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THE LINEAR NATURAL OPERATORS LIFTING 2-VECTOR FIELDS TO SOME WEIL BUNDLES

WŁODZIMIERZ M. MIKULSKI

Abstract. *All linear natural operators lifting 2-vector fields to some product preserving bundle functors are classified.*

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0. Let $F : \mathcal{M} \to \mathcal{FM}$ be a product preserving bundle functor and let $A = F(\mathbf{R})$ be its Weil algebra, [4]. We assume the following property.

(0.1) There exist a basis $a_0, \dots, a_k \in A$ and elements $b_0, \dots, b_k \in A$ such that:
(i) $a_\nu b_\nu \in A \setminus span\{a_0, \dots, a_{k-1}\}$ for $\nu = 0, \dots, k$,
(ii) $a_\mu b_\nu \in \; span\{a_0, \dots a_{k-1}\}$ for $\mu \neq \nu, \nu = 0, \dots, k$.

For example, the tangent bundle functor T^k of order k satisfies (0.1) for $a_\nu = j_0^k t^\nu$, $b_\nu = j_0^k t^{k-\nu}$. If F_1 and F_2 satisfy (0.1) for $(a^1_{\nu_1}, b^1_{\nu_1})$ and $(a^2_{\nu_2}, b^2_{\nu_2})$, then so does $F_1 \circ F_2$ for $(a^1_{\nu_1} \otimes a^2_{\nu_2}, b^1_{\nu_1} \otimes b^2_{\nu_2})$. ($F_1 \circ F_2(\mathbf{R}) = F_1(\mathbf{R}) \otimes F_2(\mathbf{R})$, see [4])

In this short note we prove that if $n \geq 2$ and F satisfies (0.1), then the vector space of all linear natural operators $T \wedge T_{|\mathcal{M}_n} \rightsquigarrow (T \wedge T)F$ lifting 2-vector fields from n-manifolds to F, in the sense of [4], has dimension $dim(F(\mathbf{R}))$. Moreover, we construct explicitly all elements from this vector space.

Thus this note is a next contribution to the theory of natural operators in differential geometry, [4].

Troughout this note the usual coordinates on $\mathbf{R}^n$ are denoted by $x^1, \dots, x^n$ and $\partial_i = \frac{\partial}{\partial x^i}$, $i = 1, \dots, n$.

All manifolds and maps are assumed to be of class C^∞.

1. The crucial point in our consideration is the following lemma.

Lemma 1 *Under the assumption (0.1) the vector space of all linear natural operators* $T \wedge T_{|\mathcal{M}_n} \rightsquigarrow (T \wedge T)F$ *has dimension* $\leq dim(F(\mathbf{R}))$.

Proof. Let $\mathcal{L} : T \wedge T_{|\mathcal{M}_n} \rightsquigarrow (T \wedge T)F$ be a linear natural operator. Let $a_0, \dots, a_k$ and $b_0, \dots, b_k$ be as in (0.1). Assume that $n \geq 2$.

At first we prove that there exist the real numbers $A_{\mu\nu} \in \mathbf{R}$ such that

$$\mathcal{L}(\partial_1 \wedge \partial_2) = \sum_{\mu,\nu=0}^{k} A_{\mu\nu} \partial_1^{(a_\mu)} \wedge \partial_2^{(a_\nu)}, \tag{1.1}$$

where for a given $a \in A$ the operation $()^{(a)}$ is the (a)-lift of vector fields to F in the sense of [1].

For proving this we write $\mathcal{L}(\partial_1 \wedge \partial_2) = \Sigma B_{(i,\mu),(j,\nu)}\ \partial_i^{(a_u)} \wedge \partial_j^{(a_\nu)}$ for some uniquely determined smooth functions $B_{(i,\mu),(j,\nu)} : F(\mathbf{R}^n) \to \mathbf{R}$, where the sum is over all (i,μ), $(j,\nu) \in \{1,\dots,n\} \times \{0,\dots,k\}$ with $(i,\mu) < (j,\nu)$. (The $\partial_i^{(a_\mu)}$ for $i = 1, \dots n$ and $\mu = 0, \dots k$ form a basis of vector fields on $F(\mathbf{R}^n)$, see [1]). By the linearity and the naturality of $\mathcal{L}$ with respect to the homotheties $tid_{\mathbf{R}^n}$, $t \neq 0$, we get $\mathcal{L}(\partial_1 \wedge \partial_2) = \Sigma\, B_{(i,\mu),(j,\nu)} \circ \frac{1}{t} id_{F(\mathbf{R}^n)} \partial_j^{(a_u)} \wedge \partial_j^{(a_\nu)}$. Then $B_{(i,\mu),(j,\nu)}$ are constants. Next, by the naturality with respect to the diffeomorphisms $(x^1, x^2, tx^3, \dots, tx^n)$, $t \neq 0$, we deduce $B_{(i,\mu),(j,\nu)} = 0$ if $i \in \{3,\dots,n\}$ or $j \in \{3,\dots,n\}$. Finally, by the naturality with respect to the diffeomorphisms $(tx^1, x^2, \dots, x^n)$ (or $(x^1, tx^2, x^3, \dots x^n)$), $t \neq 0$, we deduce $B_{(1,\mu),(1,\nu)} = 0$ (or $B_{(2,\mu),(2,\nu)} = 0$), as well.

Next, we prove that for any $c \in A$

$$\sum_{\mu,\nu=0}^{k} A_{\mu\nu} \partial_2^{(ca_\mu)} \wedge \partial_2^{(a_\nu)} = 0. \tag{1.2}$$

We can assume that $c \neq 0$. Let $c_0 = c$, $c_1, \dots, c_k \in A$ be a basis of A. Let $c_0^*, \dots, c_k^*$ be the dual basis. In [1] we proved that $(x^1\partial_2)^{(a_\mu)} = \sum_{\rho=0}^{k} (x^1)^{(c_\rho^*)}\ \partial_2^{(c_\rho a_\mu)}$ for any $\mu = 0, \dots, k$, where $()^{(\lambda)}$ is the (λ)-lift of functions to F for a linear map $\lambda : A \to \mathbf{R}$. Since $[\partial_1 + x^1\partial_2, \partial_2] = 0$, there exists a diffeomorphism $\eta : \mathbf{R}^n \to \mathbf{R}^n$ such that $\eta_*\partial_1 = \partial_1 + x^1\partial_2$ and $\eta_*\partial_2 = \partial_2$ near 0. Now, applying (1.1) and the invariancy of $\mathcal{L}$ with respect to η, we have

$$\begin{aligned} \mathcal{L}(\partial_1 \wedge \partial_2) &= \mathcal{L}((\partial_1 + x^1\partial_2) \wedge \partial_2) = \textstyle\sum_{\mu,\nu=0}^{k} A_{\mu\nu} (\partial_1 + x^1\partial_2)^{(a_\mu)} \wedge \partial_2^{(a_\nu)} = \\ &= \mathcal{L}(\partial_1 \wedge \partial_2) + \textstyle\sum_{\mu,\nu,\rho=0}^{k} A_{\mu\nu} (x^1)^{(c_\rho^*)} \partial_2^{(a_\mu c_\rho)} \wedge \partial_2^{(a_\nu)}, \end{aligned}$$

i.e. $\sum_{\mu,\nu,\rho=0}^{k} A_{\mu\nu} (x^1)^{(c_\rho^*)} \partial_2^{(a_\mu c_\rho)} \wedge \partial_2^{(a_\nu)} = 0$ over some neighbourhood of 0. Taking a point $y \in F(\mathbf{R}^n)$ over this neighbourhood with $(x^1)^{(c_0^*)}(y) \neq 0$ and $(x^1)^{(c_1^*)}(y) = \dots = (x^1)^{(c_k^*)}\ (y) = 0$, we obtain (1.2) at y. Since the vector fields $\partial_i^{(a)}$ are invariant with respect to the translations $\tau_z : A^n \to A^n$ for $z \in A^n = F(\mathbf{R}^n)$, we have (1.2).

From (1.2) for $c = 1$ it follows $A_{\mu\nu} = A_{\nu\mu}$:

Of course, the proof of Lemma will be complete after proving the following implication. If $A_{\nu k} = 0$ for $\nu = 0, \dots, k$, then $\mathcal{L} = 0$.

Assume that $A_{\nu k} = 0$ for $\nu = 0, \dots, k$.

If $\mu_o = 0, \dots k-1$ and $A_{\nu k} = 0$ for $\nu = 0, \dots, k$, then by (1.2) for $c = b_{\mu_o}$, we have $\sum_{\mu,\nu=0}^{k-1} A_{\mu\nu}\ \partial_2^{(b_{\mu_o} a_\mu)} \wedge \partial_2^{(a_\nu)} = 0$. By (0.1), the left side of this equality is $\sum_{\nu=0}^{k-1} \alpha_{\mu_o} A_{\mu_o \nu}\ \partial_2^{(a_k)} \wedge \partial_2^{(a_\nu)} + \dots$ for some $\alpha_{\mu_o} \neq 0$, where the dots is the element from $span\{\partial_2^{(a_\mu)} \wedge \partial_2^{(a_\nu)}\ |\mu,\nu = 0,\dots,k-1\}$. Hence $A_{\mu_o \nu} = 0$ for $\nu = 0, \dots, k-1$.

Then $\mathcal{L}(\partial_1 \wedge \partial_2) = 0$ because of (1.1).

Let $\alpha_1, \dots, a_n \in \mathbf{N} \cup \{0\}$ be numbers. There is a diffeomorphism $\varphi : \mathbf{R} \to \mathbf{R}$ such that $(\varphi \times id_{\mathbf{R}^{n-1}})_*\ \partial_1 = \partial_1 + (x^1)^{\alpha_1}\partial_1$ near 0. From $\mathcal{L}(\partial_1 \wedge \partial_2) = 0$ it follows that $\mathcal{L}((\partial_1 + (x^1)^{\alpha_1}\ \partial_1) \wedge \partial_2) = 0$ over 0. Then $\mathcal{L}((x^1)^{\alpha_1}\ \partial_1 \wedge \partial_2) = 0$ over 0. There is a diffeomorphism $\psi : \mathbf{R}^{n-1} \to \mathbf{R}^{n-1}$ such that $(id_{\mathbf{R}} \times \psi)_*\partial_2 = \partial_2 + (x^2)^{\alpha_2} \dots (x^n)^{\alpha_n}\ \partial_2$ near 0. Then $\mathcal{L}\ ((x^1)^{\alpha_1}\ \partial_1 \wedge (\partial_2 + (x^2)^{\alpha_2} \dots (x^n)^{\alpha_n}\ \partial_2)) = 0$ over 0. Thus $\mathcal{L}((x^1)^{\alpha_1} \dots (x^n)^{\alpha_n} \partial_1 \wedge \partial_2) = 0$ over 0.

Now, by the symmetries permuting the coordinates on $\mathbf{R}^n$, we have $\mathcal{L}((x^1)^{\alpha_1} \dots (x^n)^{\alpha_n} \partial_i \wedge \partial_j) = 0$ over 0 for any $\alpha_1, \dots, \alpha_n \in \mathbf{N} \cup \{0\}$ and any $i, j = 1, \dots, n$ with $i < j$. Hence $\mathcal{L} = 0$ over 0 because of the base extending version of Peetre theorem (see Th. 19.9 in [4]). Therefore $\mathcal{L} = 0$ because of the naturality. □

2. To present examples of linear natural operators $T \wedge T|_{\mathcal{M}_n} \rightsquigarrow (T \wedge T)F$ for F satisfying (0.1) we need the following facts from [2] holding for arbitrary product preserving bundle functors F.

(i) If $\pi : E \to M$ is a vector bundle, then $F(\pi) : F(E) \to F(M)$ is a vector bundle and an A-module bundle.

(ii) Given a vector bundle $E \to M$, we have a fibre skew-$\mathbf{R}$-bilinear map $F(\wedge) : F(E) \times_{F(M)} F(E) = F(E \times_M E) \to F(E \wedge E)$ and (by the universal factorization property) a vector bundle epimorphism $F(\wedge) : F(E) \wedge F(E) \to F(E \wedge E)$ covering $id_{F(M)}$, where $\wedge : E \times_M E \to E \wedge E$ is the standard fibre skew-bilinear map.

(iii) Given a vector bundle $E \to M$, we have an A-module bundle isomorphism $F(E^*) \to (F(E))^{*(A)} := \bigcup_{y \in F(M)} Hom_A(F(E)_y, A)$ covering $id_{F(M)}$ and corresponding to the fibre A-bilinear map $F(<,>) : F(E) \times_{F(M)} F(E^*) = F(E \times_M E^*) \to A = F(\mathbf{R})$, where $<,> : E \times_M E^* \to \mathbf{R}$ is the usual pairing.

(iv) Given an $\mathbf{R}$-linear map $\lambda : A \to \mathbf{R}$ and a vector bundle $E \to M$, we have a vector bundle homomorphism $\omega_E^\lambda : F(E^*) \to F(E)^*$ covering $id_{F(M)}$ given by the composition of $F(E^*) \tilde{=} F(E)^{*(A)}$ with the vector bundle homomorphism $F(E)^{*(A)} \to F(E)^*$, $\alpha \to \lambda \circ \alpha$. If the $\mathbf{R}$-bilinear symmetric form $A \times A \ni (a,b) \to \lambda(ab) \in \mathbf{R}$ is non-singular, then ω_E^λ is an isomorphism.

Example. Let $\lambda_o : A \to \mathbf{R}$ be a linear map such that the $\mathbf{R}$-bilinear symmetric form $A \times A \ni (a,b) \to \lambda_o(ab) \in \mathbf{R}$ is non-singular. For example, let $\lambda_o = a_k^*$, where $a_0, \dots, a_k$ is the basis as in (0.1), and $a_0^*, \dots, a_k^*$ is the dual basis. (Then $\omega_E^{\lambda_o}$ is an isomorphism for a vector bundle E, see above). Let $\lambda : A \to \mathbf{R}$ be a linear map. Given a 2-vector field Λ on an n-manifold M we define a 2-vector field $\Lambda^{(\lambda,\lambda_o)}$ on $F(M)$ to be the composition of $F(\Lambda) : F(M) \to F(TM \wedge TM) = F((TM \wedge TM)^{**})$ and the following vector bundle homomorphisms covering $id_{F(M)}$

$$\begin{aligned}
&\omega^\lambda_{(TM\wedge TM)^*} : F((TM \wedge TM)^{**}) \to (F((TM \wedge TM)^*))^*,\\
&\Phi : (F((TM \wedge TM)^*))^* \to (F((TM)^*))^* \wedge (F((TM)^*))^*,\\
&\Phi^{\lambda_o} : (F((TM)^*))^* \wedge (F((TM)^*))^* \to (F(TM))^{**} \wedge (F(TM))^{**},\\
&\Psi : (F(TM))^{**} \wedge (F(TM))^{**} \to T(F(M)) \wedge T(F(M)).
\end{aligned}$$

Here Φ is dual to $F(\wedge) : F((TM)^*) \wedge F((TM)^*) \to F((TM)^* \wedge (TM)^*) \tilde{=} F((TM \wedge TM)^*)$ modulo $(F((TM)^*))^* \wedge (F((TM)^*))^* \tilde{=} (F((TM)^*) \wedge F((TM)^*))^*$, Φ^{λ_o} is induced by $((\omega_{TM}^{\lambda_o})^*)^{-1} : (F((TM)^*))^* \to (F(TM))^{**}$ and Ψ is given by the (described in [4]) flow isomorphism $F(TM) \tilde{=} T(F(M))$ and $(F(TM))^{**} = F(TM)$.

Clearly, under the condition on λ_o, the mapping $((\omega_{TM}^{\lambda_o})^*)^{-1}$ is defined.

The correspondence $\Lambda \to \Lambda^{(\lambda,\lambda_o)}$ is a linear natural operator $T \wedge T|_{\mathcal{M}_n} \rightsquigarrow (T \wedge T)F$ in the sense of [4]. We denote this operator by $\mathcal{L}^{(\lambda,\lambda_o)}$.

Since $\Phi, \Phi^{\lambda_o}, \Psi$ are monomorphisms, the correspondence $\lambda \to \mathcal{L}^{(\lambda,\lambda_o)}$ is a linear monomorphism. Then from the lemma it follows the following theorem.

Theorem 2 *Let $F : \mathcal{M} \to \mathcal{FM}$ be a product preserving bundle functor satisfying the condition (0.1) and let A be its Weil algebra.*

If $n \geq 2$, then the vector space of all linear natural operators $T \wedge T_{\mathcal{M}_n} \rightsquigarrow (T \wedge T)F$ is of dimension $dim(F(\mathbf{R}))$.

Moreover, if $\lambda_o : A \to \mathbf{R}$ is a linear map such that the corresponding bilinear symmetric form $A \times A \ni (a,b) \to \lambda_o(ab) \in \mathbf{R}$ is non-singular (an example of such λ_o is given in Example) and $n \geq 2$, then each linear natural operator $T \wedge T|_{|\mathcal{M}_n} \rightsquigarrow (T \wedge T)F$ is equal to $\mathcal{L}^{(\lambda,\lambda_o)}$ for some linear $\lambda : A \to \mathbf{R}$.

In [3], the authors introduced the vertical lift $d_V(\Lambda)$ and the tangent lift $d_T(\Lambda)$ of a skew-symmetric multivector field Λ on M to the tangent bundle TM. Since $dim(T\mathbf{R}) = 2$, from the theorem we have

Corollary 3 *If $n \geq 2$, then all linear natural oprators $T \wedge T|_{\mathcal{M}_n} \rightsquigarrow (T \wedge T)T$ are the linear combinations of d_V and d_T with real coefficients.*

3. We give an example of a product preserving bundle functor $F : \mathcal{M} \to \mathcal{FM}$ satisfying (0.1) which is not a finite composition of higher order tangent bundle functors. We also give an example of a product preserving bundle functor $F : \mathcal{M} \to \mathcal{FM}$ not satisfying (0.1).

Example. (a) Let $k \geq 3$ be such that $k+1$ is a prime number.
Let $C^\infty_0(\mathbf{R}^{k-1})$ be the algebra of germs at 0 of maps $\mathbf{R}^{k-1} \to \mathbf{R}$ and let $y^1, \ldots, y^{k-1}$ be the usual generators of $C^\infty_0(\mathbf{R}^{k-1})$. Let $A = C^\infty_0(\mathbf{R}^{k-1})/\underline{A}$ be the factor algebra, where $\underline{A}$ is the ideal in $C^\infty_0(\mathbf{R}^{k-1})$ generated by the germs $y^i y^j$ for $i,j = 1, \ldots, k-1$ with $i \neq j$, the $(y^1)^2 - (y^l)^2$ for $l = 2, \ldots k-1$ and the $y^i y^j y^l$ for $i,j,l = 0, \ldots, k-1$. Then A is a Weil algebra. The elements $a_0 = [1]_{\underline{A}}$ $a_1 = [y^1]_{\underline{A}}, \ldots, a_{k-1} = [y^{k-1}]_{\underline{A}}$, $a_k = [(y^1)^2]_{\underline{A}}$ form a basis of A. More, the elements $a_0, \ldots, a_k$ and $b_0 = [(y^1)^2]_{\underline{A}}$, $b_1 = [y^1]_{\underline{A}}$, $\ldots$, $b_{k-1} = [y^{k-1}]_{\underline{A}}$, $b_k = [1]_{\underline{A}}$ satisfy (i) and (ii) of the condition (0.1). The Weil algebra A is not isomorphic to $T^k(\mathbf{R})$ because the length of $T^k(\mathbf{R})$ is $k \geq 3$ and the length of A is 2. Since $k+1$ is prime, A is not a tensor product of two non-trivial Weil algebra. Then the Weil bundle functor $T_A : \mathcal{M} \to \mathcal{FM}$, cf. [4], satisfies (0.1) and it is not a composition of higher order tangent bundle functors.

(b) In [1] we proved that if a Weil algebra A posses a linear map $\lambda : A \to \mathbf{R}$ such that the symmetric bilinear map $A \times A \ni (a,b) \to \lambda(ab) \in \mathbf{R}$ is non-singular, then its nilpotent ideal N is such that $dim(N^h) = 1$ for some $h \in \mathbf{N}$. Hence for $p \geq 2$ the bundle functors T^r_p of p^r-velocities do not satisfy (0.1).

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WŁODZIMIERZ M. MIKULSKI
Institute of Mathematics
Jagellonian University
Kraków, Reimonta 4
POLAND