Note di Matematica **21,** n. 1, 2002, 35–48.

On a Generalization of Wyler's Construction of Topological Projective Planes

Erwin Schörner
Mathematisches Institut der Universität München
Theresienstraße 39, D–80333 München
schoerner@lmu.de

Received: 28 August 2000; accepted: 28 August 2001.

Abstract. The purpose of this paper is to generalize the construction of topological projective planes in the sense of Salzmann given by Wyler for the case of ordered projective planes. This generalization is also applicable to projective planes having a coordinatizing ternary field which is endowed with a uniform valuation in the sense of Kalhoff with an Abelian value group.

Keywords: Topological projective plane, ordered projective plane, (multi-valued) half-ordering of a projective plane, uniform valuation of a ternary field.

MSC 2000 classification: 51H10; 12K99, 51G05.

For the projective plane $\mathfrak{E} = (\mathfrak{P}, \mathfrak{G}, \mathrel{\underline{\mathrm{I}}})$, we denote the joining line of two different points P and $Q \in \mathfrak{P}$ by PQ and the intersection point of two different lines g and $h \in \mathfrak{G}$ by $g \cap h$. Moreover, let P^* be the set of all lines through P and let g^* be the set of all points on g. Given two lines g and h and a point P with $P \notin g^*$ and $P \notin h^*$, the bijective mapping

$$g^* \ni Q \mapsto PQ \cap h \in h^*$$

is called *perspectivity* and is denoted by $g^* \xrightarrow{P} h^*$; any (finite) composition of perspectivities is a *projectivity.*

In [3], Salzmann calls $\mathfrak{E}$ a *topological projective plane,* when $\mathfrak{P}$ and $\mathfrak{G}$ are endowed with non-trivial and non-discrete topologies $\mathfrak{T}_{\mathfrak{P}}$ and $\mathfrak{T}_{\mathfrak{G}}$, respectively, such that the joining of two different points

$$\{(P,Q) \in \mathfrak{P} \times \mathfrak{P} \mid P \neq Q\} \ni (P,Q) \mapsto PQ \in \mathfrak{G}$$

and the intersection of two different lines

$$\{(g,h) \in \mathfrak{G} \times \mathfrak{G} \mid g \neq h\} \ni (g,h) \mapsto g \cap h \in \mathfrak{P}$$

are continuous mappings; here, the sets $\{(P,Q) \in \mathfrak{P} \times \mathfrak{P} \mid P \neq Q\}$ and $\{(g,h) \in \mathfrak{G} \times \mathfrak{G} \mid g \neq h\}$ carry the trace topologies of the product topologies on $\mathfrak{P} \times \mathfrak{P}$ and $\mathfrak{G} \times \mathfrak{G}$, respectively.

We consider an ordered projective plane $(\mathfrak{E}, \|)$, where $\|$ denotes the corresponding relation of separation. In [4], Wyler constructs topologies on $\mathfrak{P}$ and $\mathfrak{G}$, such that $\mathfrak{E} = (\mathfrak{P}, \mathfrak{G}, \mathrel{\underline{\perp}})$ is a topological projective plane in the sense of Salzmann. In this context, an important role is played by the segments

$$(A, B)_C = \{X \in (AB)^* \mid AB \parallel CX\}$$

for all triples of different collinear points A, B and C; since $\|$ is *perspectivity-preserving,* i.e. for four different points A, B, C and D on a line g with $AB \parallel CD$ and a perspectivity $\pi\colon g^* \to h^*$ we have $A^\pi B^\pi \parallel C^\pi D^\pi$, there is a dual relation of separation $\|^*$ for the lines with the corresponding dual segments

$$(a, b)_c = \{x \in (a \cap b)^* \mid ab \parallel^* cx\}.$$

Two dual segments $(a, b)_c$ and $(a', b')_{c'}$ with $a \cap b \neq a' \cap b'$ determine a convex quadrangle, which consists of all intersection points of lines in $(a, b)_c$ and of lines in $(a', b')_{c'}$; the set of all convex quadrangles is a base of a topology on $\mathfrak{P}$. Moreover, if $\mathfrak{G}$ is endowed with the dually constructed topology, then $\mathfrak{E}$ is a topological projective plane in the sense of Salzmann. This summary may serve as a structure of the following considerations.

The purpose of this paper is to generalize this method such that it is also applicable to projective planes endowed with an appropriate multi-valued half-ordering in the sense of Junkers, e.g. to projective planes having a coordinatizing ternary field endowed with a uniform valuation in the sense of Kalhoff with an Abelian value group.

Therefore, we consider the set $\mathfrak{T}$ of all triples of different collinear points and the set $\mathfrak{Q}$ of all quadruples (A, B, C, D) of collinear points with $(A, B, C) \in \mathfrak{T}$. Moreover, let Γ be an arbitrary set and let Δ be a non-empty proper subset of Γ. For a perspectivity-preserving mapping $\varphi\colon \mathfrak{Q} \to \Gamma$, we define the *interval*

$$(A, B)_C = \{X \in (AB)^* \mid \varphi(A, B, C, X) \in \Delta\}$$

for any $(A, B, C) \in \mathfrak{T}$, and we call φ *topological,* if the following conditions are satisfied for an $(A, B, C) \in \mathfrak{T}$:

(1) $C \notin (A, B)_C$ holds.

(2) $(A, B)_C = (A, B)_{C'}$ holds for all $(A, B, C') \in \mathfrak{T}$ with $C' \notin (A, B)_C$.

(3) For all A', $B' \in (A, B)_C$, we have $(A', B')_C \subseteq (A, B)_C$ for $A' \neq B'$, $(A, B')_C \subseteq (A, B)_C$ for $A \neq B'$ and $(A', B)_C \subseteq (A, B)_C$ for $A' \neq B$.

(4) For all $X \in (A, B)_C$, there exist $A', B' \in (A, B)_C$ with $X \neq A' \neq B' \neq X$ and $X \in (A', B')_C$.

(5) There exist (A', A'', C) and $(B', B'', C) \in \mathfrak{T}$ with $(A', A'')_C \cap (B', B'')_C = \emptyset$, $A \in (A', A'')_C$ and $B \in (B', B'')_C$.

(6) For all $(A', B', C) \in \mathfrak{T}$ and all $X \in (A, B)_C \cap (A', B')_C$, there exists $(A'', B'', C) \in \mathfrak{T}$ with $X \in (A'', B'')_C \subseteq (A, B)_C \cap (A', B')_C$.

Since φ is perspectivity-preserving, the preceding conditions are satisfied even for all $(A, B, C) \in \mathfrak{T}$. We remark that for the present concept it suffices to consider the situation $\Gamma = \{0, 1\}$ and $\Delta = \{1\}$; nevertheless, we use this more general notion with regard to easier application. Obviously, any relation of separation satisfies conditions (1) to (6).

For two triples (A, B, C) and $(A', B', C') \in \mathfrak{T}$, the corresponding intervals $(A, B)_C$ and $(A', B')_{C'}$ have the same cardinality by virtue of the bijection

$$(A, B)_C \ni X \mapsto X^\pi \in (A', B')_{C'}$$

where π denotes a projectivity with $A^\pi = A'$, $B^\pi = B'$ and $C^\pi = C'$. By (5), the intervals are non-empty, hence for $(A, B, C) \in \mathfrak{T}$ there exists $X \in (A, B)_C$, and by (4) we have A', $B' \in (A, B)_C$ with $A' \neq X$ and $X \in (A', B')_C$. Applying (5) we obtain $(A'', B'', C) \in \mathfrak{T}$ with

$$X \in (A'', B'')_C \quad \text{and} \quad A' \notin (A'', B'')_C,$$

and by (6) also $(A''', B''', C) \in \mathfrak{T}$ with

$$X \in (A''', B''')_C \subseteq (A, B)_C \cap (A'', B'')_C,$$

which yields $(A''', B''')_C \subsetneq (A, B)_C$. Consequently, every interval contains infinitely many points, but due to (1), it does not consist of all points of a line. Moreover, for all $(A, B, C) \in \mathfrak{T}$ and $X \in (A, B)_C$ with $A \neq X \neq B$, we also have $X \in (B, A)_C$: since the quadruples (A, B, X, C) and (B, A, C, X) are projective, the assumption $X \notin (B, A)_C$ yields

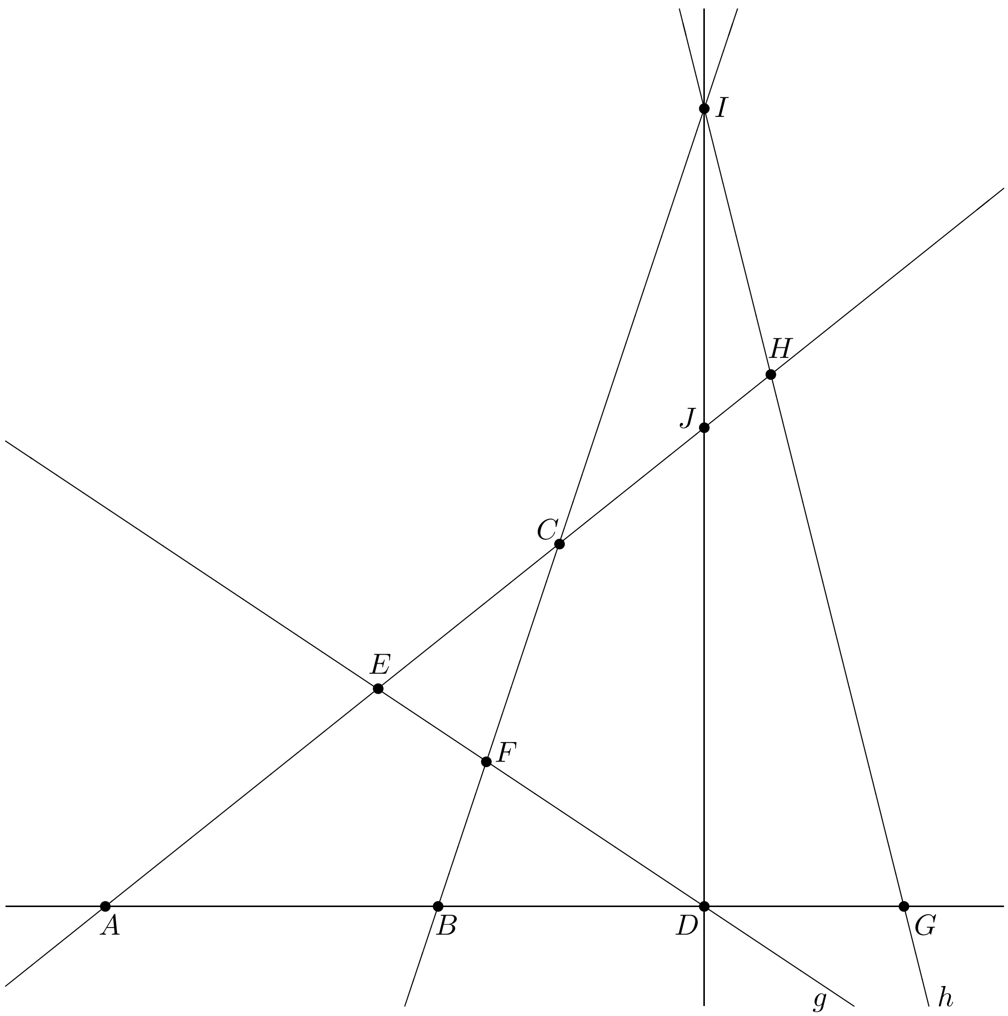
$$\varphi(A, B, X, C) = \varphi(B, A, C, X) \notin \Delta,$$

hence $C \notin (A, B)_X$ and therefore $(A, B)_C = (A, B)_X$ by (2), and by (1) we finally obtain $X \notin (A, B)_C$, a contradiction.

Since φ is perspectivity-preserving, the dual mapping φ^* for the lines is well-defined, and we also consider the intervals on the set P^* of lines through a point P determined by φ^*. For $(a, b)_c \in \mathfrak{I}_P$, we put $(a, b)_c^* = \bigcup_{x \in (a,b)_c} x^*$, i.e. $(a, b)_c^*$ denotes the set of all points lying on a line x in $(a, b)_c$.

For later use, we note a result which corresponds to the Axiom of Pasch in the class of ordered projective planes.

Lemma 1. *Let g and h be two lines and let A, B and C be three non-collinear points lying neither on g nor on h with $D = AB \cap g$, $E = AC \cap g$, $F = BC \cap g$, $G = AB \cap h$, $H = AC \cap h$ and $I = BC \cap h$. Then, $D \notin (A,B)_G$ and $E \in (A,C)_H$ imply $F \in (B,C)_I$.*



Proof. Let J denote the intersection point of AC and DI. From $D \notin (A,B)_G$ and $(A,B,D) \in \mathfrak{T}$ we can conclude $(A,B)_G = (A,B)_D$ by (2), and by (1) $\varphi(A,B,D,G) \notin \Delta$ holds. Using the perspectivity $(AC)^* \xrightarrow{I} (AB)^*$, we obtain $\varphi(A,C,J,H) \notin \Delta$ and therefore $H \notin (A,C)_J$; because of $(A,C,H) \in \mathfrak{T}$, it follows $(A,C)_H = (A,C)_J$ by (2). Then, $E \in (A,C)_H$ implies $\varphi(A,C,J,E) \in \Delta$, and the perspectivity $(AC)^* \xrightarrow{D} (BC)^*$ yields $\varphi(B,C,I,F) \in \Delta$ and finally

$F \in (B, C)_I$. $\boxed{QED}$

For a line g we consider the set

$$\mathfrak{I}_g = \{(A, B)_C \mid A, B, C \in g^* \text{ pairwisely different}\}$$

of all intervals with points on g, and we check that it is a base of a topology $\mathfrak{T}_g$ on g^*. Therefore, let $(A, B)_C$ and $(A', B')_{C'} \in \mathfrak{I}_g$ with $X \in (A, B)_C \cap (A', B')_{C'}$.

In the case $C \notin (A', B')_{C'}$, we have $(A', B')_{C'} = (A', B')_C$ by (2), and by (6) there exists $(A'', B'')_C \in \mathfrak{I}_g$ with

$$X \in (A'', B'')_C \subseteq (A, B)_C \cap (A', B')_{C'}.$$

In the case $C \in (A', B')_{C'}$, we have $C \neq C'$, $C \neq X$ and $X \neq C'$ by (1), and by (5) there exist $(A_1, B_1)_{C'}$ and $(A_2, B_2)_{C'} \in \mathfrak{I}_g$ with $C \in (A_1, B_1)_{C'}$ and $X \in (A_2, B_2)_{C'}$ and $(A_1, B_1)_{C'} \cap (A_2, B_2)_{C'} = \emptyset$. Due to (6), there is $(A'', B'')_{C'} \in \mathfrak{I}_g$ with

$$X \in (A'', B'')_{C'} \subseteq (A', B')_{C'} \cap (A_2, B_2)_{C'}.$$

Hence, $X \in (A, B)_C \cap (A'', B'')_{C'}$ holds, and because of $C \notin (A'', B'')_{C'}$ we obtain $(A''', B''')_{C'''} \in \mathfrak{I}_g$ with

$$X \in (A''', B''')_{C'''} \subseteq (A, B)_C \cap (A'', B'')_{C'} \subseteq (A, B)_C \cap (A', B')_{C'}$$

according to the first case.

The above considerations on the intervals yield that the topology $\mathfrak{T}_g$ is neither trivial nor discrete; moreover, the following lemma gives a closer description of the subsets of g^* which are open with respect to $\mathfrak{T}_g$.

Lemma 2. *Let g be a line. Then for all $\mathfrak{M} \in \mathfrak{T}_g$ and all X, $C \in g^*$ with $X \in \mathfrak{M}$ and $C \notin \mathfrak{M}$ there exists $(A, B)_C \in \mathfrak{I}_g$ with A, $B \in \mathfrak{M}$ and $X \in (A, B)_C \subseteq \mathfrak{M}$.*

Proof. Due to $\mathfrak{M} \in \mathfrak{T}_g$ there exists $(A', B')_{C'} \in \mathfrak{I}_g$ with $X \in (A', B')_{C'} \subseteq \mathfrak{M}$, and $C \notin \mathfrak{M}$ yields $(A', B')_C = (A', B')_{C'}$. By (4), there exists $(A, B)_C \in \mathfrak{I}_g$ with A, $B \in (A', B')_C$ and $X \in (A, B)_C \subseteq (A', B')_C \subseteq \mathfrak{M}$. $\boxed{QED}$

Let g be a line. A subset $\mathfrak{M}$ of $\mathfrak{P}$ with $g^* \cap \mathfrak{M} = \emptyset$ is called *g-convex* if for all A, $B \in \mathfrak{M}$ with $A \neq B$ we have $(A, B)_{AB \cap g} \subseteq \mathfrak{M}$. In this case, $\mathfrak{M}$ is also h-convex for any line h with $h^* \cap \mathfrak{M} = \emptyset$: indeed, for A, $B \in \mathfrak{M}$ with $A \neq B$ and $C' = AB \cap h$, we obtain $C' \notin (A, B)_C$ by $(A, B)_C \subseteq \mathfrak{M}$ and therefore $(A, B)_{C'} = (A, B)_C \in \mathfrak{M}$ by (2).

We now consider three non-collinear points O, U and V with $w = UV$. Let $u_1 \neq u_2$ and $v_1 \neq v_2$ be lines different from w with $U = u_1 \cap u_2$ and $V = v_1 \cap v_2$

and with $OU \in (u_1, u_2)_w$ and $OV \in (v_1, v_2)_w$. Then

$$\begin{aligned}\mathfrak{V} &= \{X \mid XU \in (u_1, u_2)_w \text{ and } XV \in (v_1, v_2)_w\} \\ &= \{x \cap y \mid x \in (u_1, u_2)_w \text{ and } y \in (v_1, v_2)_w\} \\ &= (u_1, u_2)_w^* \cap (v_1, v_2)_w^*\end{aligned}$$

is called a *convex quadrangle around O with respect to U and V*. This notion is justified, since on the one hand we obviously have $O \in \mathfrak{V}$ and on the other hand $\mathfrak{V}$ is also w-convex: $w^* \cap \mathfrak{V} = \emptyset$ is an immediate consequence of (1). Let A and $B \in \mathfrak{V}$ with $A \neq B$ and $X \in (A, B)_C$ with $C = AB \cap w$. In the case $C = U$ we have $XU = AU \in (u_1, u_2)_w$. In the case $C \neq U$ we have $U \notin (AB)^*$, and we obtain

$$\varphi^*(AU, BU, w, XU) = \varphi(A, B, C, X) \in \Delta$$

and therefore by (3) also

$$XU \in (AU, BU)_w \subseteq (u_1, u_2)_w.$$

In an analogous way it follows $XV \in (v_1, v_2)_w$, finally yielding $X \in \mathfrak{V}$.

We consider the trace of a line in a convex quadrangle.

Lemma 3. *Let $\mathfrak{V}$ be a convex quadrangle around O with respect to U and V. Then for any line g, the subset $g^* \cap \mathfrak{V}$ of g^* is open with respect to $\mathfrak{T}_g$.*

Proof. There is no loss of generality in assuming $g^* \cap \mathfrak{V} \neq \emptyset$; furthermore, let $\mathfrak{V} = (u_1, u_2)_w^* \cap (v_1, v_2)_w^*$. For $U \in g^*$, we have $g = XU \in (u_1, u_2)_w$ for any $X \in g^* \cap \mathfrak{V}$ and therefore $g^* \cap (u_1, u_2)_w^* = g^*$. For $U \notin g^*$, we consider the mapping $\pi \colon U^* \to g^*$, $h \mapsto g \cap h$, and we obtain

$$\begin{aligned}g^* \cap (u_1, u_2)_w^* = \{P \in g^* \mid \varphi^*(u_1, u_2, w, PU) \in \Delta\} = \\ = \{P \in g^* \mid \varphi(u_1^\pi, u_2^\pi, w \cap g, P) \in \Delta\} = (u_1^\pi, u_2^\pi)_{w \cap g}.\end{aligned}$$

Applying the same arguments to V and making use of (6), we finally obtain that

$$g^* \cap \mathfrak{V} = (g^* \cap (u_1, u_2)_w^*) \cap (g^* \cap (v_1, v_2)_w^*)$$

is an open subset of g^* with respect to $\mathfrak{T}_g$. $\boxed{QED}$

The following lemma guarantees that a convex quadrangle around O contains a convex quadrangle around O with respect to the same points U and V which is disjoint to a given line g with $O \notin g^*$.

Lemma 4. *Let $\mathfrak{V}$ be a convex quadrangle around O with respect to U and V, and let g be a line with $O \notin g^*$. Then there exists a convex quadrangle $\mathfrak{V}'$ around O with respect to U and V with $\mathfrak{V}' \subseteq \mathfrak{V}$ and $g^* \cap \mathfrak{V}' = \emptyset$.*

PROOF. Without loss of generality we may assume $g \neq w$ with $w = UV$ and therefore $g \notin V^*$; furthermore, let $W = g \cap w$ and $\mathfrak{V} = (u_1, u_2)^*_w \cap (v_1, v_2)^*_w$.

In the case $U \in g^*$, there exists $(u''_1, u''_2)_w \in \mathfrak{I}_U$ with $OU \in (u''_1, u''_2)_w$ and $g \notin (u''_1, u''_2)_w$ by (5) due to $g \neq OU$; by (6), there is $(u'_1, u'_2)_w \in \mathfrak{I}_U$ with

$$OU \in (u'_1, u'_2)_w \subseteq (u_1, u_2)_w \cap (u''_1, u''_2)_w.$$

Obviously,

$$\mathfrak{V}' = (u'_1, u'_2)^*_w \cap (v_1, v_2)^*_w$$

is a convex quadrangle around O with respect to U and V with $\mathfrak{V}' \subseteq \mathfrak{V}$ and $g^* \cap \mathfrak{V}' = \emptyset$.

In the case $U \notin g^*$, we consider $U' = OU \cap g$ and $V' = OV \cap g$, where $O \notin g^*$ yields $U' \neq V'$. Hence, by (5) there exist $(U_1, U_2)_W$ and $(V_1, V_2)_W$ in $\mathfrak{I}_g$ with $(U_1, U_2)_W \cap (V_1, V_2)_W = \emptyset$ and $U' \in (U_1, U_2)_W$ and $V' \in (V_1, V_2)_W$. By (6) there exist $(u'_1, u'_2)_w \in \mathfrak{I}_U$ and $(v'_1, v'_2)_w \in \mathfrak{I}_V$ with

$$OU \in (u'_1, u'_2)_w \subseteq (u_1, u_2)_w \cap (U_1U, U_2U)_w$$

and

$$OV \in (v'_1, v'_2)_w \subseteq (v_1, v_2)_w \cap (V_1V, V_2V)_w.$$

Therefore,

$$\mathfrak{V}' = (u'_1, u'_2)^*_w \cap (v'_1, v'_2)^*_w$$

is a convex quadrangle around O with respect to U and V with $\mathfrak{V}' \subseteq \mathfrak{V}$ and

$$g^* \cap \mathfrak{V}' = (g^* \cap (u'_1, u'_2)^*_w) \cap (g^* \cap (v'_1, v'_2)^*_w) \subseteq (U_1, U_2)_W \cap (V_1, V_2)_W = \emptyset,$$

proving the assertion. $\boxed{QED}$

According to the next lemma, a convex quadrangle around O contains a convex quadrangle around O with respect to a given pair of points.

Lemma 5. *Let $\mathfrak{V}$ be a convex quadrangle around O with respect to U and V, and let U' and V' be two points which are non-collinear with O. Then there exists a convex quadrangle $\mathfrak{V}'$ around O with respect to U' and V' with $\mathfrak{V}' \subseteq \mathfrak{V}$.*

PROOF. We have $O \notin g^*$ with $g = U'V'$. Therefore, by Lemma 4 there exists a convex quadrangle $\mathfrak{V}''$ around O with respect to U and V with $\mathfrak{V}'' \subseteq \mathfrak{V}$ and $g^* \cap \mathfrak{V}'' = \emptyset$. By Lemma 3, the set $(OU')^* \cap \mathfrak{V}''$ is open with respect to $\mathfrak{T}_{OU'}$, and Lemma 2 ensures the existence of $(V_1, V_2)_{U'} \in \mathfrak{I}_{OU'}$ with

$$O \in (V_1, V_2)_{U'} \subseteq (OU')^* \cap \mathfrak{V}''$$

and V_1, $V_2 \in \mathfrak{V}''$. Another application of Lemma 3 yields that for $i \in \{1,2\}$ there exists $(U_{i1}, U_{i2})_{V'} \in \mathfrak{I}_{V_i V'}$ with

$$V_i \in (U_{i1}, U_{i2})_{V'} \subseteq (V_i V')^* \cap \mathfrak{V}''.$$

Let $(U_1, U_2)_{V'} \in \mathfrak{I}_{OV'}$ and $(V_{i1}, V_{i2})_{U'} \in \mathfrak{I}_{U_i U'}$ for $i \in \{1,2\}$ be chosen in the corresponding way. By (6), we obtain $(u_1, u_2)_g \in \mathfrak{I}_{U'}$ with

$$OU' \in (u_1, u_2)_g \subseteq (U_{11}U', U_{12}U')_g \cap (U_{21}U', U_{22}U')_g$$

and $(v_1, v_2)_g \in \mathfrak{I}_{V'}$ with

$$OV' \in (v_1, v_2)_g \subseteq (V_{11}V', V_{12}V')_g \cap (V_{21}V', V_{22}V')_g \cap (V_1V', V_2V')_g.$$

Then,

$$\mathfrak{V}' = (u_1, u_2)_g^* \cap (v_1, v_2)_g^*$$

is obviously a convex quadrangle around O with respect to U' and V'; to check $\mathfrak{V}' \subseteq \mathfrak{V}''$, let $X \in \mathfrak{V}'$. For $i \in \{1,2\}$ we have

$$X_i = XU' \cap V_iV' \in (U_{i1}, U_{i2})_{V'} \subseteq \mathfrak{V}''$$

with $XV' \in (V_1V', V_2V')_g = (X_1V', X_2V')_g$; the g-convexity of $\mathfrak{V}''$ yields

$$X \in (X_1, X_2)_{U'} \subseteq \mathfrak{V}'',$$

concluding the proof. $\boxed{QED}$

The set of all convex quadrangles is a base of a topology $\mathfrak{T}_{\mathfrak{P}}$ on the set $\mathfrak{P}$ of points. Indeed, let $O \in \mathfrak{V} \cap \mathfrak{V}'$, where $\mathfrak{V}$ and $\mathfrak{V}'$ are convex quadrangles around O with respect to U and V and to U' and V', respectively. By virtue of Lemma 5, there exists a convex quadrangle $\mathfrak{V}''$ around O with respect to U and V with $\mathfrak{V}'' \subseteq \mathfrak{V}'$; let $w = UV$. With

$$\mathfrak{V} = (u_1, u_2)_w^* \cap (v_1, v_2)_w^* \quad \text{and} \quad \mathfrak{V}'' = (u_1'', u_2'')_w^* \cap (v_1'', v_2'')_w^*$$

we have $OU \in (u_1, u_2)_w \cap (u_1'', u_2'')_w$, and by (6) there is $(\widetilde{u_1}, \widetilde{u_2})_w \in \mathfrak{I}_U$ with

$$OU \in (\widetilde{u_1}, \widetilde{u_2})_w \subseteq (u_1, u_2)_w \cap (u_1'', u_2'')_w;$$

analogously, there exists $(\widetilde{v_1}, \widetilde{v_2})_w \in \mathfrak{I}_V$ with

$$OV \in (\widetilde{v_1}, \widetilde{v_2})_w \subseteq (v_1, v_2)_w \cap (v_1'', v_2'')_w.$$

Finally,

$$\widetilde{\mathfrak{V}} = (\widetilde{u_1}, \widetilde{u_2})_w^* \cap (\widetilde{v_1}, \widetilde{v_2})_w^*$$

is a convex quadrangle around O with respect to U and V with $\widetilde{\mathfrak{V}} \subseteq \mathfrak{V} \cap \mathfrak{V}''$ and therefore $O \in \widetilde{\mathfrak{V}} \subseteq \mathfrak{V} \cap \mathfrak{V}'$.

With the same arguments as for $\mathfrak{T}_g$ for a line g, we observe that the topology $\mathfrak{T}_{\mathfrak{P}}$ is neither trivial nor discrete. Moreover, let the set $\mathfrak{G}$ of lines be endowed with the topology $\mathfrak{T}_{\mathfrak{G}}$ which has been constructed in the dual way. In the following theorem we show that the projective plane $\mathfrak{E} = (\mathfrak{P}, \mathfrak{G}, \mathbin{\text{I}})$ together with these topologies is a topological projective plane in the sense of Salzmann.

Theorem 1. *If the set $\mathfrak{P}$ of points and the set $\mathfrak{G}$ of lines are endowed with the topologies $\mathfrak{T}_{\mathfrak{P}}$ and $\mathfrak{T}_{\mathfrak{G}}$, respectively, then $\mathfrak{E} = (\mathfrak{P}, \mathfrak{G}, \mathbin{\text{I}})$ is a topological projective plane in the sense of* Salzmann.

Proof. Obviously, it suffices to show that the intersection of two different lines

$$\{(g,h) \in \mathfrak{G} \times \mathfrak{G} \mid g \neq h\} \ni (g,h) \mapsto g \cap h \in \mathfrak{P}$$

is continuous. Therefore, let u_0 and v_0 be two different lines with $O = u_0 \cap v_0$, and let $O \neq U \in u_0$ and $O \neq V \in v_0$. For a neighbourhood $\mathfrak{M}$ of O, we first have $\mathfrak{M}' \in \mathfrak{T}_{\mathfrak{P}}$ with $O \in \mathfrak{M}' \subseteq \mathfrak{M}$, then a convex quadrangle $\mathfrak{V}'$ with $O \in \mathfrak{V}' \subseteq \mathfrak{M}'$ and finally by Lemma 5 a convex quadrangle $\mathfrak{V}$ around O with respect to U and V with $\mathfrak{V} \subseteq \mathfrak{V}'$.

The figure illustrates the situation in the affine plane $\mathfrak{E}_w$ with $w = UV$, where U and V are on the horizontal lines and on the vertical lines, respectively. By virtue of (4), we can assume

$$\mathfrak{V} = (u_1, u_2)_w^* \cap (v_1, v_2)_w^*$$

with $u_1 \neq u_0 \neq u_2$ and $v_1 \neq v_0 \neq v_2$; therefore by (5), there is $(u_{1i}, u_{2i})_w \in \mathfrak{I}_U$ with $u_0 \in (u_{1i}, u_{2i})_w$ and $u_i \notin (u_{1i}, u_{2i})_w$ for $i \in \{1,2\}$, hence (6) yields $(s_1, s_2)_w \in \mathfrak{I}_U$ with

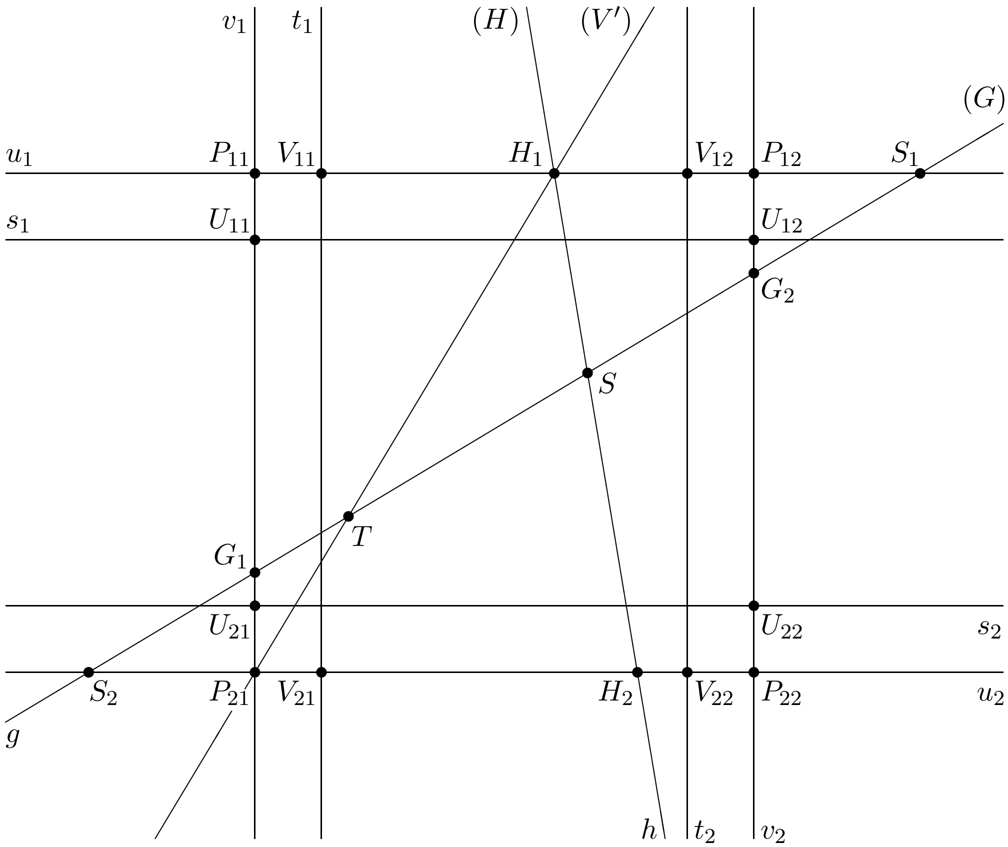
$$u_0 \in (s_1, s_2)_w \subseteq (u_1, u_2)_w \cap (u_{11}, u_{21})_w \cap (u_{12}, u_{22})_w.$$

Analogously, there exists $(t_1, t_2)_w \in \mathfrak{I}_V$ with

$$v_0 \in (t_1, t_2)_w \subseteq (v_1, v_2)_w \quad \text{and} \quad v_1,\ v_2 \notin (t_1, t_2)_w.$$

For i, $j \in \{1,2\}$, let $P_{ij} = u_i \cap v_j$ and $U_{ij} = s_i \cap v_j$ and $V_{ij} = u_i \cap t_j$. Now, u_0, v_1 and v_2 are three lines without common point with $v_1 \cap v_2 = V$, and $U_{1j} \neq U_{2j}$ are two points different from V with $v_j = U_{1j}U_{2j}$ and $u_0 \cap v_j \in (U_{1j}, U_{2j})_V$ for $j \in \{1,2\}$. Therefore

$$\mathfrak{V}_{u_0} = \{XY \mid X \in (U_{11}, U_{21})_V \text{ and } Y \in (U_{12}, U_{22})_V\}$$



is a convex quadrangle around u_0 with respect to v_1 and v_2; analogously,

$$\mathfrak{V}_{v_0} = \{XY \mid X \in (V_{11}, V_{12})_U \text{ and } Y \in (V_{21}, V_{22})_U\}$$

is a convex quadrangle around v_0 with respect to u_1 and u_2.

For all $g \in \mathfrak{V}_{u_0}$ and $h \in \mathfrak{V}_{v_0}$, we show $g \neq h$ and $S = g \cap h \in \mathfrak{V}$. First, we have $g = G_1G_2$ with $G_j \in (U_{1j}, U_{2j})_V$ for $j \in \{1,2\}$ and $h = H_1H_2$ with $H_i \in (V_{i1}, V_{i2})_U$ for $i \in \{1,2\}$, and by $g \neq w \neq h$ there exist $G = g \cap w$ and $H = h \cap w$. Because of $P_{11} \notin (U_{11}, U_{21})_V$ and $G_1 \in (U_{11}, U_{21})_V$ we have $u_1 \neq g$; moreover, we have $S_1 \notin (P_{11}, P_{12})_U$ for $S_1 = u_1 \cap g$. This is clear for $S_1 = U$ by virtue of (1). For $S_1 \neq U$ we have $G_1U \neq G_2U$ with $(G_1U, G_2U)_w \subseteq (s_1, s_2)_w$ and by construction $u_1 \notin (G_1U, G_2U)_w$ and $S_1 \notin (G_1, G_2)_G$; with the perspectivity $u_1^* \xrightarrow{V} g^*$ it follows $\varphi(P_{11}, P_{12}, U, S_1) = \varphi(G_1, G_2, G, S_1) \notin \Delta$ and therefore the assertion. In an analogous way, we obtain $u_2 \neq g$ with $S_2 = u_2 \cap g \notin (P_{21}, P_{22})_U$. In particular, we have $g \notin \mathfrak{V}_{v_0}$ and therefore $g \neq h$, and by (3) also $S_1 \notin (P_{11}, H_1)_U$ and $S_2 \notin (P_{21}, H_2)_U$.

We now apply Lemma 1 two times. On the one hand, P_{11}, H_1 and P_{21} are three non-collinear points lying neither on g nor on w with

$$S_1 = P_{11}H_1 \cap g, \quad G_1 = P_{11}P_{21} \cap g, \quad T = H_1P_{21} \cap g,$$

$$U = P_{11}H_1 \cap w, \quad V = P_{11}P_{21} \cap w \quad \text{and} \quad V' = H_1P_{21} \cap w;$$

then, $S_1 \notin (P_{11}, H_1)_U$ and $G_1 \in (P_{11}, P_{21})_V$ imply $T \in (H_1, P_{21})_{V'}$. On the other hand, H_1, P_{21} and H_2 are three non-collinear points lying neither on w nor on g with

$$V' = H_1P_{21} \cap w, \quad H = H_1H_2 \cap w, \quad U = P_{21}H_2 \cap w,$$

$$T = H_1P_{21} \cap g, \quad S = H_1H_2 \cap g \quad \text{und} \quad S_2 = P_{21}H_2 \cap g;$$

then, $T \in (H_1, P_{21})_{V'}$ and $S_2 \notin (P_{21}, H_2)_U$ imply $S \in (H_1, H_2)_H$. Hence, we obtain $SU \in (u_1, u_2)_w$ and with the corresponding arguments also $SV \in (v_1, v_2)_w$, which finally yields $S \in \mathfrak{V}$. $\boxed{QED}$

We have already remarked that the construction presented in this paper is a generalization of the method given by Wyler; apart from the ordered projective planes, we now consider a further class of projective planes which can be regarded as topological projective planes in the sense of Salzmann by virtue of the construction suggested here.

Therefore, let $\mathfrak{E} = (\mathfrak{P}, \mathfrak{G}, \mathrel{\underline{\mathrm{I}}})$ be a projective plane and let (K, T) be the coordinatizing ternary field with respect to the quadrangle (O, E, U, V); moreover, let $(G, \cdot)$ be an Abelian group with the neutral element ε; for x, $y \in K$ we define $x - y \in K$ by $(x - y) + y = x$ and $x/y \in K$ by $(x/y) \cdot y = x$ for $y \neq 0$. In [1], Junkers and Kalhoff give an algebraic characterization of the G-valued half-orderings φ of $\mathfrak{E}$; by virtue of

$$\varphi(0, 1, \infty, x) = v(x),$$

they exactly correspond with the mappings $v\colon K^* \to G$ satisfying

(i) $v(T(m, x, c) - T(m, x, d)) = v(c - d)$ for m, x, c, $d \in K$ with $c \neq d$,

(ii) $v(T(m, u, c) - T(n, u, d)) = v(m - n) \cdot v(u - x)$ for m, n, x, u, c, $d \in K$ with $m \neq n$, $x \neq u$ and $T(m, x, c) = T(n, x, d)$.

By $v(0) = 0 \notin G$ and $v(\infty) = \infty \notin G$ with $0 \neq \infty$, we extend v to the whole projective line $K \cup \{\infty\} = (OE)^*$ with $\Gamma = G \cup \{0, \infty\}$. For a multiplicatively closed subset D of G with $\Delta = D \cup \{0\}$, we consider the following properties:

(iii) $v(x \pm y) \in \Delta$ holds for all x, $y \in K$ with $v(x)$, $v(y) \in \Delta$.

(iv) $v(1-x) = v(x)$ holds for all $x \in K$ with $v(x) \notin \Delta$.

(v) There is $x_0 \in K$ with $v(x_0) \in D$ such that $v(x) \neq \varepsilon$ and $v(x \pm 1) = \varepsilon$ hold for all $x \in K$ with $v(x/x_0) \in \Delta$.

For example, these conditions are satisfied by a uniform valuation v of the ternary field K in the sense of KALHOFF (see [2]) with an Abelian value group $(G, \leq)$ and $D = \{\gamma \in G \mid \gamma \leq \varepsilon\}$.

In the sequel, we show that the properties (iii), (iv) and (v) for v ensure that the corresponding G-valued half-ordering φ is topological; therefore, we check (1) to (6) for $(0, 1, \infty) \in \mathfrak{T}$.

First, we have $v(0) = 0 \in \Delta$ and $v(1) = \varepsilon \in \Delta$, since by (iv), $v(1) \notin \Delta$ implies $v(0) = v(1)$, a contradiction. Moreover, for all x, $y \in K$ with $v(x) \in \Delta$ and $v(y) \notin \Delta$ we have $v(y/x) \notin \Delta$; hence, (iv) yields $v(1 - y/x) = v(y/x)$ and therefore $v(y) = v(1 - y/x) \cdot v(x) = v(x - y)$.

(1) is an immediate consequence of $v(\infty) = \infty \notin \Delta$.

For (2), let $c' \in K$ with $v(c') \notin \Delta$ and let $x \in K$ with $v(x) \in \Delta$. By virtue of the projectivity $\pi = \pi_1\pi_2$ with

$$\pi_1 \colon (OE)^* \xrightarrow{U} (OV)^* \quad \text{and} \quad \pi_2 \colon (OV)^* \overset{(1-c',1)}{\longrightarrow} (OE)^*,$$

we obtain

$$\varphi(0, 1, \infty, x^\pi) = \varphi(0, 1, c', x)$$

with $T(x', 1 - c', x) = 1$ and $T(x', x^\pi, x) = x^\pi$. From

$$T(x', 1 - c', x) = T(1, 1 - c', c')$$

it follows $v(x - c') = v(x' - 1) \cdot v(1 - c')$ and by $v(x - c') = v(c') = v(1 - c')$ also $v(x' - 1) = \varepsilon$. Thus, by

$$T(x', x^\pi, x) = T(1, x^\pi, 0)$$

we have $v(x) = v(x'-1) \cdot v(x^\pi) = v(x^\pi)$ and therefore $v(x^\pi) \in \Delta$ and $x \in (0, 1)_{c'}$. Hence, we obtain $(0, 1)_\infty \subseteq (0, 1)_{c'}$, and

$$\varphi(0, 1, c', \infty) = \varphi(0, 1, \infty, 1 - c')$$

with $v(1 - c') \notin \Delta$ yields equality.

For (3), let a', $b' \in K$ with $a' \neq b'$ and $v(a')$, $v(b') \in \Delta$. For all $x \in (a', b')_\infty$, we have

$$\varphi(0, 1, \infty, (x - a')/(b' - a')) = \varphi(0, b' - a', \infty, x - a') = \varphi(a', b', \infty, x) \in \Delta,$$

hence $v((x-a')/(b'-a')) \in \Delta$, and by (iii) also $v(x-a') \in \Delta$ and $v(x) \in \Delta$.

For (4), let $x \in (0,1)_\infty$. In the case $0 \neq x \neq 1$, we can put $a' = 0$ and $b' = 1$. Otherwise, we choose $x_0 \in K$ according to (v). For $x = 0$, we put $a' = x_0$ and $b' = 1$ and we have

$$\begin{aligned}\varphi(a', b', \infty, x) &= \varphi(x_0, 1, \infty, 0) = \varphi(0, x_0 - 1, \infty, x_0) = \\ &= \varphi(0, 1, \infty, x_0/(x_0 - 1)) = v(x_0) \cdot v(x_0 - 1)^{-1} = v(x_0) \in \Delta\end{aligned}$$

and therefore $x \in (a', b')_\infty$. For $x = 1$, we put $a' = 0$ and $b' = x_0 + 1$ and we have

$$\begin{aligned}\varphi(a', b', \infty, x) &= \varphi(0, x_0 + 1, \infty, 1) = \\ &= \varphi(0, 1, \infty, 1/(x_0 + 1)) = v(x_0 + 1)^{-1} = \varepsilon \in \Delta\end{aligned}$$

and again $x \in (a', b')_\infty$.

For (5), we choose $a' = 0$, $a'' = x_0$, $b' = 1$ and $b'' = x_0 + 1$; it immediately follows $0 \in (a', a'')_\infty$ and $1 \in (b', b'')_\infty$. For all $x \in (a', a'')_\infty$ we have

$$v(x/x_0) = \varphi(0, 1, \infty, x/x_0) = \varphi(0, x_0, \infty, x) = (a', a'', \infty, x) \in \Delta$$

hence, $v(x) \neq \varepsilon$ by (v); for all $y \in (b', b'')_\infty$ we have

$$\begin{aligned}v((y-1)/x_0) &= \varphi(0, 1, \infty, (y-1)/x_0) = \varphi(0, x_0, \infty, y - 1) = \\ &= \varphi(1, x_0 + 1, \infty, y) = \varphi(b', b'', \infty, y) \in \Delta,\end{aligned}$$

hence, $v(y) = \varepsilon$ by (v). Consequently, $(a', a'')_\infty \cap (b', b'')_\infty = \emptyset$ holds.

For (6), let $x \in (0,1)_\infty \cap (a', b')_\infty$, where we exemplarily consider the case $0 \neq x \neq a'$. By (3), it holds

$$x \in (0, x)_\infty \subseteq (0,1)_\infty \quad \text{and} \quad x \in (a', x)_\infty \subseteq (a', b')_\infty.$$

In the case $v(a') \in \Delta$, we have $(a', x)_\infty \subseteq (0,1)_\infty$ and therefore

$$x \in (a', x)_\infty \subseteq (0,1)_\infty \cap (a', b')_\infty.$$

In the case $v(a') \notin \Delta$, we have $v(-a') = v(a') = v(x - a')$ and therefore

$$\begin{aligned}\varphi(a', x, \infty, 0) &= \varphi(0, x - a', \infty, -a') = \\ &= \varphi(0, 1, \infty, (-a')/(x - a')) = v(-a') \cdot v(x - a')^{-1} = \varepsilon \in \Delta\end{aligned}$$

and therefore $0 \in (a', x)_\infty$; thus by (3), we obtain $(0, x)_\infty \subseteq (a', x)_\infty$ and therefore

$$x \in (0, x)_\infty \subseteq (0,1)_\infty \cap (a', b')_\infty,$$

proving the assertion.

References

[1] W. Junkers, F. Kalhoff: *Zur algebraischen Kennzeichnung mehrwertiger Halbordnungen.* J. Geometry **37** (1990), 95–104.

[2] F. Kalhoff: *Uniform Valuations on Planar Ternary Rings.* Geom. Dedicata **28** (1988), 337–348.

[3] H. Salzmann: *Topologische projektive Ebenen.* Math. Z. **67** (1957), 436–466.

[4] O. Wyler: *Order and topology in projective planes.* Amer. J. Math. **74** (1952), 656–666.