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NOTE ON SQUAREFREE INTEGERS THROUGH A SET THEORETICAL PROPERTY

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Abstract. *In this paper, we show a property of set theory, that in number theory has the following consequence: if $a_1 < a_2 < \ldots < a_n$ are squarefree integers, then the number of distinct ratios $a_i/(a_i,a_j)$ is greater than or equal to n, where (a_i,a_j) denotes the greatest common divisor of a_i and a_j.*

The cardinal of a set F is denoted by $|F|$, and the collection of differences of members of any collection $\mathbf{G}$, by $\mathcal{D}(\mathbf{G})$; obviously, if $\mathbf{G} = \emptyset$, then $\mathcal{D}(\mathbf{G}) = \emptyset$.

Let $\mathbf{F} = \{F_i\}$ be a finite collection of finite sets. If $|\mathbf{F}| \geq 2$, let $k = \min|F_i \cap F_j|$ for $F_i \neq F_j$, and let F_1, F_2 be two fixed sets for which this minimum is attained, that is, $F_1 \cap F_2 = I$, $|I| = k$.

Lemma 1 *If $\mathbf{F}$ is any finite non-empty collection of sets, then there exists a partition of $\mathbf{F}$ into disjoint subcollections $\mathbf{A}$ and $\mathbf{D}$, with $\mathbf{A} \neq \emptyset$, satisfying $|\mathcal{D}(\mathbf{F})| \geq |\mathbf{A}| + |\mathcal{D}(\mathbf{D})|$.*

Proof. If $|\mathbf{F}| = 1$, the assertion is trivial. If $|\mathbf{F}| \geq 2$, divide $\mathbf{F}$ into three disjoint subcollections $\mathbf{A}$, $\mathbf{B}$ and $\mathbf{C}$, according to the following criteria:

$$(a) \qquad \mathbf{C} = \{\textit{members of } \mathbf{F}\textit{, which do not contain } I\}.$$

The rest of the sets do contain I and we write $F_i = F_i' + I$, where $F_i' = F_i - I$, for such sets. Then,

$$(b) \qquad \mathbf{B} = \{F_i : \textit{for all } F_j \notin \mathbf{C},\ F_i' \cap F_j' \neq \emptyset\}.$$

$$(c) \qquad \mathbf{A} = \{F_i : \textit{for some } F_j \notin \mathbf{C},\ F_i' \cap F_j' = \emptyset\}.$$

It is clear that $\mathbf{A} \neq \emptyset$, since at least F_1 and F_2 are in $\mathbf{A}$. If $F_i \in \mathbf{A}$ and F_j is as in (c), then F_j is also in $\mathbf{A}$, F_i' and F_j' are disjoint and so appear in $\mathcal{D}(\mathbf{A}), (F_i - F_j = F_i')$. We can see that F_i' and F_j' do not occur in $\mathcal{D}(\mathbf{B} \cup \mathbf{C})$, as follows. That each set in $\mathbf{B}$ has a non-empty intersection with F_i' is immediate from the definition of $\mathbf{B}$.

No member Q of $\mathbf{C}$ can be disjoint from F_i'; for $|Q \cap F_i| \geq k$ and since $Q \cap F_i \neq I$ (from (a)), $Q \cap F_i' = Q \cap (F_i - I) \neq \emptyset$. If now $X, Y \in \mathbf{B} \cup \mathbf{C}$, then $X - Y \neq F_i'$ because $X - Y$ contains no element of Y while F_i' does contain some element of Y. This holds for any F_i in $\mathbf{A}$.

Then, we have found that for each member F_i of $\mathbf{A}$ there exists a difference F_i' appearing in $\mathcal{D}(\mathbf{A})$, which does not appear in $\mathcal{D}(\mathbf{B} \cup \mathbf{C})$. Clearly, $F_i \neq F_j$ implies $F_i' \neq F_j'$ and the lemma is proved. □

Theorem 2 *If $\mathbf{F}$ is a finite collection of sets, then the number of distinct differences of members of $\mathbf{F}$ is at least as large as the number of members of $\mathbf{F}$.*

Proof. We proceed by induction. Clearly, the theorem holds for collections of 1 or 2 sets. If $\mathbf{F}$ were a collection of minimal cardinal for which it failed, then, taking $\mathbf{F} = \mathbf{A} \cup \mathbf{D}$ as above, we would have $|\mathcal{D}(\mathbf{F})| \geq |\mathbf{A}| + |\mathcal{D}(\mathbf{D})|$. But, $\mathbf{A} \neq \emptyset$ so $|\mathbf{D}| < |\mathbf{F}|$, and, by induction, $|\mathcal{D}(\mathbf{D}| \geq |\mathbf{D}|$. Thus, $|\mathcal{D}(\mathbf{F}| \geq |\mathbf{A}| + |\mathbf{D}| = |\mathbf{F}|$, that is, a contradiction, and the theorem is proved. □

Remark. Let $K(n,\mathbf{F})$ denote $|\mathcal{D}(\mathbf{F})|$, for $\mathbf{F}$ a collection of n sets. We have shown that $K(n, \mathbf{F}) \geq n$ and, since $F_i \in \mathbf{F}$ implies $F_i - F_i = \emptyset$, it is clear that $K(n, \mathbf{F}) \leq n^2 - n + 1$. It can be shown that both of these bounds are attained for each n with a suitable $\mathbf{F}$. However, one can still ask which restrictions can be imposed on $\mathbf{F}$, in order to yield more precise but usefull results, e.g., $\emptyset$ and $\cup F_i \notin \mathbf{F}$.

Given n positive integers $a_1 < a_2 < \ldots < a_n$, we denote by (a_i, a_j) the greatest common divisor of a_i and a_j. The following question, naturally, arises:

$$\textit{there exist n different ratios } a_i/(a_i,a_j)? \tag{1}$$

However, this is not true in general, as shown by the following counterexample: the set of all non trivial divisors of 36. There are 7 divisors, but only 5 distinct ratios.

Obviously, the above theorem is the combinatorial analogue of (1), and immediately, we have the following

Corollary 3 *If $a_1 < a_2 < \ldots < a_n$ are squarefree integers, then the number of distinct ratios $a_i/(a_i,a_j)$ is greater than or equal to n.* □

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