(f_* e_i, f_* X)\right] \\
& - \left[\tfrac{1}{p}|df|^p g(e_i, \nabla_{e_i} X) - |df|^{p-2} h(f_* e_i, f_* \nabla_{e_i} X)\right] \\
= & \ |df|^{p-2} h(\nabla_X f_* e_i, f_* e_i) - (\nabla_{e_i}|df|^{p-2}) h(f_* e_i, f_* X) \\
& - |df|^{p-2} h(\nabla_{e_i} f_* e_i, f_* X) + |df|^{p-2} h(f_* e_i, f_* \nabla_{e_i} X) \\
& - |df|^{p-2} h\Big(f_* e_i, (\nabla_{e_i} df)(X) + f_* \nabla_{e_i} X\Big) \\
= & \ -h\Big((\nabla_{e_i}|df|^{p-2}) f_* e_i + |df|^{p-2} \nabla_{e_i} f_* e_i, f_* X)\Big) \\
= & \ -h\Big(\nabla_{e_i}(|df|^{p-2} df)(e_i), f_* X\Big) \\
= & \ -h(\tau_p(f), df(X)).
\end{aligned}$$

□

The following Corollary 3.2 and 3.3 follows immediately from Theorem 3.1.

Corollary 2 *Any p-harmonic map satisfies the p-conservation law ,i.e.,* $div\tilde{S}_f = 0$.

Corollary 3 *Let* $f : (M,g) \to (N,h)$ *be a Riemannian submersion. Then* f *satisfies the p-conservation law if and only if* f *is p-harmonic map.*

A smooth map $f : (M,g) \to (N,h)$ is called a *weakly conformal* if there exists a function λ on M such that $f^*h = \lambda^2 g$. In case of λ being constant, f is called a *homothetic map*.

Proposition 4 *Suppose* $f : (M,g) \to (N,h)$ *is a smooth map with rank one at least. Then* $\tilde{S}_f \equiv 0$ *if and only if* $\dim M = m = p$ *and* f *is weakly conformal.*

Proof. Assume that $\tilde{S}_f \equiv 0$. Then we have

$$\begin{aligned}
\tilde{S}_f \equiv 0 \quad & \Rightarrow \tfrac{1}{p}|df|^p g = |df|^{p-2} f^* h \\
& \Rightarrow \tfrac{m}{p}|df|^p = |df|^p \\
& \Rightarrow \tfrac{m-p}{p}|df|^p = 0.
\end{aligned}$$

This means that $m = p$. Also we have from $\tilde{S}_f \equiv 0$

$$\frac{1}{p}|df|^2 g = f^* h,$$

which implies that f is weakly conformal by putting $\frac{1}{p}|df|^2 =: \lambda^2$. Conversely, assume that $f^*h = \lambda^2 g$ for some function λ on M. Then we obtain $\lambda^2 = \frac{|df|^2}{m}$. So, we get $f^*h = \frac{|df|^2}{m} g$. Substituting these into $\tilde{S}_f$, we have

$$\begin{aligned}
\tilde{S}_f \ & = \tfrac{1}{p}|df|^p g - |df|^{p-2} f^* h \\
& = \tfrac{1}{p} m^{\frac{p-2}{2}} \lambda^p (m-p) g.
\end{aligned} \qquad 3.2$$

Therefore $\tilde{S}_f = 0$ if $m = p$. □

Proposition 5 *If* $\dim M = m > p$ *and* $f : (M,g) \to (N,h)$ *is a weakly conformal map, then* f *is homothetic if and only if it satisfies the p-conservation law (i.e.,* $div\tilde{S}_f \equiv 0$*).*

Proof. Choose a local orthonormal frame field $\{e_i\}_{i=1}^m$ near an arbitrary point $x \in M$ with $\nabla_{e_i} e_j|_x = 0$. Under the assumption that f is weakly conformal, the equation (3.2) holds. If f is homothetic(i.e., λ is constant), then the divergence of $\tilde{S}_f$ at x is given by

$$(div\tilde{S}_f)(X) = \frac{1}{p} m^{\frac{p-2}{2}} \lambda^p (m-p)(\nabla_{e_i} g)(e_i, X) = 0$$

for any tangent vector $X \in T_xM$. That is, f satisfies the p- conservation law. Conversely, we also obtain from (3.2)

$$\begin{aligned} 0 &= (div\tilde{S}_f)(X) = \tfrac{1}{p} m^{\frac{p-2}{2}} (m-p) g(e_i, X) \nabla_{e_i} \lambda^p \\ &= \tfrac{1}{p} m^{\frac{p-2}{2}} (m-p) \nabla_X \lambda^p, \end{aligned}$$

which gives $\nabla_X \lambda^p = 0$. Thus λ is constant. □

Proposition 6 *If the support of a vector field X on M* $(=: supp(X))$ *is compact, then*

$$\int_M (div\tilde{S}_f)(X)\, dv_g + \int_M < \tilde{S}_f, \nabla X > dv_g = 0. \qquad 3.3$$

Furthermore, if D is a compact domain in M with its smooth boundary ∂D*, then*

$$\begin{aligned} \int_{\partial D} \; & e_p(f) g(X,n)\, dv_g = \int_{\partial D} |df|^{p-2} h(f_*X, f_*n)\, dv_g \\ & + \int_D (div\tilde{S}_f)(X)\, dv_g + \int_D < \tilde{S}_f, \nabla X > dv_g, \end{aligned} \qquad 3.4$$

where **n** *is a unit vector field normal to the hypersuface* ∂D *of D,* ∇X *is a 2-tensor field defined by* $\nabla X(Y,Z) := g(\nabla_Y X, Z)$ *for any vector fields Y and Z on M and* $< \cdot, \cdot >$ *denotes the inner product on 2-tensor fields.*

Proof. Let X be any vector field on M. Then we have

$$\begin{aligned} div(e_p(f)X) &= (\nabla_{e_i} e_p(f)) g(X, e_i) + e_p(f) g(\nabla_{e_i} X, e_i) \\ &= \nabla_X e_p(f) + e_p(f) < \nabla X, g >, \end{aligned} \qquad 3.5$$

where $\{e_i\}_{i=1}^m$ is a local orthonormal frame field near a point $x \in M$ with $\nabla_{e_i} e_j|_x = 0$. Note that at x

$$\begin{aligned} \nabla_X e_p(f) = \; & |df|^{p-2} h(\nabla_X f_* e_i, f_* e_i) \\ = \; & |df|^{p-2} h((\nabla_{e_i} df)(X), f_* e_i) \\ = \; & |df|^{p-2} [\nabla_{e_i} h(f_*X, f_*e_i) - h(f_*X, \nabla_{e_i} f_* e_i) \\ & - g(\nabla_{e_i} X, e_j) h(f_* e_j, f_* e_i)] \\ = \; & |df|^{p-2} [div(h(f_*X, f_*e_i)e_i) - h(f_*X, \tau(f)) \\ & - < \nabla X, f^*h >], \end{aligned} \qquad 3.6$$

where $\tau(f)$ denotes the tension field of f.

Substituting (3.6) into (3.5), we obtain

$$\begin{aligned} div(e_p(f)X) = & \; |df|^{p-2} div(h(f_*X, f_*e_i)e_i) - |df|^{p-2} h(f_*X, \tau(f)) \\ & + < \nabla X, \tfrac{1}{p}|df|^p g - |df|^{p-2} f^*h > \\ = & \; |df|^{p-2} div(h(f_*X, f_*e_i)e_i) - |df|^{p-2} h(f_*X, \tau(f)) \\ & + < \nabla X, \tilde{S}_f > \\ = & \; |df|^{p-2} div(h(f_*X, f_*e_i)e_i) - h(f_*X, \nabla_{e_i}(|df|^{p-2} df)(e_i)) \\ & + \nabla_{e_i} |df|^{p-2} h(f_*X, f_*e_i) + < \nabla X, \tilde{S}_f > \\ = & \; div(|df|^{p-2} h(f_*X, f_*e_i)e_i) - h(f_*X, \tau_p(f)) \\ & + < \nabla X, \tilde{S}_f > . \end{aligned} \tag{3.7}$$

If $supp(X)$ is compact, integrating both sides of (3.7), then using Green's theorem and Theorem 3.1 we have the integral formula (3.3).

For the proof of the second formula (3.4) we take a local orthonormal frame field $\{e_i\}_{i=1}^m$ of M along ∂D such that $e_1, \cdots, e_{m-1}$ are tangent to ∂D and $\mathbf{n} = e_m$ is normal to ∂D. By Green's theorem

$$\int_D divX \, dv_g = \int_{\partial D} g(X, \mathbf{n}) \, dv_g,$$

integrating (3.7) again over D gives the formula (3.4). □

4 A monotonicity inequality and Liouville-type theorem for p-harmonic maps

In this section, using the integral formulas (3.3) and (3.4), we shall prove a monotonicity inequality and Liouville-type theorem for p-harmonic maps. The proofs are based on those for harmonic maps due to Y.L.Xi([4,5]).

Theorem 7 *Let (M,g) be an m-dimensional Riemannian manifold, and $B_\sigma(x)$ its geodesic ball with radius σ and centered at $x \in M$. Suppose that the distance from a point $x_0 \in M$ to its cut locus and the boundary of M is at least one. If $f : (M,g) \to (N,h)$ is a p-harmonic map, then for any $x \in B_{\frac{1}{2}}(x_0)$ and $0 < \sigma \le \rho \le \frac{1}{2}$*

$$e^{C\Lambda\sigma} \sigma^{p-m} \int_{B_\sigma(x)} e_p(f) dv_g \le e^{C\Lambda\rho} \rho^{p-m} \int_{B_\rho(x)} e_p(f) dv_g, \tag{4.1}$$

where C is a constant depending on m and Λ is a constant depending only on the bounds of the sectional curvature in $B_1(x_0)$.

Proof. Let r be the distance function in $B_{\frac{1}{2}}(x)$ from x, and $\frac{\partial}{\partial r}$ the unit radial vector field. Let

$$X = \xi(r) r \frac{\partial}{\partial r},$$

where $\xi(r)$ will be chosen later. Let us derive (4.1) by using (3.3).

Choose a local orthonormal frame field $\{e_\alpha, \frac{\partial}{\partial r}\} (\alpha = 1, \cdots, m-1)$. Then

$$\nabla_{\frac{\partial}{\partial r}} X = (\xi' r + \xi) \frac{\partial}{\partial r},$$

and

$$\begin{aligned}\nabla_{e_\alpha} X &= \xi r \nabla_{e_\alpha} \tfrac{\partial}{\partial r}\\ &= \xi r\Big[g\Big(\nabla_{e_\alpha}\tfrac{\partial}{\partial r},\tfrac{\partial}{\partial r}\Big)\tfrac{\partial}{\partial r} + g\Big(\nabla_{e_\alpha}\tfrac{\partial}{\partial r}, e_\beta\Big)e_\beta\Big]\\ &= \xi r Hess(r)(e_\alpha, e_\beta)e_\beta,\end{aligned}$$

where Hess$(\cdot)$ stands for the Hessian operator.

If the sectional curvature in $B_1(x_0)$ lies in $[a,b]$, then, by using Hessian comparison theorem, i.e.,

$$\sqrt{|b|}F(\sqrt{|b|}r) \le Hess(r)(e_\alpha, e_\alpha) \le \sqrt{|a|}F(\sqrt{|a|}r) \qquad 4.2$$

for each α, where

$$\sqrt{|c|}rF\Big(\sqrt{|c|}r\Big) = \begin{cases} \sqrt{c}r\cot(\sqrt{c}r), & c>0,\\ 1, & c=0,\\ \sqrt{-c}r\coth(\sqrt{-c}r), & c<0,\end{cases}$$

and

$$\Big|\sqrt{|c|}rF\Big(\sqrt{|c|}r\Big) - 1\Big| \le r\Lambda, \qquad 4.3$$

we have

$$\begin{aligned} divX &= g\Big(\nabla_{\frac{\partial}{\partial r}}X, \tfrac{\partial}{\partial r}\Big) + g(\nabla_{e_\alpha}X, e_\alpha)\\ &\ge \xi' r + \xi + \xi(m-1)\sqrt{|b|}rF(\sqrt{|b|}r),\end{aligned} \qquad 4.4$$

and

$$\begin{aligned} < f^*h, \nabla X > =&\ h(f_*e_\alpha, f_*e_\beta)g(\nabla_{e_\alpha}X, e_\beta)\\ &+h\Big(f_*\tfrac{\partial}{\partial r}, f_*\tfrac{\partial}{\partial r}\Big)g\Big(\nabla_{\frac{\partial}{\partial r}}X, \tfrac{\partial}{\partial r}\Big)\\ =&\ \xi r Hess(r)(e_\alpha, e_\beta)h(f_*e_\alpha, f_*e_\beta)\\ &+(\xi' r + \xi)h\Big(f_*\tfrac{\partial}{\partial r}, f_*\tfrac{\partial}{\partial r}\Big)\\ \le&\ \xi r h(f_*e_\alpha, f_*e_\alpha)\sqrt{|a|}F(\sqrt{|a|}r)\\ &+(\xi' r + \xi)h\Big(f_*\tfrac{\partial}{\partial r}, f_*\tfrac{\partial}{\partial r}\Big)\\ =&\ \xi r|df|^2\sqrt{|a|}F(\sqrt{|a|}r) + \xi' r h\Big(f_*\tfrac{\partial}{\partial r}, f_*\tfrac{\partial}{\partial r}\Big)\\ &+\xi\Big\{1 - r\sqrt{|a|}F(\sqrt{|a|}r)\Big\}h\Big(f_*\tfrac{\partial}{\partial r}, f_*\tfrac{\partial}{\partial r}\Big).\end{aligned} \qquad 4.5$$

On the other hand, the formula (3.3) and the definition of $\tilde{S}_f$ show that

$$\begin{aligned} 0 =&\ \int_M < \tilde{S}_f, \nabla X > dv_g\\ =&\ \int_M \tfrac{1}{p}|df|^p < g, \nabla X > dv_g\\ &- \int_M |df|^{p-2} < f^*h, \nabla X > dv_g.\end{aligned} \qquad 4.6$$

The first equality follows from the assumption that f is p-harmonic.

Substituting (4.4) and (4.5) into (4.6) yields

$$\begin{aligned} 0 \ge&\ \int_M \Big[|df|^p\xi' r - (p-m)|df|^p\xi + |df|^p\xi(m-1)\{\sqrt{|b|}rF(\sqrt{|b|}r) - 1\}\\ &+p\xi\{1 - \sqrt{|a|}rF(\sqrt{|a|}r)\}|df|^p - p\xi' r\Big|f_*\tfrac{\partial}{\partial r}\Big|^2|df|^{p-2}\\ &-p\xi\{1 - \sqrt{|a|}rF(\sqrt{|a|}r)\}\Big|f_*\tfrac{\partial}{\partial r}\Big|^2|df|^{p-2}\Big]dv_g.\end{aligned}$$

Noting (4.3), we have

$$\begin{aligned}&-\int_M |df|^p\xi' r dv_g+(p-m)\int_M |df|^p\xi dv_g\\&+(m+p-1)\Lambda\int_M |df|^p r\xi dv_g \geq -\int_M p\xi' r\Big|f_*\tfrac{\partial}{\partial r}\Big|^2|df|^{p-2}dv_g\\&-\int_M p\xi\{1-\sqrt{|a|}rF(\sqrt{|a|}r)\}\Big|f_*\tfrac{\partial}{\partial r}\Big|^2|df|^{p-2}dv_g.\end{aligned} \tag{4.7}$$

Choose a smooth function

$$\phi(t)=\begin{cases}1, & \text{for } t\in[0,1],\\ 0, & \text{for } t\in[1+\varepsilon,\infty),\end{cases}$$

and $\phi'\leq 0$, where $\varepsilon>0$ is a sufficently small number. For $r\in[\sigma,\rho]$ set

$$\xi(r)=\xi_\tau(r)=\phi\Big(\frac{r}{\tau}\Big).$$

Then

$$\tau\frac{\partial}{\partial\tau}(\xi_\tau(r))=-r\xi_\tau'(r),\ \xi_\tau'=\phi'\Big(\frac{r}{\tau}\Big)\frac{1}{\tau}$$

and

$$\xi_\tau r=\phi\Big(\frac{r}{\tau}\Big)r\leq\phi\Big(\frac{r}{\tau}\Big)\tau(1+\varepsilon)=\xi_\tau(1+\varepsilon)\tau.$$

Substituting these formulas into (4.7) gives

$$\tau\frac{\partial}{\partial\tau}\int_M\xi_\tau|df|^p dv_g+(p-m)\int_M\xi_\tau|df|^p dv_g+C\tau\Lambda\int_M\xi\tau|df|^p dv_g\geq 0,$$

where $C=(1+\varepsilon)(m+p-1)$, which means that

$$\frac{\partial}{\partial\tau}\Big(e^{C\Lambda\tau}\tau^{p-m}\int_M\xi_\tau|df|^p dv_g\Big)\geq 0.$$

Thus we complete the proof. □

Theorem 8 *Let M be a Cartan-Hadamard manifold (i.e., complete simply-connected Riemannian manifold of nonpositive sectional curvature) whose sectional curvature varies in a small range (the precise range will be seen in the proof), and f a p-harmonic map from M into any Riemannian manifold N with finite p-energy. If $\dim M=m>p$, then f has to be constant.*

Proof. Let $D=B_R(x_0)$ be a geodesic ball of radius R and centered at x_0. Its boundary $\partial B_R(x_0)$ is the geodesic sphere. Obviously, the square of the distance function from x_0 in $B_R(x_0)$ is smooth. Let $\frac{\partial}{\partial r}$ denote the unit radial vector field which is also the unit normal vector field $\mathbf{n}$ to $\partial B_R(x_0)$. Choosing $X=r\frac{\partial}{\partial r}$ in the formula (3.4), we have

$$\begin{aligned}&\int_{\partial B_R(x_0)}e_p(f)g(X,\mathbf{n})\,dv_g-\int_{\partial B_R(x_0)}|df|^{p-2}h(f_*X,f_*\mathbf{n})dv_g\\=\ &\int_{\partial B_R(x_0)}Re_p(f)\,dv_g-\int_{\partial B_R(x_0)}R|df|^{p-2}h\Big(f_*\tfrac{\partial}{\partial r},f_*\tfrac{\partial}{\partial r}\Big)\,dv_g\\\leq\ &\int_{\partial B_R(x_0)}Re_p(f)\,dv_g.\end{aligned} \tag{4.8}$$

On the other hand,

$$\begin{aligned}\nabla_{\frac{\partial}{\partial r}} X &= \frac{\partial}{\partial r},\\ \nabla_{e_\alpha} X &= r\nabla_{e_\alpha}\frac{\partial}{\partial r}\\ &= rHess(r)(e_\alpha, e_\beta)e_\beta,\\ divX &= 1 + rHess(r)(e_\alpha, e_\alpha),\end{aligned}$$

where $\{e_\alpha, \frac{\partial}{\partial r}\}_{\alpha=1}^{m-1}$ is a local orthonormal frame field on $B_R(x_0)$. Thus,

$$\begin{aligned}&h(f_*e_s, f_*e_t)g(\nabla_{e_s}X, e_t)\\ &= rHess(r)(e_\alpha, e_\beta)h(f_*e_\alpha, f_*e_\beta) + h\Big(f_*\frac{\partial}{\partial r}, f_*\frac{\partial}{\partial r}\Big),\end{aligned}$$

and

$$\begin{aligned}< \tilde{S}_f, \nabla X > =& < e_p(f)g - |df|^{p-2}f^*h, \nabla X >\\ =& e_p(f) < g, \nabla X > -|df|^{p-2} < f^*h, \nabla X >\\ =& e_p(f)divX - |df|^{p-2}h(f_*e_s, f_*e_t)g(\nabla_{e_s}X, e_t)\\ =& e_p(f)[1 + Hess(r)(e_\alpha, e_\alpha)]\\ & -|df|^{p-2}\Big[rHess(r)(e_\alpha, e_\beta)h(f_*e_\alpha, f_*e_\beta) + \Big|f_*\frac{\partial}{\partial r}\Big|^2\Big],\end{aligned} \tag{4.9}$$

where $s,t \in \{1,\cdots,m\}$ and $e_m = \frac{\partial}{\partial r}$.

We consider cases when the sectional curvature K of the domain manifold satisfies one of the following conditions;

(1) $-a^2 \le K \le -b^2 < 0$, a,b are positive constant,
(2) $-\frac{A}{1+r^2} \le K \le 0$, A is another constant.

Case (1). By using Hessian comparison theorem (4.2), (4.9) becomes

$$\begin{aligned}< \tilde{S}_f, \nabla X > \ge & \frac{1}{p}|df|^p\Big[1 + (m-1)(br)\coth(br)\Big]\\ & -|df|^{p-2}\Big[\Big|f_*\frac{\partial}{\partial r}\Big|^2 + (ar)\coth(ar)h(f_*e_\alpha, f_*e_\alpha)\Big]\\ = & \frac{1}{p}|df|^{p-2}\Big[|df|^2\{1 + (m-1)(br)\coth(br)\} - p\Big|f_*\frac{\partial}{\partial r}\Big|^2\\ & -p(ar)\coth(ar)h(f_*e_\alpha, f_*e_\alpha)\Big]\\ = & \frac{1}{p}|df|^{p-2}\Big[-p(ar)\coth(ar)h(f_*e_\alpha, f_*e_\alpha) - p\Big|f_*\frac{\partial}{\partial r}\Big|^2\\ & +\Big\{h(f_*e_\alpha, f_*e_\alpha) + \Big|f_*\frac{\partial}{\partial r}\Big|^2\Big\}\{1 + (m-1)(br)\coth(br)\}\Big]\\ = & \frac{1}{p}|df|^{p-2}\Big[\{1 + (m-1)(br)\coth(br)\\ & -p(ar)\coth(ar)\}h(f_*e_\alpha, f_*e_\alpha)\\ & +\{1 + (m-1)(br)\coth(br) - p\}\Big|f_*\frac{\partial}{\partial r}\Big|^2\Big]\\ \ge & \frac{1}{p}|df|^{p-2}\Big[\{1 + r\coth(br)(b(m-1) - pa)\}h(f_*e_\alpha, f_*e_\alpha)\\ & +(m-p)\Big|f_*\frac{\partial}{\partial r}\Big|^2\Big]\\ \ge & \frac{\delta}{p}|df|^p\end{aligned} \tag{4.11}$$

where $\delta > 0$, provided $b(m-1) - pa \geq 0$. Thus we have

$$\delta \int_{B_R(x_0)} e_p(f) dv_g \leq \int_{B_R(x_0)} < \nabla X, \tilde{S}_f > dv_g.$$

Therefore combining this with (3.4) and (4.8) gives

$$R \int_{\partial B_R(x_0)} e_p(f) dv_g \geq \delta \int_{B_R(x_0)} e_p(f) dv_g.$$

If the p-energy density $e_p(f)$ does not vanish identically, then there exists $R_0 > 0$ such that for $R > R_0$,

$$\int_{B_R(x_0)} e_p(f) dv_g \geq C,$$

where C is a positive constant. Hence

$$\int_{\partial B_R(x_0)} e_p(f) dv_g \geq \frac{\delta C}{R}.$$

This implies that

$$\begin{aligned} \int_M e_p(f) dv_g &= \int_0^\infty dr \int_{\partial B_r(x_0)} e_p(f) dv_g \\ &\geq \int_{R_0}^\infty dr \int_{\partial B_r(x_0)} e_p(f) dv_g \\ &\geq \int_{R_0}^\infty \frac{\delta C}{r} dr \to \infty, \end{aligned}$$

which contradicts the finiteness of the p-energy. Therefore f has to be constant.

Case(2). If the sectional curvature K satisfies $K \geq -\frac{A}{1+r^2}$, then, by Hessian comparison theorem (see [2]), i.e.,

$$\frac{1}{r}(g - dr \otimes dr) \leq Hess(r) \leq \frac{\beta}{r}(g - dr \otimes dr),$$

where $\beta = \frac{1}{2} + \frac{1}{2}(1+4A)^{\frac{1}{2}}$, we get

$$\begin{aligned} < \tilde{S}_f, \nabla X > &\geq \tfrac{m}{p}|df|^p - |df|^{p-2}\left|f_* \tfrac{\partial}{\partial r}\right|^2 - \beta |df|^{p-2} h(f_* e_\alpha, f_* e_\alpha) \\ &= \tfrac{1}{p}|df|^{p-2}\left[m|df|^2 - p\left|f_* \tfrac{\partial}{\partial r}\right|^2 - p\beta h(f_* e_\alpha, f_* e_\alpha)\right] \\ &= \tfrac{1}{p}|df|^{p-2}\left[(m-p)\left|f_* \tfrac{\partial}{\partial r}\right|^2 + (m - p\beta) h(f_* e_\alpha, f_* e_\alpha)\right] \\ &\geq \delta e_p(f) \end{aligned}$$

for some constant $\delta > 0$. Hence, for the proof of the rest part, we can argue as that of Case (1).

Therefore we complete the proof. □

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