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GEODESIC REFLECTIONS AND ALMOST PARA-HERMITIAN STRUCTURES

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Abstract. *Characterizations of locally symmetric para–Hermitian manifolds and para-complex space forms are derived by means of geodesic reflections. Also, an application to* $(J^4 = 1)$*–structures is given.*

1 Introduction

In understanding the geometry of a manifold the study of its curvature naturally appears as the basic tool. Centering our attention in general pseudo–Riemannian manifolds, and pointing out those with a *simple* curvature tensor, we must consider manifolds with constant curvature and, more generally, locally symmetric manifolds.

The initial Euclidean notion of a reflection with respect to a point or a linear subspace has motivated the definition of the geodesic reflection with respect to a submanifold in pseudo–Riemannian geometry, and special classes of pseudo–Riemannian manifolds have been characterized using them. We refer to [2], [7], [10] for results in the Riemannian and pseudo–Riemannian setting and for further references. The aim of this paper is to analyse the behavior of geodesic reflections when there exists an almost para–Hermitian structure on the manifold, deriving characterizations for some curvature properties following the directions marked in the previous references.

The paper is organized as follows. In section 2 we recall some basic notions on para–Kähler manifolds and, more generally, on almost para–Hermitian manifolds, and we briefly describe some properties about geodesic reflections on pseudo–Riemannian manifolds we will need in the other sections. In §3 we obtain a characterization of locally symmetric para–Hermitian manifolds by means of geodesic reflections with respect to points (cf. Theorem 1). In §4 the constancy of the paraholomorphic sectional curvature of para–Kähler manifolds is treated, deriving a criteria by means of geodesic reflections with respect to paraholomorphic surfaces (cf. Theorem 4). Finally, in §5 we apply the characterizations obtained in the previous sections to derive similar results for e–metric $(J^4 = 1)$–manifolds.

2 Preliminaries

In this section we will recall some basic definitions and known results we will use in what follows. From now on, (M,g) will denote a pseudo–Riemannian manifold, ∇ its Levi Civita connection and R the curvature tensor taken with the sign convention $R(X,Y) = [\nabla_X, \nabla_Y] -$

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$\nabla_{[X,Y]}$. For any given nondegenerate plane $\pi = \langle\{X,Y\}\rangle$, that is, $g(X,X)g(Y,Y) - g(X,Y)^2 \neq 0$, the sectional curvature is defined by

$$K(\pi) = \frac{R(X,Y,X,Y)}{g(X,X)g(Y,Y) - g(X,Y)^2},$$

where $R(X,Y,Z,W) = g(R(X,Y)Z,W)$.

2.1 Para–Kähler manifolds

A para–Kähler manifold is a symplectic manifold (M,Ω) locally diffeomorphic to a product of Lagrangian submanifolds. Generalizing this definition, an almost para–Hermitian manifold is an almost symplectic manifold (M,Ω) whose tangent bundle splits into a Whitney sum of Lagrangian subbundles $TM = L \oplus L'$. Induced by this decomposition, an almost para-complex structure J is defined by $J = \pi_L - \pi_{L'}$, where π_L (resp., $\pi_{L'}$) is the projection of the tangent bundle into L (resp., L').

Since both L and L' are Lagrangian, it follows that $\Omega(JX,JY) = -\Omega(X,Y)$ and thus, $g(X,Y) = \Omega(X,JY)$ is a pseudo–Riemannian metric on M satisfying

$$g(JX,JY) = -g(X,Y). \tag{1}$$

From now on, attending to their pseudo–Riemannian structure, we will refer to the triple (M,g,J) as an almost para–Hermitian manifold. Note that the Kähler condition is now expressed by the parallelizability of the paracomplex structure, $\nabla J = 0$. (We refer to [1], [5], [8] for more information and further references on para–Hermitian geometry. See [3] for a survey).

2.2 Geodesic reflections

Let (M,g) be an n–dimensional pseudo–Riemannian manifold and N a nondegenerate n_0–dimensional topologically embedded submanifold. The *geodesic reflection* φ_N with respect to the submanifold N is defined in a neighborhood of the zero section of the normal bundle ν where exp_ν is a diffeomorphism by

$$\varphi_N : p = exp_\nu(r\xi) \longmapsto \varphi(p) = exp_\nu(-r\xi), \tag{2}$$

where exp_ν denotes the exponential map of the normal bundle. Note that, if N reduces to a point, the geodesic reflection becomes the geodesic symmetry with respect to such a point.

To derive an analytic description for φ_N we shall use adapted systems of Fermi coordinates, which will play a fundamental role in what follows. We briefly describe such coordinates as follows. Let $m \in N$ and consider $\{E_1,\dots,E_n\}$ a local orthonormal frame field such that $\{E_1,\dots,E_{n_0}\}$ are tangent to the submanifold and $\{E_{n_0+1},\dots,E_n\}$ are normal vector fields of N. Let $(y^1,\dots,y^{n_0})$ be a system of coordinates in a neighborhood of m in N such that $\frac{\partial}{\partial y^i}(m) = E_i(m)$, $i = 1,\dots,n_0$. The Fermi coordinates $(x^1,\dots,x^n)$, centered in m, with respect to $(y^1,\dots,y^{n_0})$ and $\{E_1,\dots,E_n\}$ are defined by

$$x^i\left(exp_\nu\left(\sum_{\beta=n_0+1}^{n} t^\beta E_\beta\right)\right) = y^i, \qquad i=1,\dots,n_0,$$

$$x^a\left(exp_\nu\left(\sum_{\beta=n_0+1}^{n} t^\beta E_\beta\right)\right) = t^a, \qquad a=n_0+1,\dots,n.$$

Note that when N reduces to the point m, then $(x^1,\dots,x^n)$ are nothing but normal coordinates. Also note that, considering the Fermi coordinates introduced above, the geodesic reflection is now expressed by

$$(x^1,\dots,x^{n_0},x^{n_0+1},\dots,x^n) \longmapsto (x^1,\dots,x^{n_0},-x^{n_0+1},\dots,-x^n). \tag{3}$$

We refer to [10] for a method to write power series expansions of the components in a Fermi coordinate system of any tensor field along a normal geodesic to N. In what follows, we will make use of some of such expansions without computing them explicitly (see also [7]).

We finish this section by recalling some properties of geodesic reflections. If J is an almost paracomplex structure on M, φ_N is called *paraholomorphic* if $(\varphi_N)_*\circ J = J\circ(\varphi_N)_*$ and *symplectic* if it satisfies $(\varphi_N)^*\Omega=\Omega$, where Ω denotes the Kähler 2–form. Also, φ_N is called *isometric* if $(\varphi_N)^*g=g$. (See [2] for more information on isometric and symplectic geodesic reflections with respect to points and submanifolds in Riemannian geometry, and [7] for the pseudo–Riemannian case).

3 Locally symmetric para–Hermitian manifolds

Among the classes of pseudo–Riemannian manifolds, an interesting one is that of symmetric spaces. Moreover, if the manifold is assumed to be equipped with an additional structure, such kind of manifolds may be further specialize. In this sense, Kaneyuki and Kozai introduced in [8] the class of para–Hermitian symmetric spaces as *those almost para–Hermitian manifolds with isometric and paraholomorphic geodesic symmetries*. (See also [4] for a classification of para–Hermitian homogeneous structures).

In what follows we will obtain a characterization of locally symmetric para–Hermitian manifolds by means of the geodesic symmetries, which will show the existence of a certain redundancy in the previous definition.

Theorem 1 *Let (M^{2n},g,J) be an almost para–Hermitian manifold. The following conditions are equivalent:*

i) (M,g,J) is locally symmetric para–Hermitian.

ii) The local geodesic symmetries are paraholomorphic.

iii) The local geodesic symmetries are symplectic.

Proof. First, one easily gets that if the manifold is locally symmetric para–Hermitian, then both *ii*) and *iii*) are satisfied.

To show that condition *ii*) implies *i*), let us consider an orthonormal basis of T_mM $\{e_1,\dots,e_n,e_{n+1},\dots,e_{2n}\}\equiv\{\xi,e_2,\dots,e_n,J\xi,Je_2,\dots,Je_n\}$, where ξ is any unit vector in T_mM. Let $\gamma(r)=exp_m(r\xi)$ be the geodesic with initial conditions $\gamma(0)=m$, $\gamma'(0)=\xi$, and take the associated geodesic symmetry given by $S_m(exp_m(r\xi))=exp_m(-r\xi)$. Since S_m is paraholomorphic, in the following power series expansion for the components of the almost paracomplex structure J,

$$\begin{aligned} J_1^a(\gamma(r)) &= -\{\Omega_{1c}g^{ca}\}(m)-r\{g^{ca}\nabla_\xi\Omega_{1c}\}(m)+O(r^2),\ a=2,\dots,2n,\\ J_a^b(\gamma(r)) &= \{\Omega_{ca}g^{cb}\}(m)+r\{g^{cb}\nabla_\xi\Omega_{ca}\}(m)+O(r^2),\ a,b=2,\dots,n, \end{aligned}$$

the coefficients of r must vanish. Moreover, $\nabla_\xi\Omega_{11}=0$ and $\nabla_\xi\Omega_{a1}=-\nabla_\xi\Omega_{1a}$. Then, it follows that $\nabla_\xi J=0$ for all nonnull vectors ξ. Now, this condition can be extended for null vectors, and so (M,g,J) is a para–Kähler manifold. In this case, we have

$$\begin{aligned} J_1^a(\gamma(r)) &= -\{\Omega_{1c}g^{ca}\}(m)-r^2\left\{\frac{1}{6}g^{ca}R_{\xi e_c\xi J\xi}+\frac{1}{3}\varepsilon_c\varepsilon_a R_{\xi e_c\xi e_a}\Omega_{1c}\right\}(m)\\ &\quad -r^3\left\{\frac{1}{12}g^{ca}\nabla_\xi R_{\xi e_c\xi J\xi}+\frac{1}{6}\varepsilon_c\varepsilon_a\nabla_\xi R_{\xi e_c\xi e_a}\Omega_{1c}\right\}(m)+O(r^4), \end{aligned}$$

where $a=2,\dots,2n$ and $\varepsilon_i=g(e_i,e_i)$. Again the coefficient of r^3 must be zero, which implies

$$\left\{\frac{1}{12}\varepsilon_a\nabla_\xi R_{\xi e_a\xi J\xi}+\frac{1}{6}\varepsilon_{n+1}\varepsilon_a\nabla_\xi R_{\xi J\xi\xi e_a}\Omega_{1,n+1}\right\}(m)=0.$$

Then, $\nabla_\xi R_{\xi e_a\xi J\xi}=0$ for $a=2,\dots,2n$, and taken $a=n+1$ we get

$$\nabla_\xi R_{\xi J\xi\xi J\xi}=0 \tag{4}$$

for all nonnull vectors ξ.

If we suppose *iii*), proceeding in the same way and using the power series expansions

$$\begin{aligned} \Omega_{1a}(\gamma(r)) &= \Omega_{1a}(m)+r\nabla_\xi\Omega_{1a}(m)+O(r^2),\\ \Omega_{ab}(\gamma(r)) &= \Omega_{ab}(m)+r\nabla_\xi\Omega_{ab}(m)+O(r^2), \end{aligned}$$

it follows that (M,g,J) is a para–Kähler manifold. In this case, we obtain the expression

$$\Omega_{1a}(\gamma(r))=\Omega_{1a}(m)+\frac{r^2}{6}R_{\xi e_a\xi J\xi}(m)+\frac{r^3}{12}\nabla_\xi R_{\xi e_a\xi J\xi}(m)+O(r^4),$$

and since S_m is symplectic the coefficient of r^3 must vanish. Therefore, we get $\nabla_\xi R_{\xi e_a\xi J\xi}=0$, for $a=2,\dots,2n$, from which again (4) holds.

Now, the proof finishes considering (4) and the following lemma. □

Lemma 2 *Let (M,g,J) be a para–Kähler manifold. Then (M,g,J) is locally symmetric if and only if*

$$\nabla_X R_{XJXXJX} = 0, \tag{5}$$

for all unit spacelike vectors X.

Proof. Let X and Y be unit tangent vectors, X being spacelike. We consider the vector $Z = \lambda_0 X + \mu Y$, where $\lambda_0 = 1$ if $g(X,Y) = 0$ and $\lambda_0 = g(X,Y)^{-1}$ in other case, $\mu \in (0,1)$. Therefore, Z is a spacelike vector and considering the coefficient of μ^4 in (5) it follows that

$$\nabla_X R_{YJYYJY} + 4\nabla_Y R_{XJYYJY} = 0, \tag{6}$$

and replacing Y by JY in this expression we get

$$\nabla_X R_{YJYYJY} + 4\nabla_{JY} R_{XYJYY} = 0. \tag{7}$$

Now, using the second Bianchi identity (7) becomes

$$5\nabla_X R_{YJYYJY} - 4\nabla_Y R_{XJYYJY} = 0, \tag{8}$$

and, from (6) and (8) it follows $\nabla_X R_{YJYYJY} = 0$ for all unit vectors X, Y, X spacelike. Finally, after a straightforward calculation (similar to the above), it is obtained that $\nabla_X R_{XYXY} = 0$ for all unit vectors X and Y, X spacelike, which shows that (M,g) is locally symmetric in virtue of [7, Lemma 2.1]. □

4 Paracomplex space forms

Since the curvature tensor of a pseudo–Riemannian manifold with constant sectional curvature c is given by $R(X,Y)Z = c\{g(X,Z)Y - g(Y,Z)X\}$, it follows that in the particular case of para–Kähler manifolds, the constancy of the sectional curvature implies that $c = 0$. This fact motivates the definition of the paraholomorphic sectional curvature, H, as the restriction of the sectional curvature to nondegenerate paraholomorphic planes, i.e., planes which remain invariant under the action of the paracomplex structure J (see [5]).

Para–Kähler manifolds of constant paraholomorphic sectional curvature are locally symmetric. Moreover, in the nonflat case, they are irreducible spaces and thus, if complete and simply connected, they correspond to the symmetric space $SL(n+1,\mathbb{R})/(SL(n,\mathbb{R})+\mathbb{R})$, therefore being the simplest examples of para–Hermitian symmetric spaces.

An algebraic characterization of the constancy of the paraholomorphic sectional curvature in terms of the Jacobi operators can be stated as follows

Theorem 3 [1, Thm.3.1] *Let (M,g,J) be a connected para–Kähler manifold. It is a space of constant paraholomorphic sectional curvature if and only if*

$$R_X(JX) \sim JX, \qquad \textit{for all vectors } X, \tag{9}$$

where $R_X = R(\cdot,X)X$ is the Jacobi operator associated to X and $\sim$ means “is proportional to”.

Next, the purpose of this section is to obtain a geometric interpretation of (9) as follows

Theorem 4 *Let (M^{2n},g,J) be a para–Hermitian manifold, with $n>2$. Then it is a paracomplex space form if and only if the geodesic reflections with respect to any nondegenerate paraholomorphic surface is symplectic.*

Proof. First of all note that since J is a paracomplex structure on M, it is possible to construct a system of coordinates with paraholomorphic changes of coordinates (see [8]). As a consequence, for each point $m\in M$ and any two–dimensional paracomplex subspace $V\subset T_mM$ there exists a paraholomorphic surface S passing through m with tangent space $T_mS=V$.

Now, let S be a nondegenerate paracomplex surface passing through a point $m\in M$. Let $(x^1,\dots,x^n)$ be a system of Fermi coordinates in a neighborhood of m where the exponential map of the normal bundle is a diffeomorphism, and consider the analytic expression (3) for the geodesic reflection φ_S. Therefore, φ_S is symplectic if and only if

$$\Omega_{ij}(p)=\Omega_{ij}(\varphi_S(p)),\,\Omega_{ia}(p)=-\Omega_{ia}(\varphi_S(p)),\,\Omega_{ab}(p)=\Omega_{ab}(\varphi_S(p)),\qquad(10)$$

where $i,j\in\{1,2\}$ and $a,b\in\{3,\dots,2n\}$.

Let $p=exp_m(r\xi)=\gamma(r)$ be a point in a nonnull geodesic normal to S, and specialize the system of Fermi coordinates so that $\frac{\partial}{\partial x^n}(\gamma(r))=\gamma(r)$. Now, one has the power series expansion

$$\Omega_{2n,a}(p)=\Omega_{2n,a}(m)+rg(\xi,(\nabla_\xi J)E_a)(m)+O(r^2),$$

and considering the third condition in (10), it follows that $(\nabla_\xi J)\xi$ is tangent to S for each unit ξ normal to S. Since $dim\,M\geq 6$ and considering a nondegenerate paracomplex surface S' passing through m and normal to S and ξ, it follows as before that $(\nabla_\xi J)\xi$ is tangent to S', and thus $(\nabla_\xi J)\xi=0$. This shows that (M,g,J) is a nearly para–Kähler manifold and therefore para–Kähler since it is para–Hermitian by assumption (see [3] for a classification of almost para–Hermitian manifolds).

Next, we consider the power series expansion

$$\begin{aligned}\Omega_{ia}(p)&=\Omega_{ia}(m)+rg(JE_a,T(\xi)E_i-{}^t\perp(\xi)E_i)(m)\\&\quad-r^2\left\{-\frac{1}{6}R_{\xi E_a\xi JE_i}+\frac{1}{2}R_{\xi E_i\xi JE_a}\right\}(m)+O(r^3),\end{aligned}$$

where $T(\xi)$ denotes the shape operator of S ($g(T(\xi)X,Y)=-g(B(X,Y),\xi)$, B being the second fundamental form), and ${}^t\perp(\xi)$ is the operator defined by ${}^t\perp(\xi)X=\nabla^\perp_X\xi$. Once again using (10) it follows that the coefficient of r^2 vanishes, and for $a=2n-1$ we get $R(\xi,J\xi,\xi,JE_i)=0$, condition equivalent to the constancy of the paraholomorphic sectional curvature of the para–Kähler manifold M as we stated in Theorem 3.

To prove the converse, we solve the Jacobi equation using the fact that if the paraholomorphic sectional curvature of the para–Kähler manifold M is constant, say c, then its curvature tensor is given by

$$\begin{aligned}R(X,Y)Z&=\frac{c}{4}\{g(X,Z)Y-g(Y,Z)X+g(X,JZ)JY\\&\qquad-g(Y,JZ)JX+2g(X,JY)JZ\}.\end{aligned}$$

Taking $c > 0$ (for $c < 0$ or $c = 0$ is similar), proceeding as in [2], we obtain the following expressions:

$$\begin{aligned}
\Omega_{ij}(\gamma(r)) &= \left(cos\frac{r\sqrt{c}}{2}\right)^2 \Omega_{ij}(m) \\
&\quad + \left(\frac{2}{\sqrt{c}}sin\frac{r\sqrt{c}}{2}\right)^2 \Bigg\{ g(T(\xi)F_i, JT(\xi)F_j) \\
&\quad + \sum_{t,s=3}^{2n-2} g({}^t\!\perp\!\!\perp(\xi)F_i, F_t)g({}^t\!\perp\!\!\perp(\xi)F_j, F_s)\Omega_{ts} \Bigg\}(m), \\
\Omega_{ia}(\gamma(r)) &= -\frac{1}{r}\left(\frac{2}{\sqrt{c}}sin\frac{r\sqrt{c}}{2}\right)^2 \left\{\sum_{t=3}^{2n-2} g({}^t\!\perp\!\!\perp(\xi)F_i, F_t)\Omega_{ta}\right\}(m), \\
\Omega_{ab}(\gamma(r)) &= \frac{1}{r^2}\left(\frac{2}{\sqrt{c}}sin\frac{r\sqrt{c}}{2}\right)^2 \Omega_{ab}(m), \\
\Omega_{2n,i}(\gamma(r)) &= \frac{1}{\sqrt{c}}sin(r\sqrt{c})g({}^t\!\perp\!\!\perp(\xi)F_i(m), J\xi)g(J\xi, J\xi), \\
\Omega_{2n,a}(\gamma(r)) &= -\frac{1}{r}\frac{1}{\sqrt{c}}sin(r\sqrt{c})\Omega_{2n,a}(m),
\end{aligned}$$

where $\gamma(r) = exp_m(r\xi)$ is an spacelike geodesic (for a timelike geodesic, we just replace trigonometric functions by hyperbolic ones).

In any case, we get that the geodesic reflection is symplectic along any geodesic, and so the geodesic reflection with respect to each nondegenerate paraholomorphic surface is symplectic. □

Remark 5 Note that the previous theorem remains valid in dimension four, provided that the manifold is para–Kähler.

Remark 6 Since a para–Hermitian manifold admits a coordinate system consisting of paraholomorphic functions, for any 2–dimensional paraholomorphic subspace V of the tangent space T_mM there exists a paraholomorphic surface S passing through m with $T_mS = V$. At this moment, the authors do not know if such condition can be stated for general almost para–Hermitian manifolds.

5 Application to ($J^4 = 1$)–structures

An e–metric ($J^4 = 1$)–manifold is a pseudo–Riemannian manifold (M^n, g) endowed with a $(1,1)$–tensor field J such that $J^4 = 1$, and satisfying $g(X, JY) + g(JX, Y) = 0$ (g is said to be adapted in the electromagnetic sense metric). Note that the characteristic polynomial of J is given by $(x^2-1)^r(x^2+1)^s$, $2r + 2s = n$, and thus, the eigenvalues -1 and $+1$ define two distributions, $\mathfrak{D}_c$ and $\mathfrak{D}_{pc}$ respectively. We will say that a J–invariant subspace contained in $\mathfrak{D}_c$ (resp., $\mathfrak{D}_{pc}$) is complex (resp., paracomplex). Further, the Kähler condition ($\nabla J = 0$) means that the manifold is locally a product of a Kähler and a para–Kähler manifold. (We refer to [5] and the references therein for more information on *e*–metric manifolds). Next, combining the results in [7] with those of previous sections, we have the following:

Theorem 7 *Let (M,g,J) be an e–metric $(J^4 = 1)$–manifold. Then the following conditions are equivalent:*

i) *(M,g,J) is locally the product of a locally symmetric Hermitian manifold and a locally symmetric para–Hermitian manifold.*

ii) *The local geodesic symmetries are J–preserving maps.*

iii) *The local geodesic symmetries are symplectic.*

Theorem 8 *Let (M,g,J) be an integrable e–metric $(J^4 = 1)$–manifold with characteristic polynomial $(x^2-1)^r(x^2+1)^s$, $r \geq 4$. Then it is a Kähler manifold of constant J–holomorphic curvature if and only if the local geodesic reflections with respect to any nondegenerate complex and paracomplex surface are symplectic.*

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