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THE NATURAL AFFINORS ON $\otimes^k T^{(r)}$

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Abstract. *For integers $k \geq 2$, r and $n \geq k$ we prove that any natural affinor A on the k-tensor power $\otimes^k T^{(r)}$ of the linear r-tangent bundle functor $T^{(r)}$ over n-manifolds is proportional to the identity affinor.*

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0. Given a natural bundle F over n-manifolds a natural affinor A on F is a system of affinors (i.e. tensor fields of type (1,1)) $A : TFM \to TFM$ for any n-manifold M which is invariant with respect to local embeddings between n-manifolds, see [3].

In [3], Gancarzewicz and Kolář obtained a classification of all natural affinors on the extended linear r-tagent bundle functor $E^{(r)}M = (J^r(M,\mathbf{R}))^*$ over n-manifolds. From the mentioned classification one can easily deduce that any natural affinor A on the linear r-tangent bundle functor $T^{(r)}M = (J^r(M,\mathbf{R})_0)^*$ is a linear combination (with real coefficients) of the identity affinor $id_{TT^{(r)}M} : TT^{(r)}M \to TT^{(r)}M$ and the affinor being the composition $TT^{(r)}M \to T^{(r)}M \times_M TM \subset T^{(r)}M \times_M T^{(r)}M \cong VT^{(r)}M \subset TT^{(r)}M$, where the arrow is $(\pi^T, T\pi)$ $\pi^T : TT^{(r)}M \to T^{(r)}M$ is the tangent bundle projection, $\pi : T^{(r)}M \to M$ is the bundle projection and the first inclusion is given by the dualization of the jet projection $J^r(M,\mathbf{R})_0 \to J^1(M,\mathbf{R})_0$.

In this short note we prove the following theorem.

Theorem 1 *For integers $k \geq 2$, r and $n \geq k$ any natural affinor A on the k-tensor power $\otimes^k T^{(r)}$ of $T^{(r)}$ over n-manifolds is proportional (by a real number to the identity affinor.*

In Item 1, for natural numbers r, k and $n \geq k$ we present a classification of all natural transformations $\otimes^k T^{(r)} \to \otimes^k T^{(r)}$ over n-manifolds. For $k = 1$ we reobtain a result of Kolář and Vosmanská, [6]. In Item 2, using similar arguments as in Item 1, for natural numbers $r, k \geq 2$ and n we present a classification of all natural transformations $T(\otimes^k T^{(r)}) \to T$ over n-manifolds. In Item 3, using similar arguments as in Item 1, we prove that for natural numbers $r, k \geq 2$ and $n \geq k$ any linear natural transformation $T(\otimes^k T^{(r)}) \to \otimes^k T^{(r)}$ over n-manifolds is 0. In Item 4, using the results of Item 2 and 3, we prove Theorem 1. In Item 5, we formulae similar results for $\odot^k$ and $\bigwedge^k$ instead of $\otimes^k$.

Classifications of natural affinors on some other natural bundles are given in [1], [2], [7] and [8].

Natural affinors play a very importrant role in the differential geometry. For example, they can be used to define torsions of a connection, see [5].

Throughout this note the usual coordinates on $\mathbf{R}^n$ are denoted by $x^1, \ldots, x^n$ and $\partial_i = \frac{\partial}{\partial x^i}$, $i = 1, \ldots, n$.

All manifolds and maps are assumed to be of class C^∞.

1. Each permutation $\sigma = (\sigma_1, \dots, \sigma_k) \in B_k$ determines a (linear) natural transformation $A_\sigma : \otimes^k T^{(r)} \to \otimes^k T^{(r)}$, $\omega_1 \otimes \dots \otimes \omega_k \to \omega_{\sigma_1} \otimes \dots \otimes \omega_{\sigma_k}$, $\omega_1, \dots, \omega_k \in T^{(r)}_x M$, $x \in M$, M is a manifold.

Proposition 2 *For natural numbers k, r and $n \geq k$ any natural transformation $A : \otimes^k T^{(r)} \to \otimes^k T^{(r)}$ over n-manifolds is a linear combination (with real coefficients) of the A_σ for all $\sigma \in B_k$.*

Proof. Any natural transformation A as in the proposition is uniquely determined by the $< A(\omega), j^r_0\gamma_1 \otimes \dots \otimes j^r_0\gamma_k > \in \mathbf{R}$ for any $\gamma_1, \dots, \gamma_k : \mathbf{R}^n \to \mathbf{R}$ with $\gamma_1(0) = \dots = \gamma_k(0) = 0$ and any $\omega \in \otimes^k T^{(r)}_0 \mathbf{R}^n$. Since $n \geq k$, by the rank theorem $(j^r_0 x^1, \dots, j^r_0 x^k)$ has dense orbit in $\times^k (J^r_0(\mathbf{R}^n, \mathbf{R})_0)$. Then, by the naturality of A, A is uniquely determined by the $< A(\omega)$, $j^r_0 x^1 \otimes \dots \otimes j^r_0 x^k >$ for any $\omega \in \otimes^k T^{(r)}_0 \mathbf{R}^n$.

Any $\omega \otimes^k T^{(r)}_0 \mathbf{R}^n$ is a linear combination of the $(j^r_0 x^{\alpha^1})^* \otimes \dots \otimes (j^r_0 x^{\alpha^k})^*$ for all $\alpha^1, \dots, \alpha^K \in (\mathbf{N} \cup \{0\})^n$ with $1 \leq |\alpha^1| \leq r, \dots, 1 \leq |\alpha^k| \leq r$, where the $(j^r_0 x^\alpha)^* \in T^{(r)}_0 \mathbf{R}^n$ for $\alpha \in (\mathbf{N} \cup \{0\})^n$ with $1 \leq |\alpha| \leq r$ form the basis dual to the $j^r_0 x^\alpha \in J^r_0 (\mathbf{R}^n, \mathbf{R})_0$ for α as beside. By the naturality of A with respect to the homotheties $a_t = (t^1 x^1, \dots, t^n x^n)$, $t = (t^1, \dots, t^n) \in \mathbf{R}^n_+$, we have $< A(\otimes^k T^{(r)}(a_t)(\omega)),\ j^r_0 x^1 \otimes \dots \otimes j^r_0 x^k > = t^1 \dots t^k < A(\omega),\ j^r_0 x^1 \otimes \dots \otimes j^r_0 x^k >$ for any $t = (t^1, \dots, t^n) \in \mathbf{R}^n_+$. For any $t \in \mathbf{R}^n$ and any $\alpha \in (\mathbf{N} \cup \{0\})^n$ we have $T^{(r)}(a_t)\ ((j^r_0 x^\alpha)^*) = t^\alpha (j^r_0 x^\alpha)^*$. Then by the homogeneous function theorem, see [4], $< A(\omega), j^r_0 x^1 \otimes \dots \otimes j^r_0 x^k >$ depends linearly on the coefficients of ω corresponding to the $(j^r_0 x^{\sigma_1})^* \otimes \dots \otimes (j^r_0 x^{\sigma_k})^*$ for all $\sigma = (\sigma_1, \dots, \sigma_k) \in B_k$ and it is independent of the other ones.

Hence the vector space of all natural transformations $A : \otimes^k T^{(r)} \to \otimes^k T^{(r)}$ over n-manifolds has dimension $\leq card(B_k)$.

On the other hand the natural thansformations A_σ for $\sigma \in B_k$ are linearly independent.

These facts end the proof of the proposition. □

2. The tangent map $T\Pi : T(\otimes^k T^{(r)} M) \to TM$ of the bundle projection $\Pi : \otimes^k T^{(r)} M \to M$ defines a natural transformation $T\Pi : T(\otimes^k T^{(r)}) \to T$.

Proposition 3 *For natural numbers r, n and $k \geq 2$ any natural transformation $A : T(\otimes^k T^{(r)}) \to T$ over n-manifolds is proportional (by a real number) to $T\Pi$.*

Proof. Similarly as in the proof of Proposition 2, any natural transformation A as in Proposition 3 is uniquely determined by the $< A(y), d_0 x^1 >$ for any $y \in (T(\otimes^k T^{(r)} \mathbf{R}^n))_0 \tilde{=} \mathbf{R}^n \times (V(\otimes^k T^{(r)} \mathbf{R}^n))_0 \tilde{=} \mathbf{R}^n \times \otimes^k T^{(r)}_0 \mathbf{R}^n \times \otimes^k T^{(r)}_0 \mathbf{R}^n$, where $\tilde{=}$ are the standard identifications. Using the invariancy of A with respect to the homothetis $a_t = (t^1 x^1, \dots, t^n x^n)$, for $t = (t^1, \dots, t^n) \in \mathbf{R}^n_+$ and the assumption $k \geq 2$, we deduce (similarly as in the proof of Proposition 2) that $< A(y), d_0 x^1 >$ depends linearly on the first coordinate of $y \in \mathbf{R} \times \mathbf{R}^{n-1} \times \otimes^k T^{(r)}_0 \mathbf{R}^n \times \otimes^k T^{(r)}_0 \mathbf{R}^n$ and it is independent of the other ones. Then the vector space of all natural transformations as in Proposition 3 has dimension ≤ 1. This ends the proof. □

3. The crucial point in the proof of Theorem is the following proposition.

Proposition 4 *For natural numbers* $k \geq 2$, r *and* $n \geq k$ *any linear natural transformation* $A : T(\otimes^k T^{(r)}) \to \otimes^k T^{(r)}$ *over n-manifold is* 0.

We remark that the linearity of A means that A determines a linear map $T_y(\otimes^k T^{(r)} M) \to \otimes^k T^{(r)}_{\Pi(y)} M$ for any n-manifold M and any $y \in \otimes^k T^{(r)} M$.

Proof. Using similar arguments as in the proof of Proposition 2, since $n \geq k$, it is sufficient to show that $< A(y),\ j^r_0 x^1 \otimes \ldots \otimes j^r_0 x^k >= 0$ for any $y = (y_1, y_2, y_3) \in (T(\otimes^k T^{(r)}_0 \mathbf{R}^n))_0 \cong \mathbf{R}^n \times \otimes^k T^{(r)}_0 \mathbf{R}^n \times \otimes^k T^{(r)}_0 \mathbf{R}^n \times \otimes^k T^{(r)}_0 \mathbf{R}^n$.

Let $(j^r_0 x^\alpha)^* \in T^{(r)}_0 \mathbf{R}^n$ for $\alpha \in (\mathbf{N} \cup \{0\})^n$ with $1 \leq |\alpha| \leq r$ be the basis as in the proof of Proposition 1.

By the naturality of A with respect to $a_t = (t^1 x^1, \ldots, t^n x^n)$ for $t = (t^1, \ldots, t^n) \in \mathbf{R}^n_+$, $< A(T(\otimes^k T^{(r)})\ (a_t)(y)),\ j^r_0 x^1 \otimes \ldots \otimes j^r_0 x^k >= t^1 \ldots t^k < A(y),\ j^r_0 x^1 \otimes \ldots \otimes j^r_0 x^k >$ for any $y \in (T(\otimes^k T^{(r)} \mathbf{R}^n))_0$ and any $t = (t^1, \ldots, t^n) \in \mathbf{R}^n_+$. Then, using the homogeneous function theorem, we deduce easily that

$$< A(y), j^r_0 x^1 \otimes \ldots \otimes j^r_0 x^k >= \lambda y^1_1 \ldots y^k_1 + \sum_{\sigma \in B_k} \mu^\sigma y_{2\sigma} + \sum_{\sigma \in B_k} \nu^\sigma y_{3\sigma} \qquad (*)$$

for some real numbers $\lambda, \mu^\sigma, \nu^\sigma$, where $y = (y_1, y_2, y_3) \in (T(\otimes^k T^{(r)} \mathbf{R}^n))_0 \cong \mathbf{R}^n \times \otimes^k\ T^{(r)}_0\ \mathbf{R}^n \times \otimes^k\ T^{(r)}_0\ \mathbf{R}^n$, $y_1 = (y^1_1, \ldots, y^n_1) \in \mathbf{R}^n$, $y_{2\sigma}$ is the coefficient (with respect to the induced by tensoring basis of $\otimes^k T^{(r)}_0 \mathbf{R}^n$) of $y_2 \in \otimes^k T^{(r)}_0 \mathbf{R}^n$ corresponding to $(j^r_0 x^{\sigma_1})^* \otimes \ldots \otimes (j^r_0 x^{\sigma_k})^*$ and $y_{3\sigma}$ is the coefficient of $y_3 \in \otimes T^{(r)}_0\ \mathbf{R}^n$ corresponding to $(j^r_0 x^{\sigma_1})^* \otimes \ldots \otimes (j^r_0 x^{\sigma_k})^*$, $\sigma = (\sigma_1, \ldots, \sigma_k) \in B_k$.

Since A is linear, $< A(y_1, y_2, y_3),\ j^r_0 x^1 \otimes \ldots \otimes j^r_0 x^k >$ depends linearly on (y_1, y_3) for any y_2. Hence $\lambda = 0$ (as $k \geq 2$) and $\mu^\sigma = 0$ for any $\sigma \in B_k$. In particular,

$$< A(\partial^C_{1\,|\omega}), j^r_0 x^1 \otimes \ldots \otimes j^r_0 x^k >=< A(e_1, \omega, 0), j^r_0 x^1 \otimes \ldots \otimes j^r_0 x^k >= 0 \qquad (**)$$

for any $\omega \in \otimes^k T^{(r)}_0 \mathbf{R}^n$, where $(\ )^C$ is the complete lifting of vector fields to $\otimes^k T^{(r)}$.

It remains to show that $< A(0, 0, (j^r_0 x^{\sigma_1})^* \otimes \ldots \otimes (j^r_0 x^{\sigma_k})^*),\ j^r_0 x^1 \otimes \ldots \otimes j^r_0 x^k >= 0$ for any $\sigma = (\sigma_1, \ldots, \sigma_k) \in B_k$.

For showing this, for any $\sigma = (\sigma_1, \ldots, \sigma_k) \in B_k$ we prove

$$\begin{aligned}
0 &= < A((\textstyle\sum_{i=1}^n (x^i)^r \partial_i)^C_{|\omega}), j^r_0 x^1 \otimes \ldots \otimes j^r_0 x^k > \\
&= < A(0, \omega, (j^r_0 x^{\sigma_1})^* \otimes \ldots \otimes (j^r_0 x^{\sigma_k})^*), j^r_0 x^1 \otimes \ldots \otimes j^r_0 x^k > \\
&= < A(0, 0, (j^r_0 x^{\sigma_1})^* \otimes \ldots \otimes (j^r_0 x^{\sigma_k})^*), j^r_0 x^1 \otimes \ldots \otimes j^r_0 x^k >,
\end{aligned}$$

where $\omega = (j^r_0 (x^{\sigma_1})^r)^* \otimes (j^r_0 x^{\sigma_2})^* \otimes \ldots \otimes (j^r_0 x^{\sigma_k})^*$ if $r \geq 2$ and $\omega = \frac{1}{k}\ (j^r_0 x^{\sigma_1})^* \otimes \ldots \otimes (j^r_0 x^{\sigma_k})^*$ if $r = 1$.

The third equality is clear as in the formula $(*)$ λ and μ^σ are 0.

We can prove the first equality as follows. Vector fields $\partial_1 + \sum_{i=1}^n\ (x^i)^r \partial_i$ and ∂_i have the same $(r-1)$-jets at 0. Then, by the result of Zajtz [9], there exists a diffeomorphism

$\varphi : \mathbf{R}^n \to \mathbf{R}^n$ such that $j^r_0\varphi = id$ and $\varphi_*\partial_1 = \partial_1 + \sum_{i=1}^n (x^i)^r\partial_i$ near 0. Clearly, φ preserves $j^r_0x^1 \otimes \ldots \otimes j^r_0x^k$ because of the jet argument. Then, using the naturality of A with respect to φ, from $(**)$ it follows that $< A((\partial_1 + \sum_{i=1}^n (x^i)^r\partial_i)^C_\omega), j^r_0x^1 \otimes \ldots \otimes j^r_0x^k >= 0$ for any $\omega \in \otimes^k T^{(r)}_0\mathbf{R}^n$. Now, applying the linearity of A, we end the proof of the first equality.

It remains to prove the second equality. Let φ_t be the flow of $\sum(x^i)^r\partial_i$. For any $\beta^1, \ldots, \beta^k \in (\mathbf{N} \cup \{0\})^n$ with $1 \le |\beta^1| \le r$, ..., $1 \le |\beta^k| \le r$ we have

$$\begin{aligned}
&< \left(\sum(x^i)^r\partial_i\right)^C_{|\omega}, j^r_0x^{\beta^1} \otimes \ldots \otimes j^r_0x^{\beta^k} > \\
&=< \tfrac{d}{dt}_{|t=0} \otimes^k T^{(r)}(\varphi_t)(\omega), j^r_0x^{\beta^1} \otimes \ldots \otimes j^r_0x^{\beta^k} > \\
&= \tfrac{d}{dt}_{|t=0} < \otimes^k T^{(r)}(\varphi_t)(\omega), j^r_0x^{\beta^1} \otimes \ldots \otimes j^r_0x^{\beta^k} > \\
&= \tfrac{d}{dt}_{|t=0} < \omega, j^r_0(x^{\beta^1} \circ \varphi_t) \otimes \ldots \otimes j^r_0(x^{\beta^k} \circ \varphi_t) > \\
&=< \omega, \textstyle\sum_j j^r_0x^{\beta^1} \otimes \ldots \otimes j^r_0\left(\tfrac{d}{dt}_{|t=0}x^{\beta^j} \circ \varphi_t\right) \otimes \ldots \otimes j^r_0x^{\beta^k} > \\
&=< \omega, \textstyle\sum_k j^r_0x^{\beta^1} \otimes \ldots \otimes j^r_0\left(\left(\sum_j(x^i)^r\partial_i\right)x^{\beta^j}\right) \otimes \ldots \otimes j^r_0x^{\beta^k} > .
\end{aligned}$$

If $r \ge 2$, the last term is equal to $< \omega, j^r_0((\sum_i(x^i)^r\partial_i)x^{\beta^1}) \otimes j^r_0x^{\beta^2} \ldots \otimes j^r_0x^{\beta^k} >$. (It is a consequence of the definition of ω). Then the last term is equal to 1 if $j^r_0x^{\beta^1} \otimes \ldots \otimes j^r_0x^{\beta^k} = j^r_0x^{\sigma_1} \otimes \ldots \otimes j^r_0x^{\sigma_k}$ and it is equal to 0 in the other cases. Similarly, if $r = 1$, the last term is equal to 1 if $j^r_0x^{\beta^1} \otimes \ldots \otimes j^r_0x^{\beta^k} = j^r_0x^{\sigma_1} \otimes \ldots \otimes j^r_0x^{\sigma_k}$ and it is equal to 0 in the other cases. Then $(\sum(x^i)^r\partial_i)^C_{|\omega} = (j^r_0x^{\sigma_1})^* \otimes \ldots \otimes (j^r_0x^{\sigma_k})^*$ under the isomorphism $V_\omega(\otimes^k T^{(r)}\mathbf{R}^n) \cong \otimes^k T^{(r)}_0\mathbf{R}^n$. It implies the second equality. □

4. We are now in position to prove Theorem 1. Let A be a natural affinor on $\otimes^k T^{(r)}$. Then the composition $T\Pi \circ A : T(\otimes^k T^{(r)}) \to T$ is a natural transformation. By Proposition 3, there exists the real number λ such that $T\Pi \circ A = \lambda T\Pi$. Then $A - \lambda id : T(\otimes^k T^{(r)}) \to V\ (\otimes^k T^{(r)}) \cong \otimes^k T^{(r)} \times_{\mathcal{M}_n} \otimes^k T^{(r)}$. Composing this natural transformation with the projection pr_2 onto second factor we obtain a linear natural transformation $\overline{A} = pr_2 \circ (A - \lambda id) : T(\otimes^k T^{(r)}) \to \otimes^k T^{(r)}$. By Proposition 4, $\overline{A} = 0$. Then $A - \lambda id = 0$, i.e. $A = \lambda id$. □

5. Using similar proofs with $\odot^k$ and $\bigwedge^k$ (the symmetric and the skew-symmetric tensor product) instead of $\otimes^k$ one can obtain the following propositions and theorems corresponding to Proposition 2 and Theorem 1. We leave the details to the reader.

Proposition 5 *For natural numbers k, r and $n \ge k$ any natural transformation $A : \odot^k T^{(r)} \to \odot^k T^{(r)}$ over n-manifolds is proportional (by a real number) to the identity natural transformation.*

Proposition 6 *For natural numbers k, r and $n \ge k$ any natural transformation $A : \bigwedge^k T^{(r)} \to \bigwedge^k T^{(r)}$ over n-manifolds is proportional (by a real number) to the identity natural transformation.*

Theorem 7 *For integers $k \ge 2$, r and $n \ge k$ any natural affinor A on $\odot^k T^{(r)}$ over n-manifolds is proportional (by a real number) to the identity affinor.*

Theorem 8 *For integers* $k \geq 2$, r *and* $n \geq k$ *any natural affinor* A *on* $\bigwedge^k T^{(r)}$ *over* n*-manifolds in proportional (by a real number) to the identity affinor.*

References

[1] M. Doupovec, *Natural transformations between TTT^*M and TT^*TM*, "Czechoslovak Math. J.", 43 (118) 1993, 599-613.

[2] M. Doupovec, I. Kolář, *Natural affinors on time-dependent Weil bundles*, "Arch. Math." (Brno) 27 (1991), 205-209.

[3] J. Gancarzewicz, I. Kolář, *Natural affinors on the extended r-th order tangent bundles*, "Suppl. Rendiconti Circolo Mat. Palermo", 30 (1993), 95-100.

[4] I. Kolář, P.W. Michor, J. Slovák, *Natural operations in diffrential geometry*, Springer-Verlag, Berlin 1993.

[5] I. Kolář, M. Modugno, *Torsions of connections on some natural bundles*, "Diff. Geom. and Appl." 2 (1992), 1-16.

[6] I. Kolář, G. Vosmanská, *Natural transformations of higher order tangent bundles and jet spaces*, "Čas. pěst. mat." 114 (1989), 181-185.

[7] J. Kurek, *Natural affinors on higher order cotangent bundles*, "Arch. Math." (Brno) 28 (1992), 175-180.

[8] W.M. Mikulski, *Natural affinors on r-jet prolongation of the tangent bundle*, "Arch. Math." (Brno) 34 (2) (1998), 321-328.

[9] A. Zajtz, *On the order of natural operators and liftings*, "Ann. Polon. Math." 49 (1988), 169-178.

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