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SOME BOUNDS FOR THE GENUS OF $M^n \times I$[1]

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Abstract. *Starting from a (i,j)-symmetric crystallization $(\Gamma_{M^n},\gamma_{M^n})$ of a closed n-manifold M^n, we give an algorithm to build a crystallization $(\Gamma^A_{M^n},\gamma^A_{M^n})$ of $M^n \times I$. This algorithm allows to give a formula for the calculation of the regular genus of $(\Gamma^A_{M^n},\gamma^A_{M^n})$, in the cases $n = 2,3$, and some bounds for the genus of the product-manifolds represented.*

1 Introduction

This paper comes into the study of the polyhedra, represented in a combinatory way. In particular relations between the genus of the product and the genus of its factors are wished to be obtained. This problem also is of interest in the Poincaré Conjecture. In fact, as seen in [11], the following conjecture implies the (3-dimensional) Poincaré Conjecture.

Conjecture I.

$$\begin{array}{l} (a)\ \ G(S^3 \times S^2) = 8; \\ (b)\ \textit{for every closed orientable } 3-\textit{manifold } M^3 \\ \quad (b')\ \ G(M^3 \times S^2) \geq 8 \\ \quad (b'')\ G(M^3 \times S^2) = 8 \textit{ iff } M^3 = S^3. \end{array}$$

In [4] it is proved (a) and (b') and this result has suggested a new conjecture that implies the Poincaré Conjecture, for all 3-manifold M^3 with Heegard genus $H(M^3) \geq 5$.

Conjecture II. *For every closed connected orientable 3-manifold M^3, denoting by D^n the n-disk: $G(M^3 \times S^n) \geq G(M^3 \times D^n)$.*

To give a contribute in the investigation of conjecture II , in the case $n = 1$, we introduce an algorithm to construct the product between "special" contracted triangulations of n-manifolds and $D^1 = I$. Moreover we describe how to represent such products by means of coloured graphs. The obtained relations between the genus of the product-graph and the genus of the crystallization of the n-manifold M^n allows to find bounds for the genus of the product $M^n \times I$.

We conclude the paper with bounds for the genus of some products of 3-manifolds for I and with the evaluation of the genus of $RP^3 \times I$.

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2 Definitions and notations

In this paper we shall work with manifolds and maps in the PL-category, for which we refer to [15] and [10]. For graph theory, see [12] and [16]. All considered manifolds are supposed to be compact and connected unless otherwise stated.

Let $\Gamma = (V(\Gamma), E(\Gamma))$ be a finite multigraph (multiple edges are allowed, but loops are forbidden) and $\gamma: E(\Gamma) \to \Delta_n = \{i \in Z \mid 0 \leq i \leq n\}$ be a proper edge-colouring of Γ (i.e. $\gamma(e) \neq \gamma(f)$ for any two adjacent edges $e, f \in E(\Gamma)$),then the pair (Γ, γ) is a $(n+1)$*-coloured graph with boundary* and Δ_n its *colour-set.*

If $v \in V(\Gamma)$ has degree equal to $n+1$ (resp. strictly less than n+1), then v is an *internal vertex* (resp. a *boundary-vertex*). Let $\overset{\circ}{p}$ (resp. $\bar{p}$) be the number of internal vertices (resp. boundary-vertices) of Γ; if $\bar{p} = 0$, then Γ is simply called an $(n+1)$*-coloured graph.*

For each $B \subseteq \Delta_n$, we set $\Gamma_B = (V(\Gamma), \gamma^{-1}(B))$. The connected components of Γ_B are the *B-residues* of Γ and, if B has cardinality r, we say that they are *r-residues.* If $c \in \Delta_n$, set $\hat{c} = \Delta_n - \{c\}$. Γ is *regular with respect to the colour c* iff $\Gamma_{\hat{c}}$ is a regular graph of degree n. Let G_{n+1} denote the class of all graphs regular with respect to the colour n.

Let $\Gamma \in G_{n+1}$; we define the *boundary-graph* $(\partial\Gamma, {}^{\partial}\gamma)$ of Γ as follows :

i) $V(\partial\Gamma)$ is the set of the boundary-vertices of Γ;

ii) for each $u, v \in V(\partial\Gamma)$, join u and v by an edge e iff u and v lie on the same $\{i, n\}$-residue of Γ and set ${}^{\partial}\gamma(e) = i$.

Obviously, if $\bar{p} = 0$, then $\partial\Gamma = \emptyset$.

Γ is called *contracted* (resp. *∂-contracted*) if $\partial\Gamma = \emptyset$ (resp. Γ has non-empty boundary-graph) and $\Gamma_{\hat{c}}$ is connected, for each $c \in \Delta_n$ (resp. $\Gamma_{\hat{n}}$ is connected and, for each $c \neq n$, $\Gamma_{\hat{c}}$ has as many connected components as $\partial\Gamma$).

$(\Gamma, \gamma), (\Gamma', \gamma') \in G_{n+1}$ are *colour-isomorphic* iff there exists a triple (h, k, ψ) of bijections $h: V(\Gamma) \to V(\Gamma')$, $k: E(\Gamma) \to E(\Gamma')$, $\psi: \Delta_n \to \Delta_n$ such that k preserves the incidence and $\gamma' \circ k = \psi \circ \gamma$. A coloured n-complex $(K(\Gamma), \varphi)$ (see [6] and its bibliography) can be associated with any $(n+1)$-coloured graph with boundary (Γ, γ) by the following rules:

1) for each vertex $v \in V(\Gamma)$, take an n-simplex $\sigma(v)$ and label its vertices by $0, 1, \ldots, n$;

2) for each edge $e \in E(\Gamma)$ with endpoints $v, w \in V(\Gamma)$ and colour $\gamma(e) = i$, identify the $(n-1)$-faces of $\sigma(v)$ and $\sigma(w)$ opposite to the vertex labelled by i, so that equally labelled vertices are identified together.

The graph Γ will be said to *represent* $K(\Gamma)$, its space $|K(\Gamma)|$ and every homeomorphic polyhedron. This construction can be inverted in order to obtain an $(n+1)$-coloured graph with boundary $(\Gamma(K), \gamma)$ starting from a coloured n-complex (K, φ). Note that $\Gamma(K)$ represents K iff $K(\Gamma(K)) = K$ and that this condition is satisfied iff for each $\sigma \in K$ the disjont star $Std(\sigma, K)$ is strongly connected [7].

In particular a manifold M can be represented by the graph $\Gamma(K)$, for each triangulation K of M; moreover every connected component of the boundary graph $\partial\Gamma(K)$ represents a connected component of the boundary manifold ∂M.

A coloured n-complex K is said *contracted* (resp. *∂-contracted*) if it has exactly $n+1$ vertices (resp. one internal vertex coloured n and one vertex coloured c, for all $c \in \Delta_{n-1}$, on each connected component of ∂K).

A *crystallization* of a closed n-manifold (resp. n-manifold with boundary) M^n is a contracted (resp. ∂-contracted) (n+1)-coloured graph $\Gamma \in G_{n+1}$ representing M^n. The contractedness (resp. ∂-contractedness) of Γ corresponds to the same condition for $K(\Gamma)$. For existence and characterization theorems on crystallizations see [6].

From now on we shall assume the following notations: for each $r,s \in \Delta_n$ (resp. $r,s \in \Delta_{n-1}$), $\mathring{g}_{rs}$ (resp. ${}^{\partial}g_{rs}$) denotes the number of cycles of Γ (resp. of $\partial\Gamma$), with edges alternatively coloured r and s.

If Γ is a regular graph of degree $n+1$, then $\partial\Gamma = \emptyset$; in this case all cycles (r,s)-coloured are in Γ and their number is simply denoted by g_{rs}.

The *regular genus* $\rho(\Gamma)$ of a $(n+1)$-coloured graph Γ is defined in [9] as the least integer for which exists a regular embedding of Γ into a closed surface of genus $\rho(\Gamma)$. In [8] it is generalized the concept of regular genus for each $\Gamma \in G_{n+1}$, by means of a graph Γ^* for which there is a permutation $\varepsilon = (\varepsilon_0, \ldots, \varepsilon_{n-1}, \varepsilon_n = n)$ of Δ_n, such that Γ^* embeds in the orientable (resp. nonorientable) surface F_ε with $\lambda_\varepsilon(\Gamma^*) =^{\partial} g_{\varepsilon_0\varepsilon_{n-1}}$ holes and having Euler-characteristic

$$\chi_\varepsilon(\Gamma^*) = \sum_{j \in Z_{n+1}} \mathring{g}_{\varepsilon_j\varepsilon_{j+1}} + (1-n)\,\mathring{p}\ /2 + (2-n)\bar{p}/2.$$

Hence F_ε has genus $\rho_\varepsilon(\Gamma) = 1 - \chi_\varepsilon(\Gamma^*)/2 - \lambda_\varepsilon(\Gamma^*)/2$, if Γ is bipartite (resp. $\rho_\varepsilon(\Gamma) = 2 - \chi_\varepsilon(\Gamma^*) - \lambda_\varepsilon(\Gamma^*)$, if Γ is non-bipartite). Then the regular genus of Γ is $\rho(\Gamma) = \min_\varepsilon \rho_\varepsilon(\Gamma)$.

If Γ is a connected graph, its (reduced) genus is $\ddot{\rho}_\varepsilon(\Gamma) = \rho_\varepsilon(\Gamma)$, if Γ is bipartite (resp. $\ddot{\rho}_\varepsilon = \rho_\varepsilon(\Gamma)/2$, if Γ is non-bipartite); whereas if Γ is disconnected with components $\Gamma^{(1)}, \ldots, \Gamma^{(r)}$, we set

$$\ddot{\rho}_\varepsilon(\Gamma) = \sum_{i=1}^{r} \ddot{\rho}_\varepsilon(\Gamma^{(i)}).$$

In [2] it is proved that for each cyclic permutation $\varepsilon = (\varepsilon_0, \ldots, \varepsilon_{n-1}, \varepsilon_n = n)$ of Δ_n, $\ddot{\rho}_\varepsilon(\Gamma) \geq \ddot{\rho}_{\varepsilon'}(\partial\Gamma)$, where $\varepsilon' = (\varepsilon_0, \ldots, \varepsilon_{n-1})$ and moreover one define $\ddot{\rho}_\varepsilon(\partial\Gamma) = -1$ when $\partial\Gamma = \emptyset$.

We can make similar considerations for the *regular genus* of a n-manifold M^n, that is defined as follows:

$$G(M^n) = \min\{\rho(\Gamma) \mid \Gamma \textit{ represents } M^n\}.$$

In the case of $\partial M^n = \emptyset$, it is easy to show that

$$G(M^n) = \min\{\rho(\Gamma) \mid \Gamma \textit{ is a crystallization of } M^n\}.$$

This relation can be extended to manifolds with boundary in dimension less or equal than 3 [1]. As in the case of graphs we introduce the notion of reduced genus for n-manifolds by setting $\ddot{G}(M^n) = G(M^n)$ (resp. $G(M^n)/2$) if M^n is orientable (resp. nonorientable).

In [3] it is proved that each 4-manifolds, with $C \geq 1$ connected components of boundary, such that $G(M^4) = G(\partial M^4)$ (resp. $G(M^4) = G(M^4) + 1$) is homeomorphic to the connected sum of a suitable number of orientable or non orientable handlebodies (resp. of a suitable number of orientable or non orientable handlebodies and either $S^3 \times S^1$ or $S^3 \tilde{\times} S^1$). In this case we call M^4 of *type I* (resp. of *type II*).

3 The crystallization $\Gamma^A_{M^n}$

Definition 3.1. An $(n+1)$-coloured graph Γ will be called (i,j)*-symmetric*, (where $i,j \in \Delta_n, i \neq j$,) iff the subgraphs $\Gamma_{\hat{i}}$ and $\Gamma_{\hat{j}}$ are colour-isomorphic.

Note that if Γ is an (i,j)-symmetric crystallization of an n-manifold, and $K = K(\Gamma)$ is the associated complex, endowed by the vertex-colouring induced by γ, then there exists a colour-isomorphism between the disjoint links of the two vertices v_i and v_j of K, respectively coloured i and j.

We can assume that M^n is a closed n-manifold which admits a $(0,n)$-symmetric crystallization $(\Gamma_{M^n}, \gamma_{M^n})$. Let p be the order of $(\Gamma_{M^n}, \gamma_{M^n})$ and (h,k,δ) be the colour-isomorphism between $\Gamma_{\hat{0}}$ and $\Gamma_{\hat{n}}$.

We define Γ'_{M^n} be the graph isomorphic to Γ_{M^n}, by means of the colour-isomorphism (h',k',δ'), where $h' = h$ and

$$\begin{array}{lll} \delta' : \Delta_n & \to \Delta_n & \\ \quad i & \mapsto i+1 & \quad \text{with } (\mathrm{i}+1) \bmod n \end{array}$$

We set $\alpha'_i = h(\alpha_i)$. For each colour $d \in \Delta_n$, we denote with i_d the index of the vertex d-adjacent to the vertex α_i in Γ_{M^n}. Let A the following matrix[2] in which each row represents a n-simplex of a triangulation of the product of a $(n-1)$-simplex, of vertices $v_0, \ldots, v_{n-1}$, for $I = < w_0, w_1 >$.

$$A = \begin{pmatrix} (v_{n-1},w_1) & (v_{n-2},w_1) & \cdots & \cdots & (v_1,w_1) & (v_0,w_1) & (v_0,w_0) \\ (v_{n-1},w_1) & (v_{n-2},w_1) & \cdots & \cdots & (v_1,w_1) & (v_1,w_0) & (v_0,w_0) \\ \vdots & \vdots & \cdots & \cdots & \vdots & \vdots & \vdots \\ \vdots & \vdots & \cdots & \cdots & \vdots & \vdots & \vdots \\ (v_{n-1},w_1) & (v_{n-2},w_1) & (v_{n-2},w_0) & \cdots & \cdots & (v_1,w_0) & (v_0,w_0) \\ (v_{n-1},w_1) & (v_{n-1},w_0) & (v_{n-2},w_0) & \cdots & \cdots & (v_1,w_0) & (v_0,w_0) \end{pmatrix}$$

Let a_k, $1 \leq k \leq n$, be the vertex corresponding to the n-simplex η^n_k whose vertices are listed in the k-th row ω_k of the matrix A. We call $(\Gamma^A_{M^n}, \gamma^A_{M^n})$ the $(n+2)$-coloured graph with boundary obtained in this way:

i) $V(\Gamma^A_{M^n}) = \{\alpha_0, \ldots, \alpha_{p-1}, \alpha'_0, \ldots, \alpha'_{p-1}, a^0_1, \ldots, a^{p-1}_1, a^0_2, \ldots, a^{p-1}_2, \ldots, a^0_n, \ldots, a^{p-1}_n\}$

ii) take the graphs $(\Gamma_{M^n})_{\hat{n}}$ and $(\Gamma'_{M^n})_{\hat{0}}$

iii) for $i = 0, \ldots, p-1$ and $k = 1, \ldots, n$ and for each colour $d \in \Delta_n$, delete from ω_k the unique element (v_r, w_s) such that $r + s = d$ yielding the subsequence $\omega_k(\hat{d})$; subsequently follow the rules:

[2] The columns of the matrix are listed by the lexicographic order.

(a) if in $\omega_k(\hat{d})$ there are both v_r and w_s as components of others vertices, then there exists an other row ω_h such that $\omega_k(\hat{d}) = \omega_h(\hat{d})$, and we join a_k^i and a_h^i with a d-coloured edge;

(b) if v_r is missing in $\omega_k(\hat{d})$, then join a_k^i and $a_k^{i_r}$ with a d-coloured edge;

(c) if w_0 (resp. w_1) is missing in $\omega_k(\hat{d})$, then join a_k^i and α_i' (resp. α_i) with a d-coloured edge.

iv) for $i = 0, \ldots, p-1$ and $k = 1, \ldots, n$, join the vertex a_k^i with $a_k^{i_n}$ by means of an $(n+1)$-coloured edge.

Proposition 3.1. *With the above notations, $(\Gamma_{M^n}^A, \gamma_{M^n}^A)$ is a crystallization of $M^n \times I$, whose boundary graph $\partial\Gamma_{M^n}^A$ has two connected components both isomorphic to Γ_{M^n}.*

Proof: Let (K_{M^n}, φ_{M^n}) be the pseudocomplex associated to the crystallization Γ_{M^n} of the closed n-manifold M^n, then $K_{M^n} \times \{0\}$ and $K_{M^n} \times \{1\}$ are contracted triangulations of $M^n \times \{0\}$ and $M^n \times \{1\}$. Each vertex $\alpha_i \in V(\Gamma_{M^n})$ corresponds to an n-simplex $\sigma_i^n =< v_0^{(i)}, v_1^{(i)}, \ldots, v_n^{(i)} >$ of K_{M^n}, in which the vertex $v_j^{(i)}$ is j-coloured, for $j = 0, \ldots, n$.

Let $\sigma_i^{n-1} =< v_0^{(i)}, \ldots, v_{n-1}^{(i)} >$ be the (n-1)-face of σ_i^n opposite to the n-coloured vertex. For $k = 1, \ldots, n$, the elements of the row ω_k of the matrix A, are the vertices of an n-simplex $\eta_k^n \in \sigma_i^{n-1} \times \tau^1$ (in the sense of [5]); the complex T_i constituted by these simplexes and by their faces, is a triangulation of the n-ball $|\sigma_i^{n-1} \times \tau^1|$.

Let $L_{M^n}^A$ be the pseudocomplex formed by all simplexes of $K_{M^n} \times \{0\}$, of $K_{M^n} \times \{1\}$ and of the triangulations T_i for $i = 0, \ldots, p-1$.

The join $K_{M^n}^A$ between $L_{M^n}^A$ and an internal vertex u of the $(n+1)$-manifold $M^n \times I$, is a pseudodissection of such manifold. The boundary $\partial K_{M^n}^A$ is formed by the two connected components $K_{M^n} \times \{0\}$ and $K_{M^n} \times \{1\}$, in fact every n-simplex σ^n of $K_{M^n} \times \{0\}$ is exactly face of the only $(n+1)$-simplex $< u > *\sigma^n$ of $K_{M^n}^A$; likewise for $K_{M^n} \times \{1\}$.

Since Γ_{M^n}, is supposed to be $(0,n)$-symmetric, there exists a colour-isomorphism f between the disjoint links of the vertices v_0 and v_n of K_{M^n}, respectively coloured 0 and n.

Then the bases of the prism $|\sigma_i^{n-1} \times I|$ are the $(n-1)$-simplexes $f(\sigma_i^{n-1}) \times \{1\} \in Lkd(< v_0^{(i)} >, K_{M^n}) \times \{1\}$ and $\sigma_i^{n-1} \times \{0\} \in Lkd(< v_n^{(i)} >, K_{M^n}) \times \{0\}$. Hence we can give on $K_{M^n}^A$ a vertex-colouring $\psi_{M^n}^A$, that assigns the colours n and 0 respectively to the vertices $(v_n^{(i)}, 0)$ e $(v_n^{(i)}, 1)$:

$$
\begin{array}{lll}
\psi_{M^n}^A((v_j^{(i)}, 0)) = \varphi_{M^n}(v_j^{(i)}) & j = 0, \ldots, n & i = 0, \ldots, p-1 \\
\psi_{M^n}^A((v_j^{(i)}, 1)) = \delta' \circ \varphi_{M^n}(v_j^{(i)}) & j = 0, \ldots, n & i = 0, \ldots, p-1 \\
\psi_{M^n}^A(u) = n+1. & &
\end{array}
$$

Since $K_{M^n}^A$ has the only vertex u of colour $n+1$, and the connected components of its boundary are contracted complexes, the $(n+2)$-coloured graph $\Gamma_{M^n}^A$, associated to the coloured complex $K_{M^n}^A$, is a crystallization of $M^n \times I$. Obviously the components of the boundary graph $\partial\Gamma_{M^n}^A$ are respectively Γ_{M^n} and Γ_{M^n}', that, by construction, are isomorphic. □

Corollary. For each $r,s \in \Delta_{n-1}$ we have

$$^{\partial}g^A_{rs} = g_{rs} + g_{[(\delta')^{-1}(r)][(\delta')^{-1}(s)]}$$

where $^{\partial}g^A_{rs}$ is the number of cycles of $\partial\Gamma^A_{M^n}$ relative to the colours r and s. □

4 The regular genus of $\Gamma^A_{M^n}$

In this section we calculate the regular genus of the graph $(\Gamma^A_{M^n}, \gamma^A_{M^n})$ for $n = 2,3$. Moreover bounds for the regular genus of the product of a closed 3-manifolds for I, are obtained.

From now on we denote with $\mathcal{C}_{(r,s)}(u_1,u_2,\dots,u_h)$ the cycle of $\Gamma^A_{M^n}$ whose vertices are $u_1,u_2,\dots,u_h$ and whose edges are alternatively coloured with the colours $r,s \in \Delta_{n+2}$ by colouring with the colour r the edge whose endpoints are u_1 and u_2. Moreover $\overset{\circ A}{g}_{rs}$ denotes the number of these cycles.

With the symbol $\mathcal{C}_{(r,s)}(a^{i_{hk}}_j)$ we indicate the cycle $\mathcal{C}_{(r,s)}(a^i_j, a^{i_h}_j, a^{i_{h_k}}_j, \dots)$ where the labels i of the vertices a^i_j are the same as the vertices α_i involved in a cycle of Γ_{M^n} relative to the colours h and k. Then the number of the cycles $\mathcal{C}_{(r,s)}(a^{i_{hk}}_j)$ is exactly g_{hk}.

CASE $n = 2$

Proposition 4.1. *Let Γ_{M^2} be a $(0,2)$-symmetric crystallization of order p of a closed 2-manifold M^2. Then $\ddot{\rho}(\Gamma^A_{M^2}) = p/2 - 1$.*

Proof: Obviously on $(\Gamma_{M^2})_{\hat{2}}$ (resp. $(\Gamma'_{M^2})_{\hat{0}}$) there are g_{01} cycles relative to the pair of colours $(0,1)$ (resp. $(\delta'(0),\delta'(1)) = (1,2)$). Furthermore, on the graph $\Gamma^A_{M^2}$ there are also the following cycles :

- $\mathcal{C}_{(0,3)}(a^{i_{02}}_2)$, $\mathcal{C}_{(2,3)}(a^{i_{12}}_1)$;
- for $i = 0,\dots,p-1$

  $\mathcal{C}_{(0,1)}(\alpha'_i, a^i_1, a^i_2, a^{i_0}_2, a^{i_0}_1, \alpha'_{i_0}), \quad i < i_0$

  $\mathcal{C}_{(0,2)}(\alpha'_i, a^i_1, a^{i_1}_1, \alpha'_{i_1}), \quad i < i_1$

  $\mathcal{C}_{(2,0)}(\alpha_i, a^i_2, a^{i_0}_2, \alpha_{i_0}), \quad i < i_0$

  $\mathcal{C}_{(2,1)}(\alpha_i, a^i_2, a^i_1, a^{i_1}_1, a^{i_1}_2, \alpha_{i_1}), \quad i < i_1$

  $\mathcal{C}_{(3,1)}(a^i_1, a^{i_2}_1, a^{i_2}_2, a^i_2), \quad i < i_2.$

Hence we have:

$$\overset{\circ A}{g}_{01} = g_{01} + p/2$$

$$\overset{\circ A}{g}_{02} = p$$

$$\overset{\circ A}{g}_{12} = g_{01} + p/2$$

$$\overset{\circ A}{g}_{03} = g_{02}$$

$$\overset{\circ A}{g}_{13} = p/2$$

$$\overset{\circ A}{g}_{23} = g_{12}$$

Since Γ_{M^2} is a crystallization of a closed 2-manifold, then, by definition, it follows that $g_{01} = g_{02} = g_{12} = 1$. In the calculations of the regular genus we have the minimum genus for the permutation $\varepsilon = (1,0,2,3)$, its value implies the assert. □

Corollary I. $\ddot{\rho}(\Gamma^A_{M^2}) = 2\ddot{\rho}(\Gamma_{M^2})$.

Proof: For each crystallization Γ_{M^2} of a closed 2-manifold, and for each permutation
$\varepsilon = (\varepsilon_0, \varepsilon_1, \varepsilon_2)$ of Δ_2
$\chi_\varepsilon = g_{\varepsilon_0\varepsilon_1} + g_{\varepsilon_1\varepsilon_2} + g_{\varepsilon_2\varepsilon_0} - 3/2p + p = 3 - p/2$
$\ddot{\rho}_\varepsilon(\Gamma_{M^2}) = 1 - \chi_\varepsilon/2 = p/4 - 1/2$.

Since $\ddot{\rho}(\Gamma^A_{M^2}) = p/2 - 1$ and since, if M^2 is orientable (resp. non orientable), then $M^2 \times I$ is orientable (resp. non orientable), we have the assert. □

As a consequence of the above results we find again the following relation.

Corollary II. $G(M^2 \times I) = 2g(M^2)$.

Proof: Since $\partial(M^2 \times I)$ has two connected components $M^2 \times \{0\}$ and $M^2 \times \{1\}$, it follows that $G(\partial(M^2 \times I)) = 2\, G(M^2)$.

Then $G(M^2 \times I) \geq G(\partial(M^2 \times I)) = 2\, G(M^2)$. If Γ_{M^2} is a crystallization of M^2 such that $G(M^2) = \rho(\Gamma_{M^2})$, then $\rho(\Gamma^A_{M^2}) = 2\, G(M^2)$.

Hence $G(M^2 \times I) = 2\, G(M^2)$ because $M^2 \times I$ admits $\Gamma^A_{M^2}$ as crystallization of minimum genus (hence of regular genus).

Since the regular genus of a closed surface M^2 equals its genus $g(M^2)$ [9], we have $G(M^2 \times I) = 2\, G(M^2) = 2\, g(M^2)$. □

CASE $n = 3$

Proposition 4.2. *Let Γ_{M^3} be a $(0,3)$-symmetric crystallization of a closed 3-manifold M^3 of order p, then $\ddot{\rho}(\Gamma^A_{M^3}) = p - 2g_{02}$.*

Proof: On $(\Gamma_{M^3})_{\hat{3}}$ (resp. $(\Gamma'_{M^3})_{\hat{0}}$) there are the cycles relative to the pair of colours $(0,1),(0,2),(1,2)$ (resp. $(1,2),(1,3),(2,3)$) which are respectively g_{01}, g_{02}, g_{12}. The others cycles of $\Gamma^A_{M^3}$ are the following:

- $\mathcal{C}_{(0,1)}(a_3^{i_{01}})$, $\mathcal{C}_{(0,3)}(a_2^{i_{02}})$, $\mathcal{C}_{(2,3)}(a_1^{i_{12}})$

  $\mathcal{C}_{(0,4)}(a_2^{i_{03}})$, $\mathcal{C}_{(0,4)}(a_3^{i_{03}})$

  $\mathcal{C}_{(1,4)}(a_3^{i_{13}})$, $\mathcal{C}_{(2,4)}(a_1^{i_{13}})$

  $\mathcal{C}_{(3,4)}(a_1^{i_{23}})$, $\mathcal{C}_{(3,4)}(a_2^{i_{23}})$

- for $i = 0, \ldots, p-1$

  $\mathcal{C}_{(0,1)}(\alpha'_i, a_1^i, a_2^i, a_2^{i_0}, a_1^{i_0}, \alpha'_{i_0}), \quad i < i_0$

  $\mathcal{C}_{(0,2)}(\alpha'_i, a_1^i, a_1^{i_1}, \alpha'_{i_1}), \quad i < i_1$

  $\mathcal{C}_{(2,0)}(a_2^i, a_3^i, a_3^{i_0}, a_2^{i_0}), \quad i < i_0$

  $\mathcal{C}_{(0,3)}(\alpha'_i, a_1^i, a_1^{i_2}, \alpha'_{i_2}), \quad i < i_2$

  $\mathcal{C}_{(3,0)}(\alpha_i, a_3^i, a_3^{i_0}, \alpha_{i_0}), \quad i < i_0$

  $\mathcal{C}_{(1,2)}(a_1^i, a_2^i, a_3^i, a_3^{i_1}, a_2^{i_1}, a_1^{i_1}), \quad i < i_1$

  $\mathcal{C}_{(1,3)}(a_1^i, a_2^i, a_2^{i_2}, a_1^{i_2}), \quad i < i_2$

  $\mathcal{C}_{(1,3)}(a_3^i, a_3^{i_1}, \alpha_{i_1}, \alpha_i), \quad i < i_1$

  $\mathcal{C}_{(2,3)}(a_2^i, a_3^i, \alpha_i, \alpha_{i_2}, a_3^{i_2}, a_2^{i_2}), \quad i < i_2$

  $\mathcal{C}_{(1,4)}(a_1^i, a_2^i, a_2^{i_3}, a_1^{i_3}), \quad i < i_3$

  $\mathcal{C}_{(2,4)}(a_2^i, a_3^i, a_3^{i_3}, a_2^{i_3}), \quad i < i_3.$

Hence we have :

$$\overset{\circ A}{g}_{01} = 2\, g_{01} + p/2$$

$$\overset{\circ A}{g}_{02} = g_{02} + p$$

$$\overset{\circ A}{g}_{03} = g_{02} + p$$

$$\overset{\circ A}{g}_{12} = g_{12} + g_{01} + p/2$$

$$\overset{\circ A}{g}_{13} = g_{02} + p$$

$$\overset{\circ A}{g}_{23} = 2\, g_{12} + p/2$$

$$\overset{\circ A}{g}_{04} = 2\, g_{03}$$

$$\overset{\circ A}{g}_{14} = g_{13} + p/2$$

$$\overset{\circ A}{g}_{24} = g_{13} + p/2$$

$$\overset{\circ A}{g}_{34} = 2\, g_{23}$$

An easy calculation shows that, for the permutation $\varepsilon = (1,3,0,2,4)$ of Δ_4, we obtain the minimun value $\ddot{\rho}_\varepsilon(\Gamma^A_{M^3}) = p - 2\, g_{02}$. □

From now on, we denote by $O'(M^3)$ the minimun order of a (i,j)-symmetric crystallization representing M^3.

Proposition 4.3. *For each 3-manifold M^3 admitting a (i,j)-symmetric crystallization:*

$$2\ddot{G}(M^3) \leq \ddot{G}(M^3 \times I) \leq O'(M^3) - 2\ddot{G}(M^3) - 2.$$

Proof: Let Γ_{M^3} a (i,j)-symmetric crystallization of M^3 of minimum order $O'(M^3)$, then

$$\ddot{G}(M^3 \times I) \leq \ddot{\rho}(\Gamma^A_{M^3}) = O'(M^3) - 2(g_{02} - 1) - 2.$$

Since $g_{02} - 1 \geq \ddot{G}(M^3)$, it follows the assert. □

Remark. Since, for the crystallizations Γ_{M^3} of a closed 3-manifold, the following relation is satisfied

$$g_{01} + g_{02} + g_{03} = 2 + p/2 \qquad [7]$$

and $\ddot{\rho}(\Gamma_{M^3}) = \min\{g_{0j} - 1 : j = 1,2,3\}$, we have that

$$\ddot{\rho}(\Gamma^A_{M^3}) = 2(g_{01} - 1) + 2(g_{03} - 1) \geq 4\ddot{\rho}(\Gamma_{M^3}).$$

Note that the (0,3)-symmetry of Γ_{M^3} implies $g_{01} = g_{03}$; hence we have the equality iff $g_{01} = g_{02}$.

5 Some applications

Let Γ_{S^3}, $\Gamma_{L(h,k)}$ and $\Gamma_{S^1 \times S^2}$ be the standard crystallizations respectively of S^3, of the lens space of type (h,k) and of $S^1 \times S^2$, represented in figures 1, 2 and 3 with the respective graphs $\Gamma^A_{S^3}$, $\Gamma^A_{L(h,k)}$ (for $h = 2$ and $k = 1$) and $\Gamma^A_{S^1 \times S^2}$.

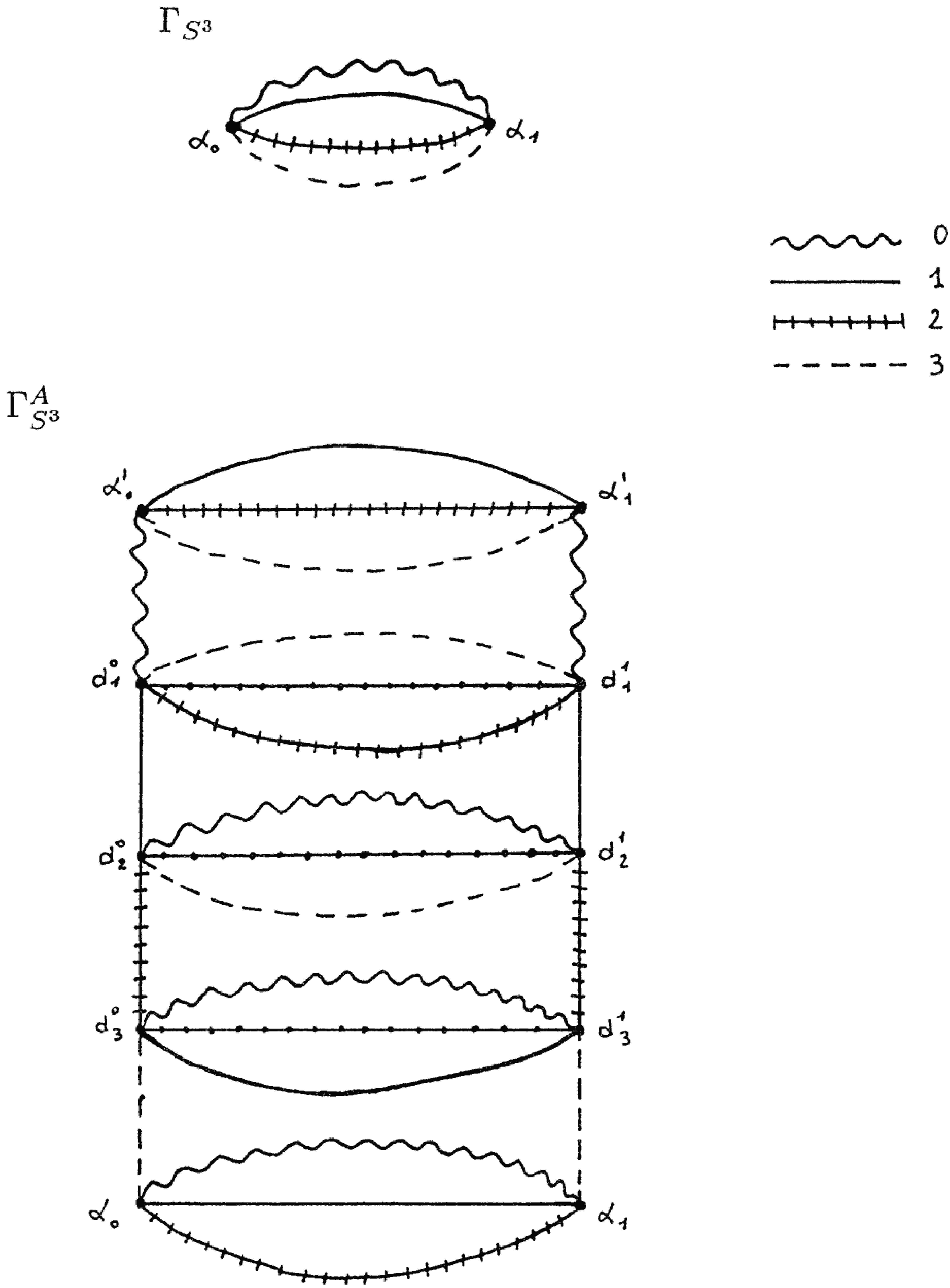

Γ_{S^3}
α_0
α_1
0
1
2
3
$\Gamma^A_{S^3}$
α'_0
α'_1
d^0_1
d^1_1
d^0_2
d^1_2
d^0_3
d^1_3
α_0
α_1

Figure 1

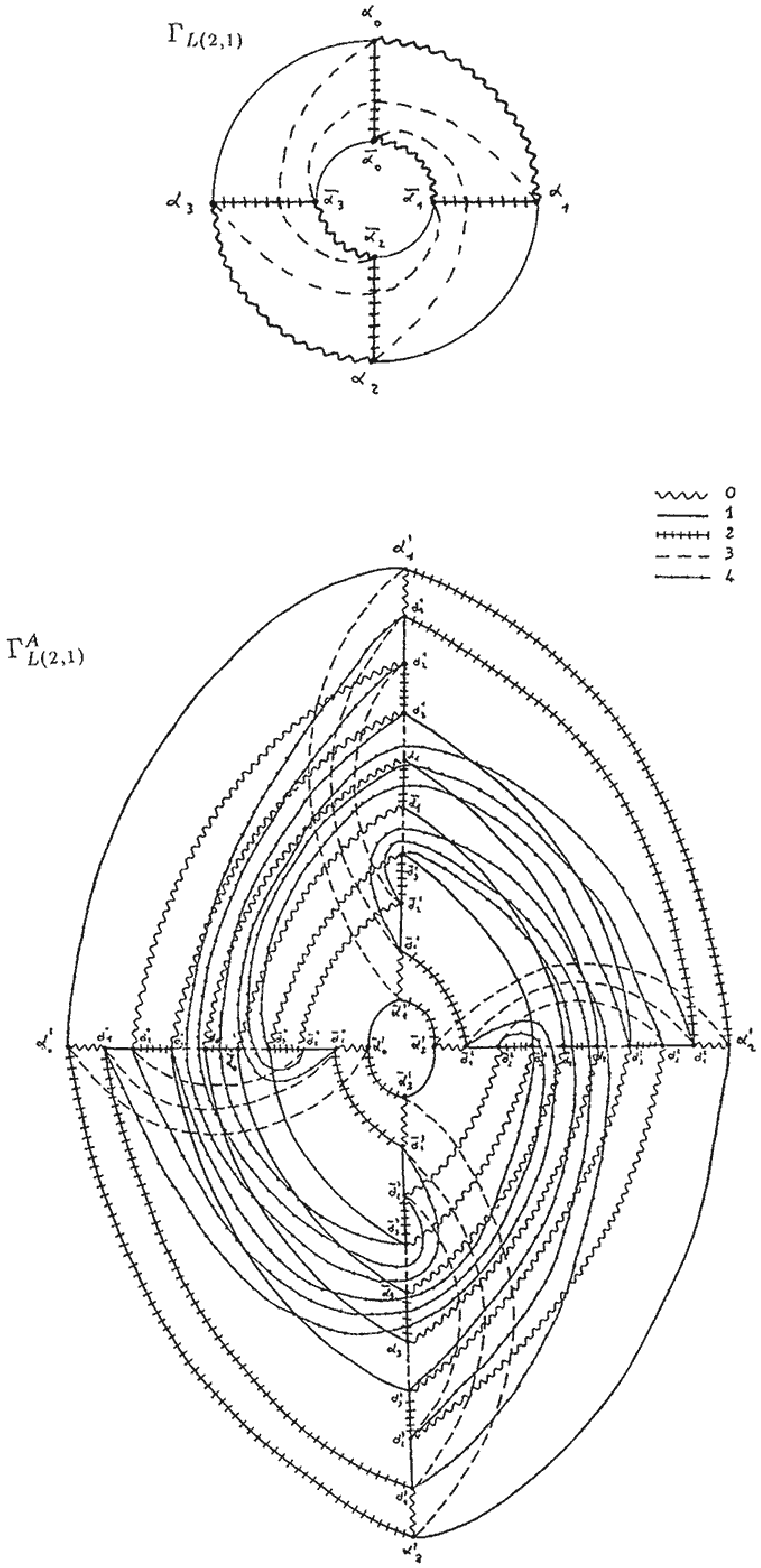
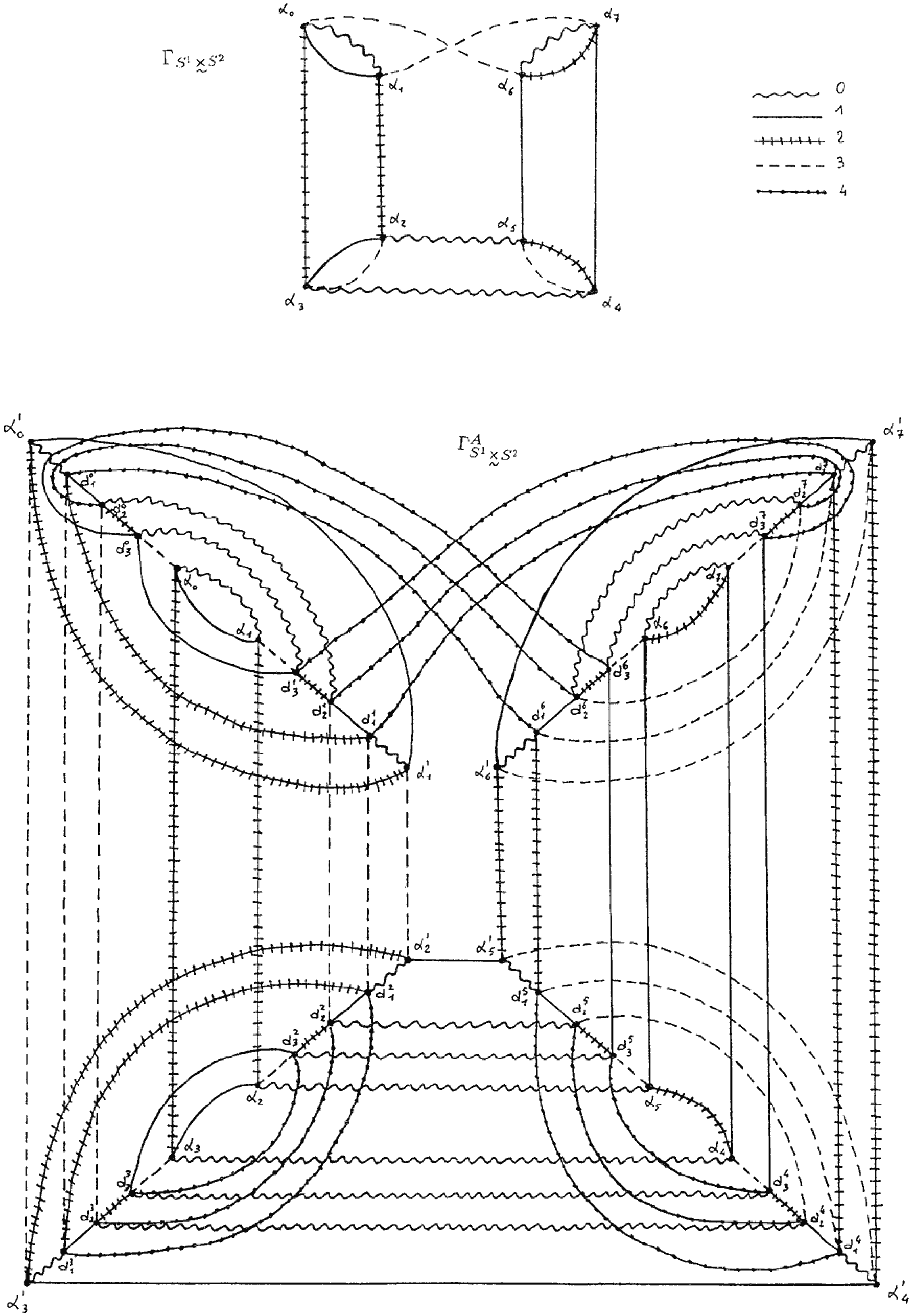


Figure 2



$\Gamma_{S^1 \underset{\sim}{\times} S^2}$
0
1
2
3
4
$\Gamma^A_{S^1 \underset{\sim}{\times} S^2}$

Figure 3

Let Γ_{T^3} be the crystallization of $T^3 = S^1 \times S^1 \times S^1$, shown in figure 4. Moreover let Σ_3 the Poincare' Sphere and Γ_{DESF} its crystallization showed in figure 5 and built in [14] as DESF and in [13] as R24/2.

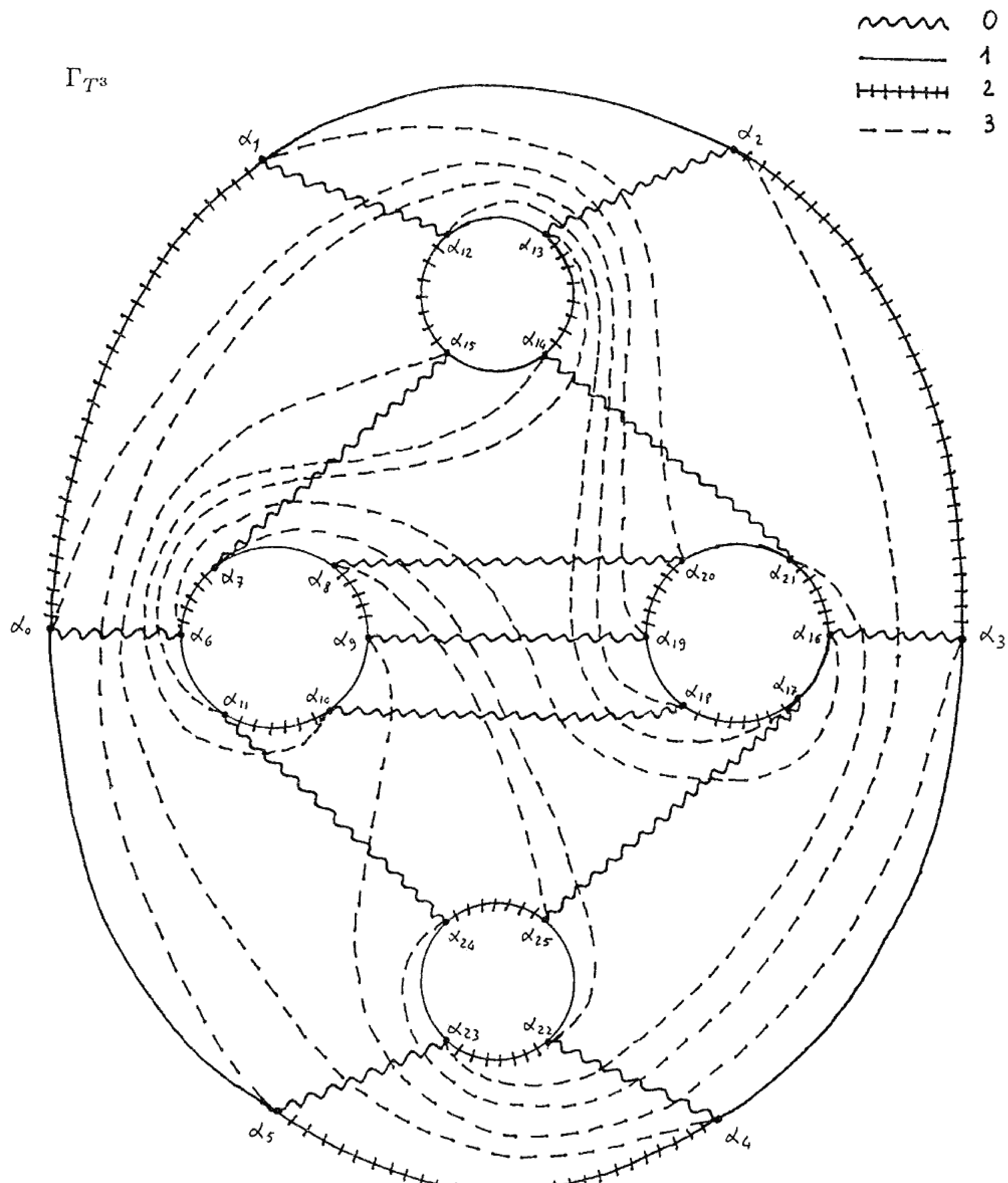


Figure 4

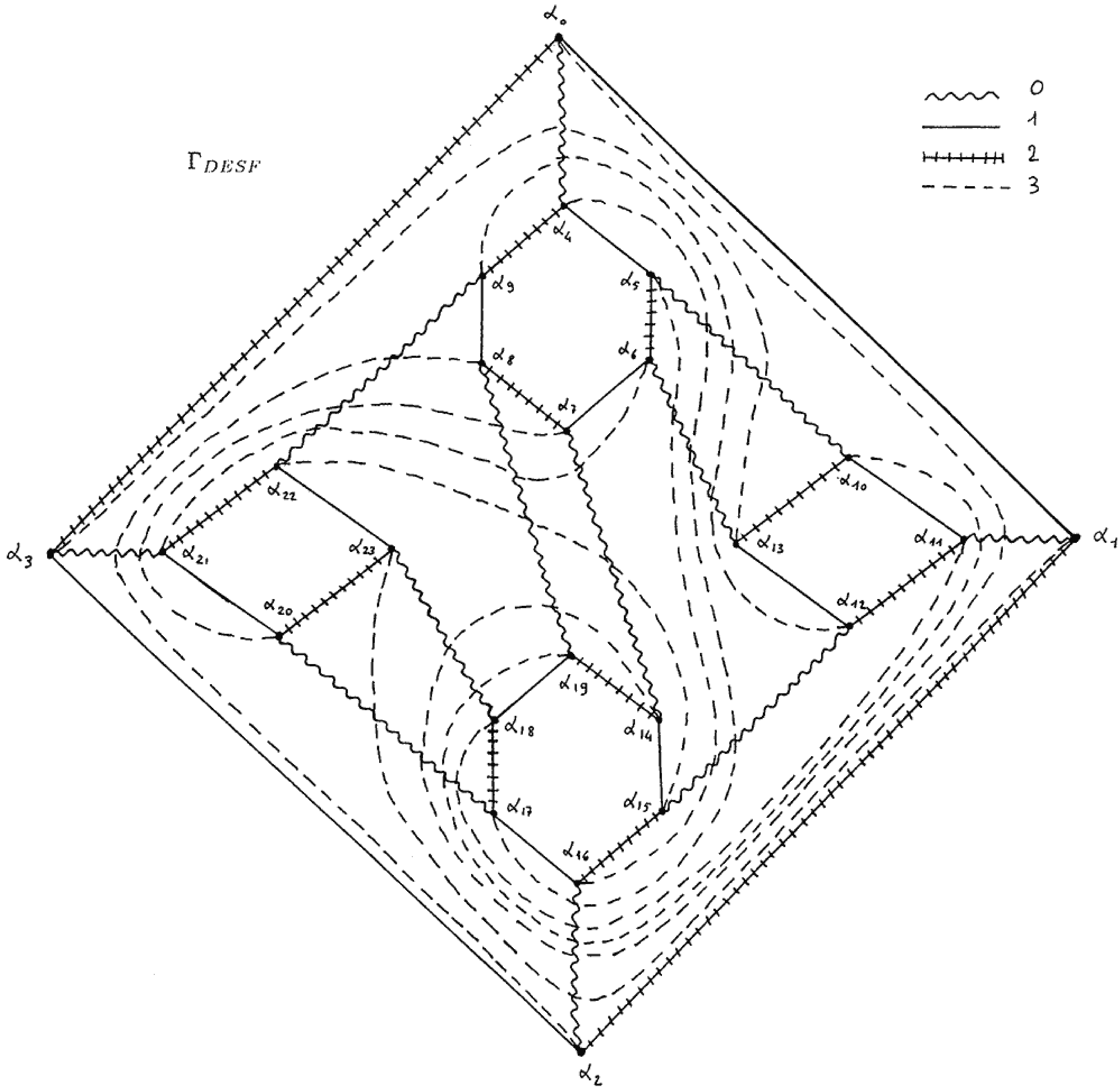


Figure 5

All these crystallizations are $(0,3)$-symmetric, then, the values of the regular genus of the crystallizations $\Gamma^A_{M^3}$, obtained starting from the above-mentioned graphs, give an upper bound for the regular genus of the respective 4-manifolds $M^3 \times I$. Moreover they represent 3-manifolds M^3 whose product for I is neither of type I nor of type II, then the gap between $G(M^3 \times I)$ and $G(\partial(M^3 \times I))$ is at least 2, and so $G(M^3 \times I) \geq 2G(M^3 \times I) + 2$. Hence we state the following:

Proposition 5.1.

(1) $G(S^3 \times I) = 0$

(2) $4 \leq G(L_{(h,k)} \times I) \leq 2h$

(3) $8 \leq G((S^1 \times S^1 \times S^1) \times I) \leq 16$

(4) $4 \leq G((S^1 \overset{\times}{\sim} S^2) \times I) \leq 8$

(5) $6 \leq G(\Sigma^3 \times I) \leq 14.$

$\square$

Remark. Note that the relation (2), for $h = 2$ and $k = 1$, directly gives

$$G(RP^3 \times I) = 4.$$

One of the authors has elaborated a program in language C that, given the graph Γ_{M^3}, it builds the graph $\Gamma^A_{M^3}$ in the cases $n = 2, 3$ and calculates its regular genus.

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