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ON THE PARABOLIC CLASS NUMBERS OF SOME FUCHSIAN GROUPS

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Abstract. *In this paper, we calculate parabolic class numbers of some Fuchsian groups and we give a conjecture related to the subject.*

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1 Introduction

By a Fuchsian group Λ we will mean a finitely generated discrete subgroup of $PSL(2,\mathbb{R})$, the group of conformal homeomorphisms of the upper-half plane. The most general presentation for Λ is

Generators;

$$\begin{array}{ll} a_1,b_1,\cdots\cdots\cdots,a_g,b_g & \text{(Hyperbolic)} \\ x_1,x_2,\cdots\cdots\cdots,x_r & \text{(Elliptic)} \\ p_1,p_2,\cdots\cdots\cdots,p_s & \text{(Parabolic).} \end{array}$$

Relations;

$$x_1^{m_1}=x_2^{m_2}=\ldots x_r^{m_r}=\prod_{i=1}^{g}[a_i,b_i]\prod_{j=1}^{r}x_j\prod_{k=1}^{s}p_k=1.$$

We then say Λ has signature (see [7])

$$(g,m_1,m_2,\ldots,m_r;s).$$

Let m be a sequare free positive number and let $H(\sqrt{m})$ denote the discrete subgroup of $PSL\ (2,\mathbb{R})$, which consists of all those mappings of the form

$$\begin{array}{ll} \text{i)} & T(z)=\frac{az+b\sqrt{m}}{c\sqrt{m}z+d},a,b,c,d,\in\mathbb{Z},ad-bcm=1, \\ \text{ii)} & T(z)=\frac{a\sqrt{m}z+b}{cz+d\sqrt{m}},a,b,c,d\in\mathbb{Z},adm-bc=1. \end{array}$$

When $m=2$ or $m=3$, the resulting groups are the Hecke groups ([3,4,9]). In [4], we dealt with $H(\sqrt{m})$ where $m=2$ or $m=3$. Here, we are interested in $H(\sqrt{m})$ in case m is a prime number or m is a product of different prime numbers.

2 Parabolic Class number

Let Λ be a discrete subgroup of $PSL(2,\mathbb{R})$. When $x\in R\cup\{\infty\}$ is a fixed point of a parabolic element of Λ, we say that x is a parabolic point of Λ. We also call a parabolic point of Λ a cusp of Λ.

We now give a theorem from [8], which is related to the cusps of Λ

Theorem 1 *Let x be a cusp of Λ, and $\Lambda_x = \{T \in \Lambda : T(x) = x\}$. Then Λ_x is an infinite cyclic group. Moreover, any element of Λ_x is either identity or parabolic.*

The parabolic subgroups of Λ are defined to be those non-identity cyclic subgroups $C \subset \Lambda$ which consist of parabolic elements (together with the identity) and which are maximal with respect to this property. The parabolic class number s of Λ is the number of conjugacy classes of parabolic subgroups of Λ (see [2]).

Let

$$\sqrt{m}\hat{\mathbb{Q}} = \left\{\frac{r}{s}\sqrt{m} : \frac{r}{s} \in \mathbb{Q}\right\} \cup \{\infty\}.$$

Then any parabolic point of $H(\sqrt{m})$ is in $\sqrt{m}\hat{\mathbb{Q}}$. Moreover, any point in $\sqrt{m}\hat{\mathbb{Q}}$ is a parabolic point of $H(\sqrt{m})$. To see this, consider the following mapping

$$T(z) = \frac{(1-rsm)z + r^2m\sqrt{m}}{-s^2\sqrt{m}z + 1 + rsm}.$$

It is clear that T is a parabolic mapping and $\frac{r}{s}\sqrt{m}$ is a fixed point of T where we represent ∞ as $\frac{1}{0}\sqrt{m}$. Thus $\sqrt{m}\hat{\mathbb{Q}}$ is the set of cusps of $H(\sqrt{m})$.

Let Γ be the modular group, and let $\Gamma_0(n)$ be the subgroups of Γ such that $c \equiv 0 \pmod{n}$. Then,

Lemma 1 *[6]* $|\Gamma : \Gamma_0(n)| = n\prod_{p|n}(1+\frac{1}{p})$.

Let $n \in \mathbb{N}$. Define

$$H_0^m(n) = \{T \in H(\sqrt{m}) : c \equiv 0 (\operatorname{mod} n)\}.$$

Then $H_0^m(n)$ is a subgroup of $H(\sqrt{m})$ and we have

Lemma 2 *Let $(m,n) = 1$, then $|H(\sqrt{m}) : H_0^m(n)| = n\prod_{p|n}(1+\frac{1}{p})$. If m is not prime to n, then $|H(\sqrt{m}) : H_0^m(n)| = 2n\prod_{p|n}(1+\frac{1}{p})$ where $p \nmid m$.*

Proof. Let

$$H = \left\{T \in H(\sqrt{m}) : T(z) = \frac{az + b\sqrt{m}}{c\sqrt{m}z + d}\right\}$$

and let

$$H_0 = \{T \in H : c \equiv 0 (\operatorname{mod} n)\}.$$

Then $H_0 \subset H$ and $H_0 \subset H_0^m(n)$. Moreover, it can be seen that H is conjugate to $\Gamma_0(m)$ and that H_0 is conjugate to $\Gamma_0(mn)$. This follows from the fact that $H = S^{-1}\Gamma_0(m)S$ and $H_0 = S^{-1}\Gamma_0(mn)S$ where $S(z) = \frac{1}{\sqrt{m}}z$. It is clear that $|H(\sqrt{m}) : H| = |H_0^m(n) : H_0| = 2$. Thus we have

$$|H(\sqrt{m}) : H_0^m(n)| = \frac{|H(\sqrt{m}) : H_0|}{|H_0^m(n) : H_0|} = \frac{|H(\sqrt{m}) : H||H : H_0|}{|H_0^m(n) : H_0|} = |H : H_0|.$$

Since $|H : H_0| = |S^{-1}\Gamma_0(m)S : S^{-1}\Gamma_0(mn)S| = |\Gamma_0(m) : \Gamma_0(mn)|$, we see that

$$|H(\sqrt{m}) : H_0^m(n)| = n\prod_{p|n}\left(1+\frac{1}{p}\right).$$

If m is not prime to n, then the proof is similar. □

Theorem 2 *Let $\Lambda = H_0^m(n)$. Then the parabolic class number of Λ is the number of orbits of Λ on $\sqrt{m}\hat{\mathbb{Q}}$.*

Proof. Let $x = \frac{r}{s}\sqrt{m}$ be a point in $\sqrt{m}\hat{\mathbb{Q}}$. Then x is a cusp of $H(\sqrt{m})$. That is, $S(x) = x$ for some parabolic element of $H(\sqrt{m})$. Since Λ is a subgroup of finite index in $H(\sqrt{m})$, we see that $S^k \in \Lambda$ for some positive number k. Then S^k is parabolic mapping and $S^k(x) = x$. Thus x is a cusp of Λ. More generally, the mapping

$$T(z) = \frac{(1 - rsnm)z + r^2nm\sqrt{m}}{-s^2n\sqrt{m}z + 1 + rsnm}$$

is a parabolic element of Λ and $T(x) = x$. By the above theorem, Λ_x is infinite cyclic group and any non-identity element of Λ_x is parabolic element. It can be seen that Λ_x is maximal with respect to this property. That is, Λ_x is a parabolic subgroup. Now let C be a parabolic subgroup of Λ. Then every element of C has a fixed point $x \in \sqrt{m}\hat{\mathbb{Q}}$. Therefore, $C \subset \Lambda_x$, and hence $C = \Lambda_x$. In addition, x and y lie in the same orbit if and only if Λ_x and Λ_y are conjugate in Λ. Then the proof follows. □

3 Main theorems

Theorem 3 *$H(\sqrt{m})$ is finitely generated.*

Since $\Gamma_0(m)$ is a subgroup of finite index in Γ, $\Gamma_0(m)$ is finitely generated and so H is finitely generated. In view of the fact that $|H(\sqrt{m}) : H| = 2$, it follows that $H(\sqrt{m})$ is finitely generated.

By virtue of the above theorems, it is seen that $H_0^m(n)$ is a finitely generated discrete subgroup of $H(\sqrt{m})$. Thus $H_0^m(n)$ is a Fuchsian group. Now we will give a theorem whose proof is clear and can be found in [4]. We sometimes represent $\frac{r}{s}\sqrt{m}$ as $r/s\sqrt{m}$.

Lemma 3 *Let m be a prime number. Then $H(\sqrt{m})$ acts transitively on $\sqrt{m}\hat{\mathbb{Q}}$.*

This shows that the parabolic class number of $H(\sqrt{m})$ is 1. If m is a product of two different prime numbers, we will see later that this number is 2.

Lemma 4 *Let $(m,n) = 1$, and $r/s\sqrt{m} \in \sqrt{m}\hat{\mathbb{Q}}$ with $m|s$. Then we can find some $T \in H_0^m(n)$ such that $T(r/s\sqrt{m}) = r_1/s_1\sqrt{m}$ with $(m,s_1) = 1$.*

Proof. Since $(m,n) = 1$, there exist some $a,b \in \mathbb{Z}$ such that $1 = ma - nb$. Let

$$T(z) = \frac{a\sqrt{m}z + b}{nz + \sqrt{m}}.$$

Then $T \in H_0^m(n)$, and

$$T\left(\frac{r}{s}\sqrt{m}\right) = \frac{arm + bs}{(rn+s)\sqrt{m}} = \frac{ar + bs/m}{rn+s}\sqrt{m}.$$

It can be easily shown that $(m, rn+s) = 1$. If we take $r_1 = ar + bs/m$, and $s_1 = rn + s$, then $T(r/s\sqrt{m}) = r_1/s_1\ \sqrt{m}$ with $(m, s_1) = 1$.

In the following, we will give some lemmas. The proofs are similar to those given in [4]. But, we include the proofs for the sake of completeness. $\square$

Lemma 5 *Let $(m,n) = 1$, and $k/s\sqrt{m} \in \sqrt{m}\hat{\mathbb{Q}}\backslash\{\infty\}$ with $(k,s) = 1$. If $(m,s) = 1$, then there exist some $T \in H_0^m(n)$ such that $T(k/s\sqrt{m}) = k_1/s_1\sqrt{m}$ with $s_1|n$.*

Proof. Since $(k,s) = (m,s) = 1$, we see that $(km,s) = 1$. Let $s_1 = (s,n)$. Then $s_1 = (s,n) = (s,kmn)$. Therefore there exist some integers c_1, d_1 such that

$$kmnc_1 + sd_1 = s_1.$$

Since $\left(d_1, \frac{kmn}{s_1}\right) = 1$, there exists an integer k_0 such that $\left(d_1 - \frac{kmn}{s_1}k_0, mn\right) = 1$. Let $d = d_1 - \frac{kmn}{s_1}k_0$ and $c = c_1 + \frac{s}{s_1}k_0$. Then

$$kmnc + sd = s_1.$$

On the other hand, $(d, cmn) = 1$, since $(d, mn) = (d, c) = 1$. Hence, we can find some integers x, y such that $xd - ycmn = 1$. If we take

$$T(z) = \frac{xz + y\sqrt{m}}{cn\sqrt{m}z + d},$$

then we have $T(r/s\sqrt{m}) = k_1/s_1\sqrt{m}$ where $k_1 = xk + ys$ and $s_1 = cnmk + ds$. It is obvious that $T \in H_0^m(n)$. On the other hand, it can be seen that $(k_1, s_1) = 1$. $\square$

Lemma 6 *Let $(m,n) = 1$, $d_1|n$, and $(a_1,d_1) = (a_2,d_1) = 1$. Then $a_1/d_1\sqrt{m}$ is conjugate to $a_2/d_1\sqrt{m}$ under $H_0^m(n)$ if and only if $a_1 \equiv a_2 \pmod{t}$ where $t = (d_1, n/d_1)$.*

Proof. Let $a_1 \equiv a_2 (\bmod\, t)$ and $n_1 = n/d_1$. Then $t = (d_1, n_1)$, and $(a_1a_2, d_1) = 1$. Furthermore, $(m, d_1) = 1$ since $(m,n) = 1$. Therefore, $(a_1a_2m, d_1) = 1$, and thus $(n_1a_1a_2m, d_1) = t$. Since $t|a_1 - a_2$, $mn_1a_1a_2x + d_1y = a_2 - a_1$ has a solution. That is, there exist some integers k, s such that $mn_1a_1a_2k + a_1 + d_1s = a_2$. Hence, we obtain $a_1(1 + mn_1a_2k) + d_1s = a_2$. If we take $a = 1 + mn_1a_2k$ and $b = s$, then we have $aa_1 + bd_1 = a_2$. On the other hand, if we take $c = n_1d_1k$ and $d = 1 - mn_1a_1k$, then we obtain $mca_1 + dd_1 = d_1$.

Furthermore,

$$ad - bcm = a(1 - mn_1a_1k) - bmn_1d_1k = a - (aa_1 + bd_1)mn_1k = 1.$$

Let

$$T(z) = \frac{az + b\sqrt{m}}{c\sqrt{m}z + d}.$$

Then it is clear that $T \in H_0^m(n)$ and $T(a_1/d_1\sqrt{m}) = a_2/d_1\sqrt{m}$.

Now let $a_1/d_1\sqrt{m}$ be equivalent to $a_2/d_1\sqrt{m}$ by some $T \in H_0^m(n)$. Then it is easily seen that

$$T(z) = \frac{az + b\sqrt{m}}{cn\sqrt{m}z + d} \quad \text{where } ad - bcmn = 1.$$

Hence, we obtain

$$\frac{aa_1 + bd_1}{cna_1m + dd_1} = \frac{a_2}{d_1}$$

Since

$$d(aa_1 + bd_1) - b(cna_1m + dd_1) = a_1,$$

and

$$a(cna_1m + dd_1) - cnm(aa_1 + bd_1) = d_1,$$

we have $(aa_1 + bd_1, cna_1m + dd_1) = 1$. Therefore, there exists some $u = \mp 1$ such that

$$aa_1 + bd_1 = ua_2,$$

and

$$cna_1m + dd_1 = ud_1.$$

From the above equations and from the fact that $ad - bcm = 1$, it follows that $a_1 \equiv a_2 \pmod{t}$. □

Theorem 4 *Let $(m,n) = 1$ and let m be a prime number. Then the parabolic class number of $H_0^m(n)$ is*

$$\sum_{d|n} \varphi((d, \frac{n}{d})).$$

Proof. It suffices to calculate the number of orbits of $H_0^m(n)$ on $\sqrt{m}\hat{\mathbb{Q}}$. Then from Lemma 4, Lemma 5, and Lemma 6, the number of orbits of $H_0^m(n)$ on $\sqrt{m}\hat{\mathbb{Q}}$ is $\sum_{d|n} \varphi((d, \frac{n}{d}))$ where φ is Euler's function.

We can deduce the following. □

Lemma 7 *Let $k/s \in \hat{\mathbb{Q}}$ with $(k,s) = 1$. Then there exists some $T \in \Gamma_0(n)$ such that $T(k/s) = k_1/s_1$ with $s_1|n$ where we represent ∞ as $\frac{1}{0}$.*

Lemma 8 *If $d_1|n$ and $(a_1,d_1) = (a_2,d_1) = 1$, then a_1/d_1 is conjugate to a_2/d_1 under $\Gamma_0(n)$ if and only if $a_1 \equiv a_2 \pmod{t}$ where $t = (d_1, n/d_1)$.*

Then it is easily seen that the number of orbits of $\Gamma_0(n)$ on $\hat{\mathbb{Q}}$ is

$$\sum_{d|n} \varphi((d, \frac{n}{d})),$$

which is the parabolic class number of $\Gamma_0(n)$.

Theorem 5 *If $m|n$, then the parabolic class number of $H_0^m(n)$ is*

$$\sum_{d|n} \varphi((d, \frac{nm}{d})).$$

Proof. If m is not prime to n, then

$$H_0^m(n) = H_0 = \{T \in H : c \equiv 0 (\bmod n)\}.$$

Therefore, $H_0^m(n)$ is conjugate to $\Gamma_0(mn)$. Then the proof follows. □

Lemma 9 *Let $m = m_1 m_2$ and $(m,n) = 1$ where m_1 and m_2 are different prime numbers, and let $\frac{k}{s}\sqrt{m} \in \sqrt{m}\hat{\mathbb{Q}} \setminus \{\infty\}$ with $m_1|s$ and $(m_2,s) = 1$. Then $T\left(\frac{k}{s}\sqrt{m}\right) = \frac{k_1}{m_2 s_1}\sqrt{m}$ with $(m_1,s_1) = 1$ for some $T \in H_0^m(n)$.*

Proof. Let T be as in Lemma 4. Then

$$T(r/s\sqrt{m}) = \frac{arm + bs}{(rn+s)\sqrt{m}} = \frac{arm + btm_1}{(rn+s)m}\sqrt{m} = \frac{arm_1m_2 + btm_1}{(rn+s)m_1m_2}\sqrt{m} = \frac{k_1}{s_1 m_2}\sqrt{m}.$$

where $k_1 = arm_2 + bt$ and $s_1 = rn + s$. It is clear that $(m_1,s_1) = 1$ and $(m_2,k_1) = 1$. □

Lemma 10 *Let $(m,n) = 1$, and $\frac{k}{sm_2}\sqrt{m} \in \sqrt{m}\hat{\mathbb{Q}} \setminus \{\infty\}$ with $(k,sm_2) = (m_1,s) = 1$. Then there exist some $T \in H_0^m(n)$ such that $T\left(\frac{k}{sm_2}\sqrt{m}\right) = \frac{k_1}{s_1 m_2}\sqrt{m}$ with $s_1|n$.*

Proof. Let $s_1 = (s,n)$. Then $s_1 = (s,n) = (s, km_1 n)$. Therefore, there exist some integers c_1, d_1 such that

$$km_1nc_1 + sd_1 = s_1.$$

Since $\left(d_1, \frac{km_1n}{s_1}\right) = 1$, there exists an integer t_0 such that $\left(d_1 - \frac{km_1n}{s_1}t_0, mn\right) = 1$. Let $d = d_1 - \frac{km_1n}{s_1}t_0$ and $c = c_1 + \frac{s}{s_1}t_0$. Then

$$km_1nc + sd = s_1.$$

On the other hand, $(d, cmn) = 1$, since $(d,mn) = (d,c) = 1$. Hence, we can find some integers x,y such that $xd - ycmn = 1$. If we take

$$T(z) = \frac{xz + y\sqrt{m}}{cn\sqrt{m}z + d},$$

then we have

$$T\left(\frac{k}{sm_2}\sqrt{m}\right) = \left(\frac{k_1}{(cnkm + ds)}\right)\sqrt{m} = \left(\frac{k_1}{m_2(cnkm_1 + ds)}\right)\sqrt{m} = \frac{k_1}{m_2 s_1}\sqrt{m}$$

where $k_1 = xk + ysm_2$ and $s_1 = cnm_1k + ds$. It is clear that $T \in H_0^m(n)$ and $s_1|n$. On the other hand, it can be seen that $(k_1,s_1) = (m_2,k_1) = 1$. □

Lemma 11 *Let $(m,n)=1$ and let $d_1|n$, and $(a_1,d_1)=(a_2,d_1)=1$. Then $\frac{a_1}{d_1m_2}\sqrt{m}$ is conjugate to $\frac{a_2}{d_1m_2}\sqrt{m}$ under $H_0^m(n)$ if and only if $a_1\equiv a_2 \pmod{t}$ where $t=(d_1,n/d_1)$.*

Proof. Let $a_1\equiv a_2 \pmod{t}$ and $n_1=n/d_1$. Then, $t=(d_1,n_1)$, and $(a_1a_2,d_1)=1$. Therefore, $(a_1a_2m_1,d_1m_2)=1$ and thus $(a_1a_2m_1n_1,d_1m_2)=t$. Since $t|a_1-a_2$, $m_1n_1a_1a_2x+d_1m_2y=a_2-a_1$ has a solution. That is, there exist some integers k,s such that $m_1n_1a_1a_2k+a_1+d_1m_2s=a_2$. Hence, we obtain $a_1(1+m_1n_1a_2k)+d_1m_2s=a_2$. If we take $a=1+m_1n_1a_2k$ and $b=s$, then we have $aa_1+bd_1m_2=a_2$. On the other hand, if we take $c=n_1d_1k$ and $d=1-m_1n_1a_1k$, then we obtain $mca_1+dd_1m_2=d_1m_2$.

Furthermore,

$$ad-bcm=a(1-m_1n_1a_1k)-bmn_1d_1k=a-(aa_1+bd_1m_2)m_1n_1k=1.$$

Let

$$T(z)=\frac{az+b\sqrt{m}}{c\sqrt{m}z+d}.$$

Then it is clear that $T\in H_0^m(n)$ and

$$T\left(\frac{a_1}{d_1m_2}\sqrt{m}\right)=\frac{aa_1+bd_1m_2}{mca_1+dd_1m_2}\sqrt{m}=\frac{a_2}{d_1m_2}\sqrt{m}.$$

If $\frac{a_1}{d_1m_2}\sqrt{m}$ is equivalent to $\frac{a_2}{d_1m_2}\sqrt{m}$ by some $T\in H_0^m(n)$. Then the proof is similar to that given in Lemma 6. □

Theorem 6 *Let $m=m_1m_2$ and $(m,n)=1$ where m_1 and m_2 are the different prime numbers. Then the parabolic class number of $H_0^m(n)$ is*

$$2\sum_{d|n}\varphi((d,\frac{n}{d})).$$

Proof. The proof follows from Lemma 4, 5, and 6 and Lemma 9, 10, and 11. □

Lemma 12 *Let $m=m_1m_2m_3$ and $(m,n)=1$. Given $k/s\sqrt{m}\in\sqrt{m}\hat{\mathbb{Q}}\setminus\{\infty\}$ with $(m_j,s)=1$, and $(m/m_j,s)=m/m_j$, we can find some $T\in H_0^m(n)$ such that*

$$T\left(\frac{k}{s}\sqrt{m}\right)=\frac{k_j}{s_jm_j}\sqrt{m}$$

with $(m_j,s_j)=1$ where $j=1,2,3$.

Lemma 13 *Let $(m,n)=1$, and $\frac{k}{sm_j}\sqrt{m}\in\sqrt{m}\hat{\mathbb{Q}}\setminus\{\infty\}$ with $(k,sm_j)=(m_j,s)=1$. Then there exist some $T_j\in H_0^m(n)$ such that*

$$T_j\left(\frac{k}{sm_j}\sqrt{m}\right)=\frac{k_j}{d_jm_j}\sqrt{m}$$

with $d_j|n$ where $j=1,2,3$.

Lemma 14 *Let* $(m,n)=1$ *and let* $d_1|n$*, and* $(a_1,d_1)=(a_2,d_1)=1$*. Then* $\frac{a_1}{d_1m_j}\sqrt{m}$ *is conjugate to* $\frac{a_2}{d_1m_j}\sqrt{m}$ *under* $H_0^m(n)$ *if and only if* $a_1 \equiv a_2 \pmod{t}$ *where* $t=(d_1,n/d_1)$ *and* $j=1,2,3$.

Theorem 7 *Let* $m=m_1m_2m_3$ *and* $(m,n)=1$ *where* m_1,m_2*, and* m_3 *are the different prime numbers. Then the parabolic class number of* $H_0^m(n)$ *is*

$$4\sum_{d|n}\varphi((d,\frac{n}{d})).$$

If m is a product of four different prime numbers where $(m,n)=1$. Then it can be shown that the parabolic class number of $H_0^m(n)$ is

$$8\sum_{d|n}\varphi((d,\frac{n}{d})).$$

At this point, it seems that

Conjecture 1 *Let m be a product of k different prime numbers and let m be prime to n. Then the parabolic class number of* $H_0^m(n)$ *is*

$$2^{k-1}\sum_{d|n}\varphi((d,\frac{n}{d})).$$

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