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ON PACKING OF FOUR AND FIVE SQUARES INTO A RECTANGLE[1]

P. NOVOTNÝ

Abstract. *It is proved in this paper that any system of four or five squares with total area 1 may be packed into a rectangle whose area is at most* $\frac{2+\sqrt{3}}{3}$.

L. Moser [3] posed the following question: What is the smallest number S such that any system of squares with total area 1 may be (parallelly) packed into a rectangle of area S? This problem is mentioned in [4], too. Moon and Moser [2] found first results for the upper bound. They proved that any system of squares with total area 1 may be packed into a square of area 2. Some further results were published by Kleitman and Krieger [1]. It follows from their paper that $S \leq \sqrt{\frac{8}{3}}$. Novotný [5] proved the inequality $S < 1.53$. On the other hand, if we denote S_n the smallest number such that any system of n squares of total area 1 may be packed into a rectangle of area S_n, then $S = \lim S_n$ and the sequence (S_n) is nondecreasing. Trivially, $S_1 = 1$ and

$$S_2 = \frac{1+\sqrt{2}}{2} \tag{1}$$

Novotný [6] proved that

$$S_3 \doteq 1.227759. \tag{2}$$

The aim of this paper is to prove the equalities $S_4 = S_5 = \frac{2+\sqrt{3}}{3}$.

Theorem 1 *Any system of four squares with total area* 1 *may be packed into a rectangle whose area is at most* $\frac{2+\sqrt{3}}{3}$*; this number is the least possible.*

Proof. The square of side $\sqrt{\frac{1}{2}} + \sqrt{\frac{1}{6}}$ or the rectangle of size $\left(\sqrt{\frac{1}{2}} + 2\sqrt{\frac{1}{6}}\right) \times \left(2\sqrt{\frac{1}{6}}\right)$ (both of them with area $\frac{2+\sqrt{3}}{3} \doteq 1.244016936$) is necessary for packing a square of side $\sqrt{\frac{1}{2}}$ and three squares of side $\sqrt{\frac{1}{6}}$. We prove that the area $\frac{2+\sqrt{3}}{3}$ is always sufficient.

We denote the sides of four squares $x_1 \geq x_2 \geq x_3 \geq x_4$ and we shall pack the squares in dependence upon x_1 and x_4. Evidently, $3x_1^2 + x_4^2 \geq 1$ and $x_1^2 + 3x_4^2 \leq 1$.

I. Let $[x_1, x_4] \in M_1$ (Fig. 1), i.e. $x_1 \geq 0.82$. If $x_2 + x_3 + x_4 \leq x_1$, then it follows from (1) that a rectangle of area at most $\frac{1+\sqrt{2}}{2}$ is sufficient for packing (the two smallest squares need no more space). Thus we assume $x_2 + x_3 + x_4 \geq x_1$. We use the rectangle of area

$$A_1 = (x_1 + x_2)(x_2 + x_3 + x_4) = (x_1 + x_2)\left(x_2 + \sqrt{1 - x_1^2 - x_2^2 - x_4^2} + x_4\right)$$

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(Fig. 3) for packing. Choosing x_1 and x_2 fixly, A_1 will be maximal if $x_3 = x_4 = \sqrt{\frac{1-x_1^2-x_2^2}{2}}$; then $A_1 = (x_1+x_2)\left(x_2+\sqrt{2}\sqrt{1-x_1^2-x_2^2}\right)$. Since $\frac{\partial A_1}{\partial x_1} = x_2+2x_4-(x_1+x_2)\,\frac{x_1}{x_4} < 0$ in M_1, A_1 is maximal for $x_1 = 0.82$. We verify easily that $A_1 < 1.24$ for $x_1 = 0.82$ and for every $x_2 \in \left\langle\sqrt{\frac{1-x_1^2}{3}}, \sqrt{1-x_1^2}\right\rangle$.

II. Let $[x_1, x_4] \in M_2$ (Fig. 1). We pack the squares by Fig. 4 into a rectangle of area $A_2 = x_1(x_1+x_2+x_3+x_4)$. A_2 is maximal if $x_2 = x_3 = \sqrt{\frac{1-x_1^2-x_4^2}{2}}$; then $A_2 = x_1\left(x_1+x_4+\sqrt{2}\sqrt{1-x_1^2-x_4^2}\right)$. Since $\frac{\partial A_2}{\partial x_1} = x_1+x_4+2x_2+x_1\left(1-\frac{x_1}{x_2}\right) > 0$ in M_2, $\frac{\partial A_2}{\partial x_4} = x_1\left(1-\frac{x_4}{x_2}\right) \geq 0$, A_2 is maximal at some from the right upper corners of M_2 (the case $x_1^2+3x_4^2 = 1$, i.e. $x_2 = x_3 = x_4$, is simple); we verify easily that this maximum is less than 1.244.

III. Let $[x_1, x_4] \in M_3$ (Fig. 1, Fig. 2). In the case $3x_1^2+2x_4^2 \geq 2$ the inequality $x_2+x_3 \leq x_1$ holds and we can pack the squares by Fig. 5 into a rectangle of area $A_3 = x_1\,(x_1+x_3+x_4)$ (we need not consider the case $x_3+x_4 < x_2$ in regard of (2)). A_3 is maximal if x_3 is maximal, i.e. $x_2 = x_3 = \sqrt{\frac{1-x_1^2-x_4^2}{2}}$; then $A_3 = x_1\left(x_1+x_4+\frac{1}{\sqrt{2}}\sqrt{1-x_1^2-x_4^2}\right)$. Since $\frac{\partial A_3}{\partial x_4} = x_1\left(1-\frac{x_4}{2x_2}\right) > 0$, A_3 is maximal for $x_4 = \sqrt{\frac{1-x_1^2}{3}}$. Then $A_3 = x_1\left(x_1+\frac{2}{\sqrt{3}}\sqrt{1-x_1^2}\right)$ and it follows from $\frac{dA_3}{dx_1} > 0$ that A_3 is maximal for $x_1 = 0.82$, $x_4 = \sqrt{\frac{1-x_1^2}{3}}$; this maximum is less than 1.24.

If $3x_1^2+2x_4^2 < 2$, then $x_2+x_3 > x_1$ can be fulfilled and we pack the squares by Fig. 6 (the case $x_2+x_3 < x_1$ is not important) into a rectangle of area $A_4 = (x_1+x_3+x_4)\,(x_2+x_3)$. A_4 is maximal if $x_2 = x_3 = \sqrt{\frac{1-x_1^2-x_4^2}{2}}$; then

$$A_4 = \sqrt{2}\sqrt{1-x_1^2-x_4^2}\left(x_1+x_4+\sqrt{\frac{1-x_1^2-x_4^2}{2}}\right).$$

Since $\frac{\partial A_4}{\partial x_1} = -\frac{x_1}{x_2}(x_1+x_2+x_4)+2x_2\left(1-\frac{x_1}{2x_2}\right) < 0$, A_4 is maximal on some from the abscissae which form the boundary of M_3 from the left; we verify easily that the maximum is less than 1.244.

IV. Let $[x_1, x_4] \in M_4$ (Fig. 1, Fig. 2). If $3x_1^2-4x_1x_4+3x_4^2 \geq 1$, then $x_1 \geq x_3+x_4$ is fulfilled and we can pack the squares by Fig. 7 into a rectangle of area $A_5 = x_1\,(x_1+x_2+x_3)$. This area is maximal if $x_2 = x_3 = \sqrt{\frac{1-x_1^2-x_4^2}{2}}$; then $A_5 = x_1\left(x_1+\sqrt{2}\sqrt{1-x_1^2-x_4^2}\right)$. Since $\frac{\partial A_5}{\partial x_1} = x_1+2x_2+x_1\left(1-\frac{x_1}{x_2}\right) > 0$ in M_4, $\frac{\partial A_5}{\partial x_4} < 0$, A_5 is maximal at some from the right lower corners of M_4 and the maximum is less than 1.244.

If $3x_1^2-4x_1x_4+3x_4^2 \leq 1$, it is sufficient to consider the case $x_3+x_4 \geq x_1$ and we can pack the squares by Fig. 8 into a rectangle of area

$$A_6 = (x_1 + x_2 + x_3)(x_3 + x_4).$$

A_6 is maximal if $x_2 = x_3 = \sqrt{\frac{1-x_1^2-x_4^2}{2}}$; then

$$A_6 = \left(x_1 + \sqrt{2}\sqrt{1 - x_1^2 - x_4^2}\right)\left(x_4 + \frac{1}{\sqrt{2}}\sqrt{1 - x_1^2 - x_4^2}\right).$$

Since $\frac{\partial A_6}{\partial x_1} < 0$, A_6 is maximal on some from the abscissae which form the boundary of M_4 from the left; the maximum is less than 1.244.

V. Let $[x_1, x_4] \in M_5$ (Fig. 1). If $x_2 \leq 0.57$, we pack the squares by Fig. 9 or by Fig. 10 (this is possible only if $3x_1^2 + 2x_1x_4 + 3x_4^2 \leq 2$). The area of the rectangle from Fig. 9 is $A_7 = (x_1 + x_2)\ (x_1 + x_4)$ and it is maximal at some from the right upper corners of M_5 for $x_2 = 0.57$. The maximum is less than 1.244.

The area of the rectangle from Fig. 10 is

$$A_8 = (x_1 + x_2)\left(x_2 + \sqrt{1 - x_1^2 - x_2^2 - x_4^2}\right).$$

Since $\frac{\partial A_8}{\partial x_4} < 0$, $\frac{\partial A_8}{\partial x_1} < 0$, $\frac{\partial A_8}{\partial x_2} = x_2 + x_3 + (x_1 + x_2)\left(1 - \frac{x_2}{x_3}\right) > 0$ in M_5, A_8 is maximal at some from the left lower corners of M_5 for $x_2 = 0.57$; the maximum is less than 1.244.

If $x_2 \geq 0.57$, we pack the squares by Fig. 8 or by Fig. 7. The area of the rectangle from Fig. 8 is

$$A_6 = \left(x_1 + x_2 + \sqrt{1 - x_1^2 - x_2^2 - x_4^2}\right)\left(x_4 + \sqrt{1 - x_1^2 - x_2^2 - x_4^2}\right).$$

It follows from $\frac{\partial A_6}{\partial x_1} < 0$, $\frac{\partial A_6}{\partial x_2} < 0$ that A_6 is amximal on the left side of M_5 and for $x_2 = 0.57$. The maximum is less than 1.244.

Since $x_1 + x_2 + x_3 \leq \sqrt{3}$, the area of the rectangle from Fig. 7 is $x_1(x_1 + x_2 + x_3) < 1.24$ for $x_1 \leq 0.71$.

VI. Let $[x_1, x_4] \in M_6$ (Fig. 1; the lower part of the boundary is the stright line $l : x_4 = \sqrt{\frac{1}{6}} + (2 - \sqrt{3})\left(x_1 - \sqrt{\frac{1}{2}}\right)$). We pack the squares by Fig. 9 into a rectangle of area $A_7 = (x_1 + x_2)\ (x_1 + x_4)$. This area is maximal if x_2 is maximal, i.e. x_3 is minimal, thus $x_3 = x_4$, $x_2 = \sqrt{1 - x_1^2 - 2x_4^2}$ (the equality $x_2 = x_1$ is impossible in M_6 since $x_1^2 + x_4^2 > \frac{1}{2}$). Then $A_7 = \left(x_1 + \sqrt{1 - x_1^2 - 2x_4^2}\right)(x_1 + x_4)$. Since $\frac{\partial A_7}{\partial x_4} < 0$, $\frac{\partial A_7}{\partial x_1} > 0$ in M_6, A_7 is maximal on the stright line l. But we have

$$\frac{dA_7}{dx_1} = \frac{\partial A_7}{\partial x_1} + \left(2 - \sqrt{3}\right)\frac{\partial A_7}{\partial x_4} = \left(1 - \frac{x_1}{x_2}\right)(x_1 + x_4)$$

$$+ (x_1 + x_2) + \left(2 - \sqrt{3}\right)\left(-\frac{2x_4}{x_2}(x_1 + x_4) + (x_1 + x_2)\right)$$

$$\geq (x_1+x_4)\left(1-\frac{x_1}{x_4}+1+\left(2-\sqrt{3}\right)(-2+1)\right)=(x_1+x_4)\left(\sqrt{3}-\frac{x_1}{x_4}\right)\geq 0$$

on l. Hence A_7 is maximal for $x_1=\sqrt{\frac{1}{2}}, x_4=\sqrt{\frac{1}{6}}$ and the maximal value of A_7 is $A=\frac{2+\sqrt{3}}{3}$.

VII. Let $[x_1,x_4]\in M_7$ (Fig. 1; $x_1=\sqrt{\frac{1}{2}}$ on the right part of the boundary). If $(x_1+x_2)(x_1+x_4)\leq A$, then we pack the squares by Fig. 9 (we have $x_1+x_4>x_2+x_3$ in M_7). Thus let $(x_1+x_2)\,(x_1+x_4)>A$, i.e. $x_2>\frac{A}{x_1+x_4}-x_1$. If $x_3+x_4<x_1$, we pack the squares by Fig. 7. If $x_3+x_4\geq x_1$, we pack them by Fig. 8 into a rectangle of area

$$A_6=\left(x_1+x_2+\sqrt{1-x_1^2-x_2^2-x_4^2}\right)\left(x_4+\sqrt{1-x_1^2-x_2^2-x_4^2}\right).$$

Since $\frac{\partial A_6}{\partial x_2}<0$, we have

$$A_6<\left(\frac{A}{x_1+x_4}+\sqrt{1-x_1^2-\left(\frac{A}{x_1+x_4}-x_1\right)^2-x_4^2}\right)$$

$$\times\left(x_4+\sqrt{1-x_1^2-\left(\frac{A}{x_1+x_4}-x_1\right)^2-x_4^2}\right)=B.$$

If we denote $u=\frac{A}{x_1+x_4}-x_1$, $v=\sqrt{1-x_1^2-u^2-x_4^2}$, then (using $x_1>v$, $(x_1+x_4)^2\leq A$ and hence $u\geq x_4$)

$$\frac{\partial B}{\partial x_4}=\left(-\frac{A}{(x_1+x_4)^2}+\frac{\frac{Au}{(x_1+x_4)^2}-x_4}{v}\right)(x_4+v)$$

$$+\left(\frac{A}{x_1+x_4}+v\right)\left(1+\frac{\frac{Au}{(x_1+x_4)^2}-x_4}{v}\right)\geq v+\frac{A}{(x_1+x_4)^2}(x_1-v)\geq x_1>0$$

and hence B has a maximum on l. Further, on l we have (using $u\geq v$ and $\frac{u-x_1}{v}\geq 1-\sqrt{3}$)

$$\frac{dB}{dx_1}=\frac{\partial B}{\partial x_1}+\left(2-\sqrt{3}\right)\frac{\partial B}{\partial x_4}=\left(\frac{-A}{(x_1+x_4)^2}+\frac{u\left(\frac{A}{(x_1+x_4)^2}+1\right)-x_1}{v}\right)$$

$$\times(x_4+v)+\left(\frac{A}{x_1+x_4}+v\right)\frac{u\left(\frac{A}{(x_1+x_4)^2}+1\right)-x_1}{v}+\left(2-\sqrt{3}\right)\frac{\partial B}{\partial x_4}$$

$$\geq\left(\frac{2u-x_1}{v}-\frac{A}{(x_1+x_4)^2}\right)(x_4+v)+\frac{2u-x_1}{v}\left(\frac{A}{x_1+x_4}+v\right)+\left(2-\sqrt{3}\right)x_1$$

$$=\frac{u-x_1}{v}(x_4+2v+x_1+u)+\frac{u}{v}(x_4+2v)+\frac{A}{(x_1+x_4)^2}\left(\frac{u}{v}(x_1+x_4)-x_4-v\right)$$

$$+\left(2+\sqrt{3}\right)x_1 \geq \left(1-\sqrt{3}\right)(x_4+x_1+u)+2u-2x_1+x_4+2u+x_1-u$$

$$+\left(2+\sqrt{3}\right)x_1 = \left(2-2\sqrt{3}\right)x_1+\left(2-\sqrt{3}\right)x_4+\left(4-\sqrt{3}\right)u$$

$$\geq \left(2-2\sqrt{3}\right)x_1+\left(6-2\sqrt{3}\right)x_4 = \left(6-2\sqrt{3}\right)\left(x_4-\frac{\sqrt{3}}{3}x_1\right) \geq 0.$$

Hence B has the maximum for $x_1 = \sqrt{\frac{1}{2}}, x_4 = \sqrt{\frac{1}{6}}$ and this maximum has the value $A = \frac{2+\sqrt{3}}{3}$.

VIII. Let $[x_1, x_4] \in M_8$ (Fig. 1), i.e. $\sqrt{\frac{1}{2}} \leq x_1 \leq 0.71$. As in **VII**, if $(x_1+x_2)\,(x_1+x_4) \leq A$, we pack the squares by Fig. 9. If $(x_1+x_2)\,(x_1+x_4) > A$, we pack them by Fig. 7 into a rectangle of area less than 1.24 or by Fig. 8 into a rectangle of area

$$A_6 = \left(x_1+x_2+\sqrt{1-x_1^2-x_2^2-x_4^2}\right)\left(x_4+\sqrt{1-x_1^2-x_2^2-x_4^2}\right).$$

We have $A_6 < B$ again. Further, $\frac{\partial B}{\partial x_4} > 0$ in M_8 and $\frac{\partial B}{\partial x_1} < 0$ for $x_4 = \sqrt{\frac{1}{6}}$. It means that max $B = A$ in M_8.

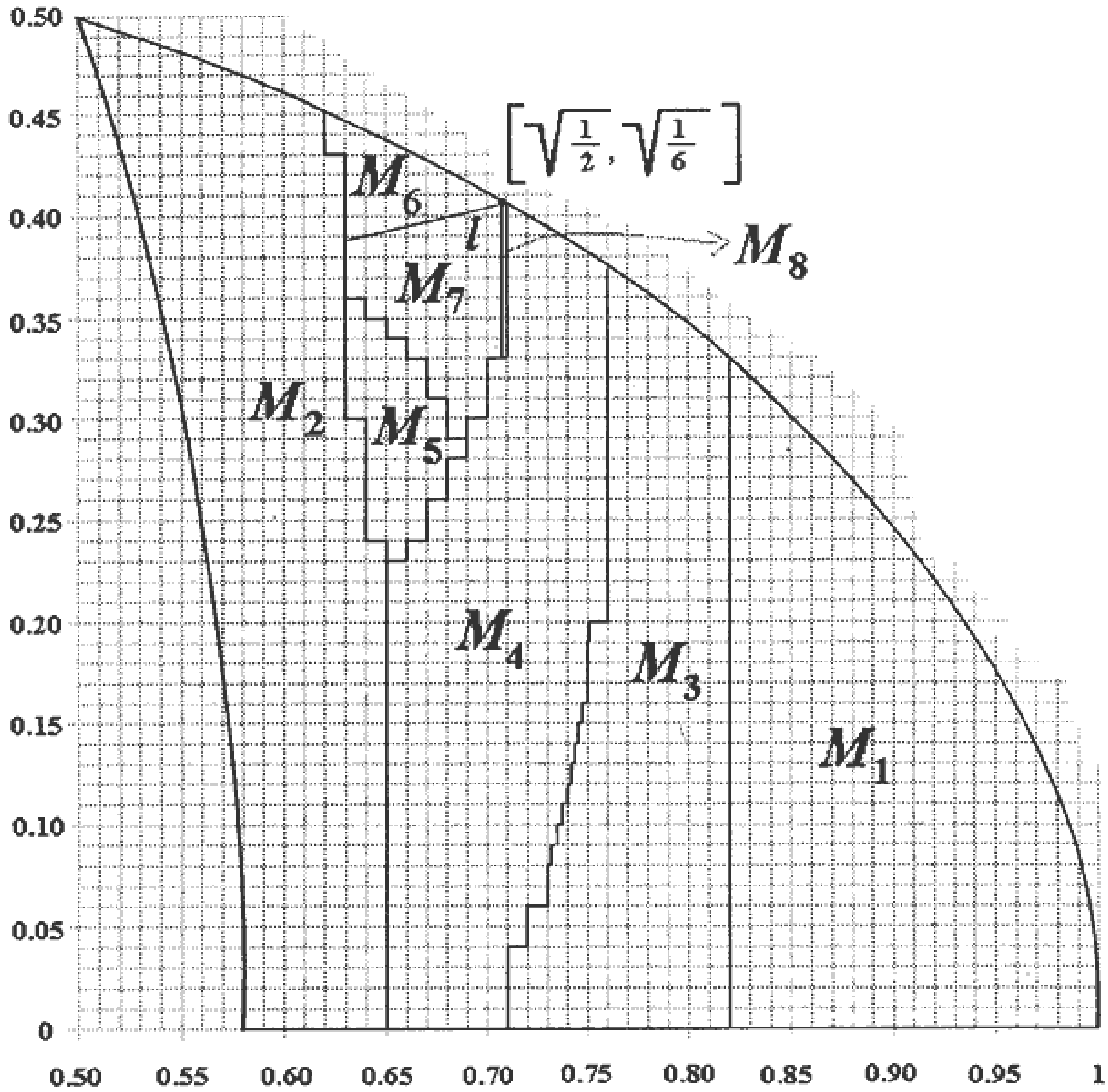
The proof is completed. □

Theorem 2 *Any system of five squares with total area* 1 *may be packed into a rectangle whose area is at most* $\frac{2+\sqrt{3}}{3}$.

Proof. We denote the sides of the squares $x_1 \geq x_2 \geq x_3 \geq x_4 \geq x_5$.

Let $x_5 \leq 0.12$. It is possible to pack the four largest squares into a rectangle R of area $S_4 = \frac{2+\sqrt{3}}{3} > 1.244$ by Theorem 1. Since $x_1^2+x_2^2+x_3^2+x_4^2 \leq 1$, we have $x_1+x_2+x_3+x_4 \leq 2$. We can construct R so that the free space in it consists of at most four rectangles (Fig. 11); one side is x_1, x_2, x_3, x_4 one after the other. Since the area of the free space is $A > 0.244$, at least one from the rectangles has the other side greater than $\frac{A}{x_1+x_2+x_3+x_4} > 0.12$. It means that there is plenty of space for the smallest square.

Let now $x_5 \geq 0.12$. We cover the domain $D = \{[x_1, x_2, x_4, x_5]\}$ (x_3 is determined by the condition $\sum x_i^2 = 1$) of the possible lengths of the sides by small hypercubes $H : x_i \in \langle a_i, a_i+d\rangle$ for $i \in \{1,2,4,5\}$ of edge d. We consider the maximal possible lengths of the sides in any hupercube, i.e. $x_1 = a_1+d, x_2 = a_2+d$, $x_4 = a_4+d$, $x_5 = a_5+d$, $x_3 = \min\left\{a_2+d, \sqrt{1-a_1^2-a_2^2-a_4^2-a_5^2}\right\}$. The total area of the squares is greater than 1 but we may permit it if d is small because we are far away from the critical point (we have $x_5 = 0$ at the critical point and we assume $x_5 \geq 0.12$). A computer verified that for $d = 0.004$ some packing into a rectangle of area less than 1.244 is possible for any mentioned hypercube. □



0.50
0.45
0.40
0.35
0.30
0.25
0.20
0.15
0.10
0.05
0
0.50
0.55
0.60
0.65
0.70
0.75
0.80
0.85
0.90
0.95
1
$\left[\sqrt{\frac{1}{2}}, \sqrt{\frac{1}{6}}\right]$
M_8
M_6
l
M_7
M_2
M_5
M_4
M_3
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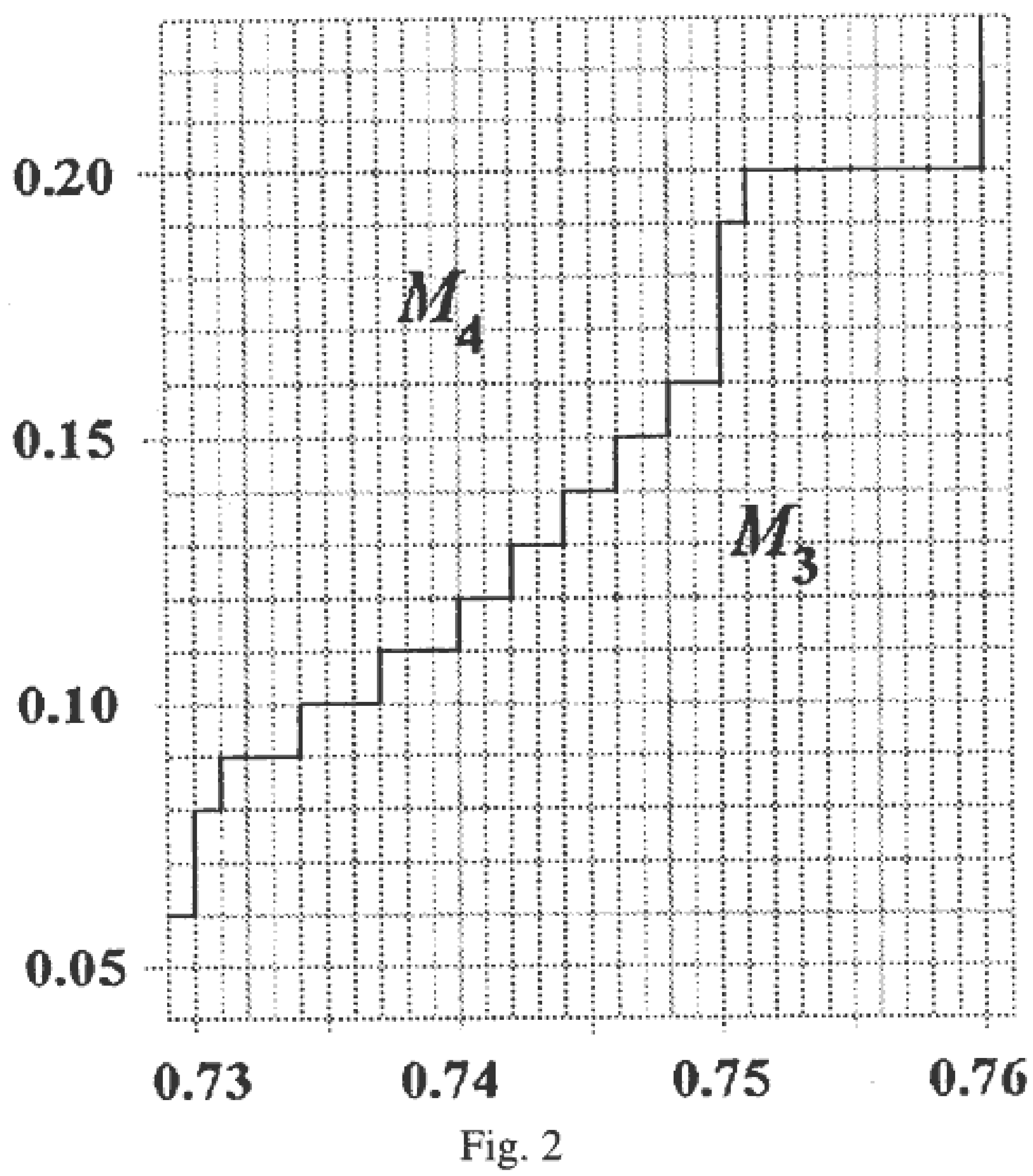
Fig. 1



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0.15
0.10
0.05
0.73
0.74
0.75
0.76
M_4
M_3

Fig. 2

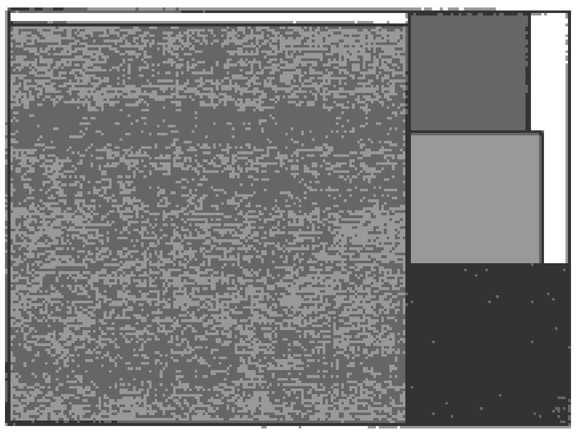

Fig. 3

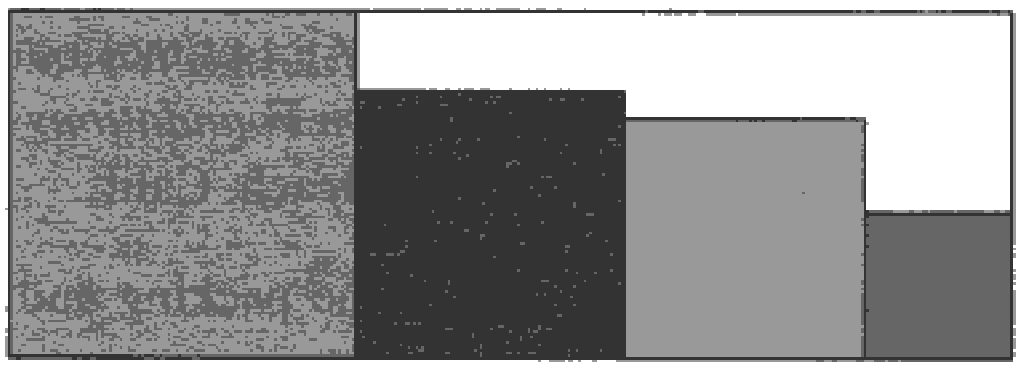

Fig. 4

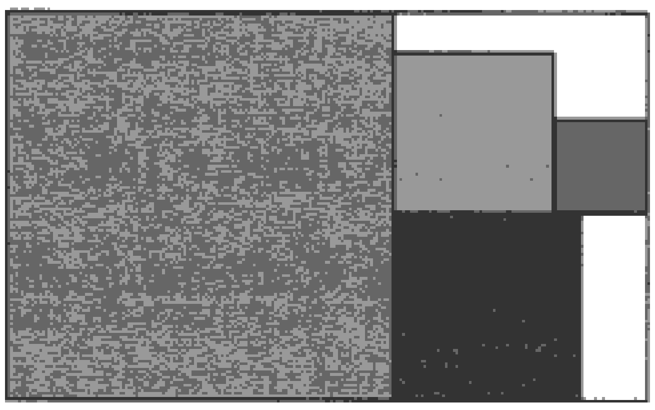

Fig. 5

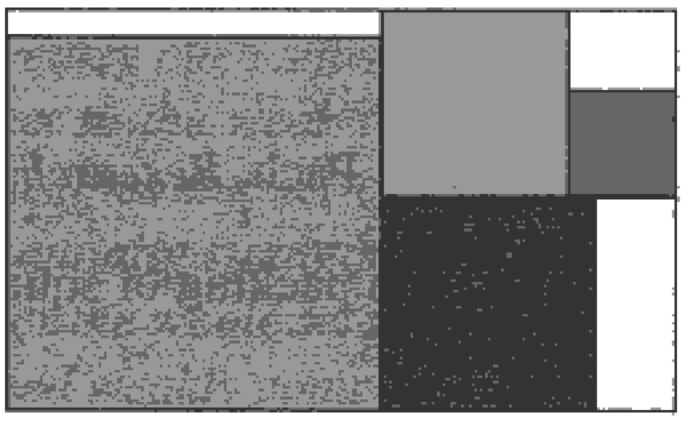

Fig. 6

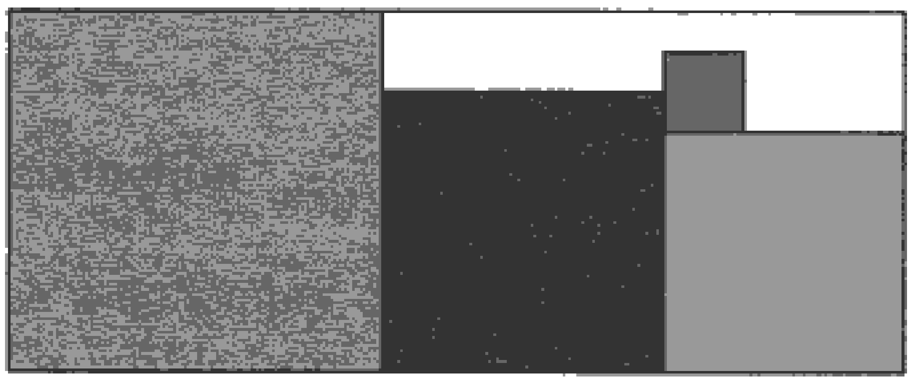

Fig. 7

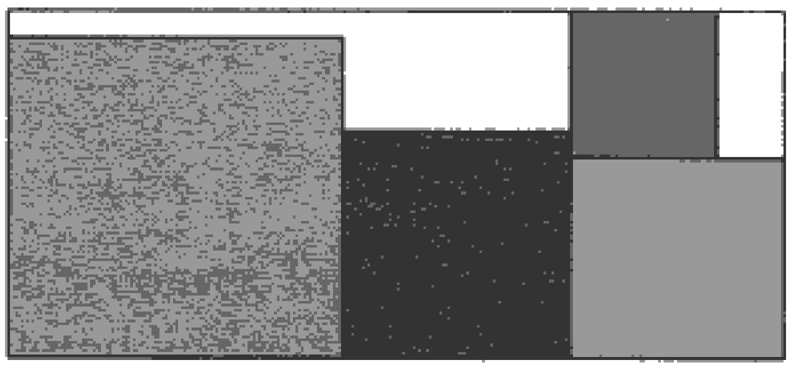

Fig. 8

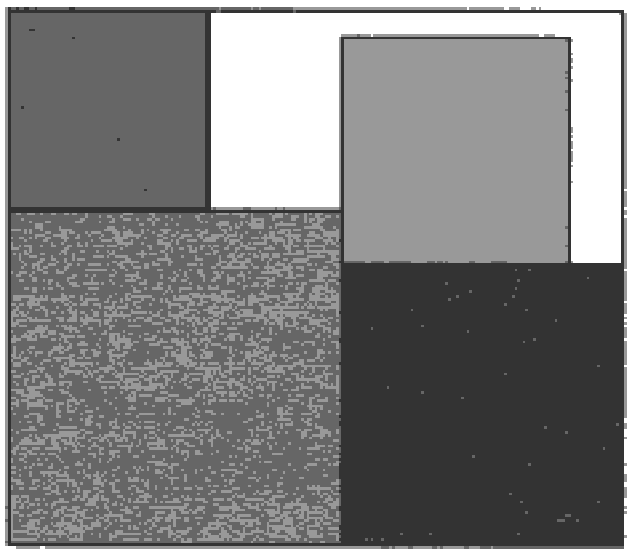

Fig. 9

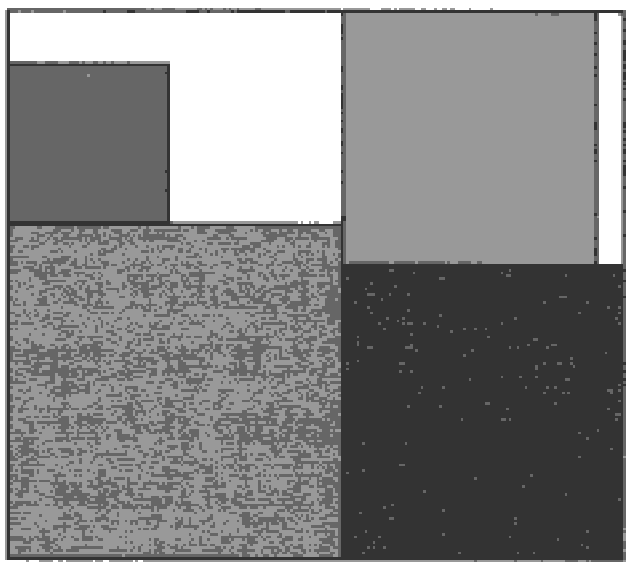

Fig. 10

Fig. 11

References

[1] D. Kleitman and M. Krieger, "An optimal bound for two dimensional bin packing", *Proc. 16-th Annual Symposium on Foundations of Comp. Sci.*, Berkeley, 1975, 163-168, IEEE Computer Society, Long Beach, Calif.

[2] J.W. Moon and L. Moser, "Some packing and covering theorems", *Colloq. Math.* **17** (1967), 103-110.

[3] L. Moser, *Poorly formulated unsolved problems of combinatorial geometry*, mimeographed.

[4] W. Moser and J. Pach, *Research Problems in Discrete Geometry* 1989, # 108.

[5] P. Novotný, "On packing of squares into a rectangle", *Archivum Mathematicum* (Brno) **32** (1996), 75-83.

[6] P. Novotný, "A note on a packing of squares", *Studies of University of Transport and Communications in Žilina, Math.-Phys. series* **10** (1995), 35-39.

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P. NOVOTNÝ
Department of Mathematics, FPEDaS
University of Žilina
010 26 Žilina, Slovakia
novotny@fpedas.utc.sk