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Higher order valued reduction theorems for general linear connections

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Abstract. The reduction theorems for general linear and classical connections are generalized for operators with values in higher order gauge-natural bundles. We prove that natural operators depending on the s_1-jets of classical connections, on the s_2-jets of general linear connections and on the r-jets of tensor fields with values in gauge-natural bundles of order $k \geq 1$, $s_1 + 2 \geq s_2$, $s_1, s_2 \geq r - 1 \geq k - 2$, can be factorized through the $(k-2)$-jets of both connections, the $(k-1)$-jets of the tensor fields and sufficiently high covariant differentials of the curvature tensors and the tensor fields.

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Introduction

It is well known that natural operators of classical (linear and symmetric) connections on manifolds and of tensor fields with values in natural bundles of order one can be factorized through the curvature tensors, the tensor fields and their covariant differentials. These theorems are known as the first (operators on classical connections only) and the second reduction theorems, [6, 8, 10]. In [6] the reduction theorems are proved by using methods of natural bundles and operators, [6, 7, 9, 11].

In [4] the reduction theorems were generalized for general linear connections on vector bundles. In this gauge-natural situation we need auxiliary classical connections on the base manifolds. It is proved that natural operators with values in gauge-natural bundles of order (1,0) defined on the space of general linear connections on a vector bundle, on the space of classical connections on the base manifold and on certain tensor bundles can be factorized through the curvature tensors of linear and classical connections, the tensor fields and their covariant differentials with respect to both connections.

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In [5] another generalization of the classical reduction theorems was presented. Namely, the reduction theorems were proved for operators with values in higher order natural bundles. It was proved that an r-th order natural operator on classical connections with values in natural bundles of order $k \geq 1$, $r+2 \geq k$, can be factorized through the $(k-2)$-jets of connections and sufficiently high covariant differentials of the curvature tensor.

In this paper we combine both possible generalizations of the reduction theorems and we prove the reduction theorems for general linear connections on vector bundles for operators with values in higher order gauge-natural bundles. In this situation we shall use the name *higher order valued reduction theorems* for general linear connections.

All manifolds and maps are assumed to be smooth. The sheaf of (local) sections of a fibered manifold $p : \boldsymbol{Y} \to \boldsymbol{X}$ is denoted by $C^\infty(\boldsymbol{Y})$, $C^\infty(\boldsymbol{Y}, I\!R)$ denotes the sheaf of (local) functions.

1 Gauge-natural bundles

Let $\mathcal{M}_m$ be the category of m-dimensional C^∞-manifolds and smooth embeddings. Let $\mathcal{FM}_m$ be the category of smooth fibered manifolds over m-dimensional bases and smooth fiber manifold maps over embeddings of bases and $\mathcal{PB}_m(G)$ be the category of smooth principal G-bundles with m-dimensional bases and smooth G-bundle maps (φ, f), where the map $f \in \mathrm{Mor}\mathcal{M}_m$.

1 Definition. A *G-gauge-natural bundle* is a covariant functor F from the category $\mathcal{PB}_m(G)$ to the category $\mathcal{FM}_m$ satisfying

i) for each $\pi : \boldsymbol{P} \to \boldsymbol{M}$ in $\mathrm{Ob}\,\mathcal{PB}_m(G), \pi_{\boldsymbol{P}} : F\boldsymbol{P} \to \boldsymbol{M}$ is a fibered manifold over $\boldsymbol{M}$,

ii) for each map (φ, f) in $\mathrm{Mor}\,\mathcal{PB}_m(G), F\varphi = F(\varphi, f)$ is a fibered manifold morphism covering f,

iii) for any open subset $\boldsymbol{U} \subseteq \boldsymbol{M}$, the immersion $\iota : \pi^{-1}(\boldsymbol{U}) \hookrightarrow \boldsymbol{P}$ is transformed into the immersion $F\iota : \pi_{\boldsymbol{P}}^{-1}(\boldsymbol{U}) \hookrightarrow F\boldsymbol{P}$.

Let $(\pi : \boldsymbol{P} \to \boldsymbol{M}) \in \mathrm{Ob}\,\mathcal{PB}_m(G)$ and $W^r\boldsymbol{P}$ be the space of all r-jets $j^r_{(0,e)}\varphi$, where $\varphi : I\!R^m \times G \to \boldsymbol{P}$ is in $\mathrm{Mor}\,\mathcal{PB}_m(G), 0 \in I\!R^m$ and e is the unit in G. The space $W^r\boldsymbol{P}$ is a principal fiber bundle over the manifold $\boldsymbol{M}$ with the structure group W^r_mG of all r-jets $j^r_{(0,e)}\Psi$ of principal fiber bundle isomorphisms $\Psi : I\!R^m \times G \to I\!R^m \times G$ covering the diffeomorphisms $\psi : I\!R^m \to I\!R^m$ such that $\psi(0) = 0$. The group W^r_mG is the semidirect product of $G^r_m = \mathrm{inv}\, J^r_0(I\!R^m, I\!R^m)_0$ and $T^r_mG = J^r_0(I\!R^m, G)$ with respect to the action of G^r_m on T^r_mG given by the jet composition, i.e. $W^r_mG = G^r_m \rtimes T^r_mG$. If $(\varphi : \boldsymbol{P} \to \bar{\boldsymbol{P}}) \in \mathrm{Mor}\,\mathcal{PB}_m(G)$, then we can define the principal bundle morphism $W^r\varphi : W^r\boldsymbol{P} \to W^r\bar{\boldsymbol{P}}$ by the jet composition. The rule transforming any $\boldsymbol{P} \in \operatorname{Ob}\mathcal{PB}_m(G)$ into $W^r\boldsymbol{P} \in \operatorname{Ob}\mathcal{PB}_m(W^r_m G)$ and any $\varphi \in \operatorname{Mor}\mathcal{PB}_m(G)$ into $W^r\varphi \in \operatorname{Mor}\mathcal{PB}_m(W^r_m G)$ is a G-gauge-natural bundle.

The gauge-natural bundle functor W^r plays a fundamental role in the theory of gauge-natural bundles. We have, [1, 6],

2 Theorem. *Every gauge-natural bundle is a fiber bundle associated to the bundle W^r for a certain order r.*

The number r from Theorem 2 is called the *order of the gauge-natural bundle.* So if F is an r-order gauge-natural bundle, then

$$F\boldsymbol{P} = (W^r\boldsymbol{P}, S_F), \quad F\varphi = (W^r\varphi, \mathrm{id}_{S_F}),$$

where S_F is a left $W^r_m G$-manifold called the *standard fiber* of F.

If (x^λ, z^a) is a local fiber coordinate chart on $\boldsymbol{P}$ and (y^i) a coordinate chart on S_F, then (x^λ, y^i) is the fiber coordinate chart on $F\boldsymbol{P}$ which is said to be *adapted.*

Let F be a G-gauge-natural bundle of order s and let $r \leq s$ be a minimal number such that the action of $W^s_m G = G^s_m \rtimes T^s_m G$ on S_F can be factorized through the canonical projection $\pi^s_r : T^s_m G \to T^r_m G$. Then r is called the *gauge-order* of F and we say that F is of order (s, r). We shall denote by $W^{(s,r)}_m G = G^s_m \rtimes T^r_m G$ the Lie group acting on the standard fiber of an (s, r)-order G-gauge-natural bundle. Then there is a one-to-one, up to equivalence, correspondence of smooth left $W^{(s,r)}_m G$-manifolds and G-gauge-natural bundles of order (s, r), [1]. So any (s, r)-order G-gauge-natural bundle can be represented by its standard fiber with an action of the group $W^{(s,r)}_m G$.

If F is an (s, r)-order G-gauge-natural bundle, then $J^k F$ is an $(s + k, r + k)$-order G-gauge-natural bundle with the standard fiber $T^k_m S_F = J^k_0(\mathbb{R}^m, S_F)$.

The class of G-gauge-natural bundles contains the class of natural bundles in the sense of [6, 7, 9, 11]. Namely, if F is an r-order natural bundle, then F is the $(r, 0)$-order G-gauge-natural bundle with trivial gauge structure.

Let F be a G-gauge-natural bundle and $(\varphi, f) : \boldsymbol{P} \to \bar{\boldsymbol{P}}$ be in the category $\mathcal{PB}_n(G)$. Let σ be a section of $F\boldsymbol{P}$. Then we define the section $\varphi^*_F\sigma = F\varphi \circ \sigma \circ f^{-1}$ of $F\bar{\boldsymbol{P}}$. Let H be another gauge-natural bundle.

3 Definition. A *natural differential operator* from F to H is a collection $D = \{D(\boldsymbol{P}), \boldsymbol{P} \in \operatorname{Ob}\mathcal{PB}_n(G)\}$ of differential operators from $C^\infty(F\boldsymbol{P})$ to $C^\infty(H\boldsymbol{P})$ satisfying $D(\bar{\boldsymbol{P}}) \circ \varphi^*_F = \varphi^*_H \circ D(\boldsymbol{P})$ for each map

$$(\varphi, f) \in \operatorname{Mor}\mathcal{PB}_n(G), \quad \varphi : \boldsymbol{P} \to \bar{\boldsymbol{P}}.$$

D is of order k if all $D(\boldsymbol{P})$ are of order k. Let D be a natural differential operator of order k from F to H. For any $\boldsymbol{P} \in \operatorname{Ob}\mathcal{PB}_n(G)$ we have the associated map $\mathcal{D}(\boldsymbol{P}) : J^k F\boldsymbol{P} \to H\boldsymbol{P}$, over $\boldsymbol{M}$, defined by $\mathcal{D}(\boldsymbol{P})(j^k_x\sigma) = D(\boldsymbol{P})\sigma(x)$ for all $x \in \boldsymbol{M}$ and any section $\sigma : \boldsymbol{M} \to F\boldsymbol{P}$. From the naturality of D it follows that $\mathcal{D} = \{\mathcal{D}(\boldsymbol{P}), \boldsymbol{P} \in \mathrm{Ob}\,\mathcal{PB}_n(G)\}$ is a natural transformation of J^kF to H. The following theorem is due to Eck, [1].

4 Theorem. *Let F and H be G-gauge-natural bundles of order $\leq (s,r)$, $s \geq r$. Then we have a one-to-one correspondence between natural differential operators of order k from F to H and $W^{(s+k,r+k)}_n G$-equivariant maps from $T^k_m S_F$ to S_H.*

So according to Theorem 4 a classification of natural operators between G-gauge-natural bundles is equivalent to the classification of equivariant maps between standard fibers. Very important tool in classifications of equivariant maps is the *orbit reduction theorem*, [6, 7]. Let $p : G \to H$ be a Lie group epimorphism with the kernel K, M be a left G-space, Q be a left H-space and $\pi : M \to Q$ be a p-equivariant surjective submersion, i.e., $\pi(gx) = p(g)\pi(x)$ for all $x \in M$, $g \in G$. Having p, we can consider every left H-space N as a left G-space by $gy = p(g)y$, $g \in G$, $y \in N$.

5 Theorem. *If each $\pi^{-1}(q)$, $q \in Q$ is a K-orbit in M, then there is a bijection between the G-maps $f : M \to N$ and the H-maps $\varphi : Q \to N$ given by $f = \varphi \circ \pi$.*

2 Linear connections on vector bundles

In what follows let $G = GL(n, \mathbb{R})$ be the group of linear automorphisms of $\mathbb{R}^n$ with coordinates (a^i_j). Let us consider the category $\mathcal{VB}_{m,n}$ of vector bundles with m-dimensional bases, n-dimensional fibers and local fibered linear diffeomorphisms. Then any vector bundle $(p : \boldsymbol{E} \to \boldsymbol{M}) \in \mathrm{Ob}\,\mathcal{VB}_{m,n}$ can be considered as a zero order G-gauge-natural vector bundle (the associated vector bundle) $\mathcal{PB}_m(G) \to \mathcal{VB}_{m,n}$.

Local linear fiber coordinate charts on $\boldsymbol{E}$ will be denoted by (x^λ, y^i). The induced local bases of sections of $\boldsymbol{E}$ or $\boldsymbol{E}^*$ will be denoted by $\mathcal{E}b_i$ or $\mathcal{E}b^i$, respectively, and the induced local bases of sections of $T\boldsymbol{E}$ or $T^*\boldsymbol{E}$ will be denoted by $(\partial_\lambda, \partial_i)$ or (d^λ, d^i), respectively.

We define a *linear connection* on $\boldsymbol{E}$ to be a linear splitting

$$K : \boldsymbol{E} \to J^1\boldsymbol{E}\,.$$

Considering the contact morphism $J^1\boldsymbol{E} \to T^*\boldsymbol{M} \otimes T\boldsymbol{E}$ over the identity of $T\boldsymbol{M}$, a linear connection can be regarded as a $T\boldsymbol{E}$-valued 1-form

$$K : \boldsymbol{E} \to T^*\boldsymbol{M} \otimes T\boldsymbol{E}$$

projecting onto the identity of $T\boldsymbol{M}$.

The coordinate expression of a linear connection K is of the type

$$K = d^\lambda \otimes \big(\partial_\lambda + K_j{}^i{}_\lambda \, y^j \, \partial_i\big) \,, \qquad \text{with} \qquad K_j{}^i{}_\lambda \in C^\infty(\boldsymbol{M}, I\!R) \,.$$

Linear connections can be regarded as sections of a (1,1)-order G-gauge-natural bundle $\mathrm{Lin}\,\boldsymbol{E} \to \boldsymbol{M}$, [1,6]. The standard fiber of the functor Lin will be denoted by $R = I\!R^{n*} \otimes I\!R^n \otimes I\!R^{m*}$, elements of R will be said to be *formal linear connections*, the induced coordinates on R will be said to be *formal symbols* of formal linear connections and will be denoted by $(K_j{}^i{}_\lambda)$. The action $\beta :$ $W^{(1,1)}_m G \times R \to R$ of the group $W^{(1,1)}_m G = G^1_m \rtimes T^1_m G$ on the standard fiber R is given in coordinates by

$$(K_j{}^i{}_\lambda) \circ \beta = a^i_p \, (K_q{}^p{}_\rho \tilde{a}^q_j \tilde{a}^\rho_\lambda - \tilde{a}^p_{j\lambda}) \,,$$

where $(a^\lambda_\mu, a^i_j, a^i_{j\lambda})$ are coordinates on $W^{(1,1)}_m G$ and $\tilde{}$ denotes the inverse element.

6 Note. Let us note that the action β gives, in a natural way, the action

$$\beta^r : W^{(r+1,r+1)}_m G \times T^r_m R \to T^r_m R$$

determined by the r-jet prolongation of the action β.

7 Remark. Let us consider the group epimorphism

$$\pi^{r+1,r+1}_{r,r} : W^{(r+1,r+1)}_m G \to W^{(r,r)}_m G$$

and its kernel $B^{r+1,r+1}_{r,r} G \overset{\text{def}}{=} \mathrm{Ker}\, \pi^{r+1,r+1}_{r,r}$. On $B^{r+1,r+1}_{r,r} G$ we have the induced coordinates $(a^\lambda_{\mu_1 \dots \mu_{r+1}}, a^i_{j\mu_1 \dots \mu_{r+1}})$. Then the restriction $\bar{\beta}^r$ of the action β^r to $B^{r+1,r+1}_{r,r} G$ has the following coordinate expression

$$\begin{aligned} &(K_j{}^i{}_{\mu_1}, \dots, K_j{}^i{}_{\mu_1,\mu_2 \dots \mu_{r+1}}) \circ \bar{\beta}^r \\ &\quad = (K_j{}^i{}_{\mu_1}, \dots, K_j{}^i{}_{\mu_1,\mu_2 \dots \mu_r}, K_j{}^i{}_{\mu_1,\mu_2 \dots \mu_{r+1}} - \tilde{a}^i_{j\mu_1 \dots \mu_{r+1}}) \,, \end{aligned} \tag{1}$$

where $(K_j{}^i{}_{\mu_1}, K_j{}^i{}_{\mu_1,\mu_2}, \dots, K_j{}^i{}_{\mu_1,\mu_2 \dots \mu_{r+1}})$ are the induced jet coordinates on $T^r_m R$.

The curvature of a linear connection K on $\boldsymbol{E}$ turns out to be the vertical valued 2–form

$$R[K] = -[K, K] : \boldsymbol{E} \to V\boldsymbol{E} \otimes \bigwedge^2 T^*\boldsymbol{M} \,,$$

where $[,]$ is the Frölicher-Nijenhuis bracket. The coordinate expression is

$$\begin{aligned} R[K] &= R[K]_j{}^i{}_{\lambda\mu} \, y^j \, \partial_i \otimes d^\lambda \wedge d^\mu \\ &= -2(\partial_\lambda K_j{}^i{}_\mu + K_j{}^p{}_\lambda K_p{}^i{}_\mu) \, y^j \, \partial_i \otimes d^\lambda \wedge d^\mu \,. \end{aligned}$$

If we consider the identification $V\boldsymbol{E} = \boldsymbol{E} \underset{\boldsymbol{M}}{\times} \boldsymbol{E}$ and linearity of $R[K]$, the curvature $R[K]$ can be considered as the curvature tensor field $R[K] : \boldsymbol{M} \to \boldsymbol{E}^* \otimes \boldsymbol{E} \otimes \bigwedge^2 T^*\boldsymbol{M}$ and

$$R[K] : C^\infty(\operatorname{Lin}\boldsymbol{E}) \to C^\infty(\boldsymbol{E}^* \otimes \boldsymbol{E} \otimes \bigwedge^2 T^*\boldsymbol{M})$$

is a natural operator which is of order one, i.e., we have the associated $W^{(2,2)}_m G$-equivariant map, called the *formal curvature map* of formal linear connections,

$$\mathcal{R}_L : T^1_m R \to \mathcal{U}$$

with the coordinate expression

$$(u_j{}^i{}_{\lambda\mu}) \circ \mathcal{R}_L = K_j{}^i{}_{\lambda,\mu} - K_j{}^i{}_{\mu,\lambda} + K_j{}^p{}_\mu K_p{}^i{}_\lambda - K_j{}^p{}_\lambda K_p{}^i{}_\mu \,, \tag{2}$$

where $(u_j{}^i{}_{\lambda\mu})$ are the induced coordinates on the standard fiber $\mathcal{U} \overset{\text{def}}{=} I\!R^{n*} \otimes I\!R^n \otimes \bigwedge^2 I\!R^{m*}$ of $\boldsymbol{E}^* \otimes \boldsymbol{E} \otimes \bigwedge^2 T^*\boldsymbol{M}$.

We define a *classical connection* on $\boldsymbol{M}$ to be a linear symmetric connection on the tangent vector bundle $p_{\boldsymbol{M}} : T\boldsymbol{M} \to \boldsymbol{M}$ with the coordinate expression

$$\Lambda = d^\lambda \otimes \big(\partial_\lambda + \Lambda_\nu{}^\mu{}_\lambda\, \dot{x}^\nu\, \dot{\partial}_\mu\big)\,, \quad \Lambda_\mu{}^\lambda{}_\nu \in C^\infty(\boldsymbol{M}, I\!R), \quad \Lambda_\mu{}^\lambda{}_\nu = \Lambda_\nu{}^\lambda{}_\mu\,.$$

Classical connections can be regarded as sections of a 2nd order natural bundle $\operatorname{Cla}\boldsymbol{M} \to \boldsymbol{M}$, [6]. The standard fiber of the functor Cla will be denoted by $Q = I\!R^m \otimes S^2 I\!R^{m*}$, elements of Q will be said to be *formal classical connections*, the induced coordinates on Q will be said to be *formal Christoffel symbols* of formal classical connections and will be denoted by $(\Lambda_\mu{}^\lambda{}_\nu)$. The action $\alpha : G^2_m \times Q \to Q$ of the group G^2_m on Q is given in coordinates by

$$(\Lambda_\mu{}^\lambda{}_\nu) \circ \alpha = a^\lambda_\rho\, (\Lambda_\sigma{}^\rho{}_\tau \tilde{a}^\sigma_\mu \tilde{a}^\tau_\nu - \tilde{a}^\rho_{\mu\nu})\,.$$

8 Note. Let us note that the action α gives, in a natural way, the action

$$\alpha^r : G^{r+2}_m \times T^r_m Q \to T^r_m Q$$

determined by the r-jet prolongation of the action α.

9 Remark. Let us consider the group epimorphism $\pi^{r+2}_{r+1} : G^{r+2}_m \to G^{r+1}_m$ and its kernel $B^{r+2}_{r+1} \overset{\text{def}}{=} \operatorname{Ker}\pi^{r+2}_{r+1}$. We have the induced coordinates $(a^\lambda_{\mu_1\dots\mu_{r+2}})$ on B^{r+2}_{r+1}. Then the restriction $\bar{\alpha}^r$ of the action α^r to B^{r+2}_{r+1} has the following coordinate expression

$$\begin{aligned}(\Lambda_{\mu_1}{}^\lambda{}_{\mu_2}, \dots, \Lambda_{\mu_1}{}^\lambda{}_{\mu_2,\mu_3\dots\mu_{r+2}}) \circ \bar{\alpha}^r \\ = (\Lambda_{\mu_1}{}^\lambda{}_{\mu_2}, \dots, \Lambda_{\mu_1}{}^\lambda{}_{\mu_2,\mu_3\dots\mu_{r+1}}, \Lambda_{\mu_1}{}^\lambda{}_{\mu_2,\mu_3\dots\mu_{r+2}} - \tilde{a}^\lambda_{\mu_1\dots\mu_{r+2}})\,,\end{aligned} \tag{3}$$

where $(\Lambda_{\mu_1}{}^{\lambda}{}_{\mu_2}, \Lambda_{\mu_1}{}^{\lambda}{}_{\mu_2,\mu_3}, \dots, \Lambda_{\mu_1}{}^{\lambda}{}_{\mu_2,\mu_3\dots\mu_{r+2}})$ are the induced jet coordinates on $T^r_m Q$.

The curvature tensor of a classical connection is a natural operator

$$R[\Lambda] : C^\infty(\mathrm{Cla}\,\boldsymbol{M}) \to C^\infty(T^*\boldsymbol{M} \otimes T\boldsymbol{M} \otimes \bigwedge^2 T^*\boldsymbol{M})$$

which is of order one, i.e., we have the associated G^3_m-equivariant map, called the *formal curvature map* of formal classical connections,

$$\mathcal{R}_C : T^1_m Q \to S_{T^*\otimes T\otimes \bigwedge^2 T^*}$$

with the coordinate expression

$$(w_\nu{}^\rho{}_{\lambda\mu}) \circ \mathcal{R}_C = \Lambda_\nu{}^\rho{}_{\lambda,\mu} - \Lambda_\nu{}^\rho{}_{\mu,\lambda} + \Lambda_\nu{}^\sigma{}_\mu \Lambda_\sigma{}^\rho{}_\lambda - \Lambda_\nu{}^\sigma{}_\lambda \Lambda_\sigma{}^\rho{}_\mu \,,$$

where $(w_\nu{}^\rho{}_{\lambda\mu})$ are the induced coordinates on the standard fiber

$$\mathcal{W} \overset{\text{def}}{=} S_{T^*\otimes T\otimes \bigwedge^2 T^*} = I\!R^{m*} \otimes I\!R^m \otimes \bigwedge^2 I\!R^{m*}.$$

Let us denote by $\boldsymbol{E}^{p,r}_{q,s} \overset{\text{def}}{=} \otimes^p \boldsymbol{E} \otimes \otimes^q \boldsymbol{E}^* \otimes \otimes^r T\boldsymbol{M} \otimes \otimes^s T^*\boldsymbol{M}$ the tensor product over $\boldsymbol{M}$ and recall that $\boldsymbol{E}^{p,r}_{q,s}$ is a vector bundle which is a G-gauge-natural bundle of order $(1,0)$.

A classical connection Λ on $\boldsymbol{M}$ and a linear connection K on $\boldsymbol{E}$ induce the linear tensor product connection $K^p_q \otimes \Lambda^r_s \overset{\text{def}}{=} \otimes^p K \otimes \otimes^q K^* \otimes \otimes^r \Lambda \otimes \otimes^s \Lambda^*$ on $\boldsymbol{E}^{p,r}_{q,s}$

$$K^p_q \otimes \Lambda^r_s : \boldsymbol{E}^{p,r}_{q,s} \to T^*\boldsymbol{M} \underset{\boldsymbol{M}}{\otimes} T\boldsymbol{E}^{p,r}_{q,s}$$

which can be considered as a linear splitting

$$K^p_q \otimes \Lambda^r_s : \boldsymbol{E}^{p,r}_{q,s} \to J^1\boldsymbol{E}^{p,r}_{q,s} \,.$$

Then we define, [3], the *covariant differential of a section* $\Phi : \boldsymbol{M} \to \boldsymbol{E}^{p,r}_{q,s}$ *with respect to the pair of connections* (K,Λ) as a section of $\boldsymbol{E}^{p,r}_{q,s} \otimes T^*\boldsymbol{M}$ given by

$$\nabla^{(K,\Lambda)}\Phi = j^1\Phi - (K^p_q \otimes \Lambda^r_s) \circ \Phi \,.$$

In what follows we set $\nabla = \nabla^{(K,\Lambda)}$ and $\phi^{i_1\dots i_p\lambda_1\dots\lambda_r}_{j_1\dots j_q\mu_1\dots\mu_s;\nu} = \nabla_\nu \phi^{i_1\dots i_p\lambda_1\dots\lambda_r}_{j_1\dots j_q\mu_1\dots\mu_s}$.

We have the following relations between the covariant differentials and the curvatures, [3].

10 Proposition. *The curvature*

$$R[K^p_q \otimes \Lambda^r_s] \overset{def}{=} -[K^p_q \otimes \Lambda^r_s, K^p_q \otimes \Lambda^r_s] : \boldsymbol{E}^{p,r}_{q,s} \to \boldsymbol{E}^{p,r}_{q,s} \otimes \bigwedge^2 T^*\boldsymbol{M}$$

is determined by the curvatures $R[K]$ and $R[\Lambda]$.

11 Theorem. *(The generalized Bianchi identity) We have*

$$R[K]_j{}^i{}_{\lambda\mu;\nu} + R[K]_j{}^i{}_{\mu\nu;\lambda} + R[K]_j{}^i{}_{\nu\lambda;\mu} = 0\,.$$

12 Theorem. *Let $\Phi \in C^\infty(\boldsymbol{E}^{p,r}_{q,s})$. Then we have*

$$\operatorname{Alt} \nabla^2\Phi = -\frac{1}{2}\, R[\Lambda^p_q \otimes K^r_s] \circ \Phi \in C^\infty(\boldsymbol{E}^{p,r}_{q,s} \otimes \bigwedge^2 T^*\boldsymbol{M})\,,$$

where Alt *is the antisymmetrization.*

13 Remark. From the above Theorem 12 and the expression of $R[K^p_q \otimes \Lambda^r_s]$, [3], it follows, that $\operatorname{Alt}\nabla^2\Phi$ is a $\boldsymbol{E}^{p,r}_{q,s}$-valued 2-form which is a quadratic polynomial in $R[K], R[\Lambda], \Phi$. Especially, we have

$$\operatorname{Alt} \nabla^2 R[K] : \boldsymbol{M} \to \boldsymbol{E}^* \otimes \boldsymbol{E} \otimes \bigwedge^2 T^*\boldsymbol{M} \otimes \bigwedge^2 T^*\boldsymbol{M}\,,$$

given in coordinates by

$$\begin{aligned} \operatorname{Alt} \nabla^2 R[K] = -\frac{1}{2}\big(& R[K]_p{}^i{}_{\nu_1\nu_2}\, R[K]_j{}^p{}_{\lambda\mu} - R[K]_j{}^p{}_{\nu_1\nu_2}\, R[K]_p{}^i{}_{\lambda\mu} \\ & - R[\Lambda]_\lambda{}^\omega{}_{\nu_1\nu_2}\, R[K]_j{}^i{}_{\omega\mu} - R[\Lambda]_\mu{}^\omega{}_{\nu_1\nu_2}\, R[K]_j{}^i{}_{\lambda\omega}\big) \\ & \mathsf{b}^j \otimes \mathsf{b}_i \otimes d^\lambda \wedge d^\mu \otimes d^{\nu_1} \wedge d^{\nu_2}\,. \end{aligned}$$

14 Remark. Let us note that for classical connections we have the first and the second Bianchi identities

$$R[\Lambda]_{(\nu}{}^\rho{}_{\lambda\mu)} = 0 \quad \text{and} \quad R[\Lambda]_\nu{}^\rho{}_{(\lambda\mu;\sigma)} = 0\,,$$

respectively, where $(\dots)$ denotes the cyclic permutation. Moreover, we have the antisymmetrization of the second order covariant differential of the curvature tensor which is a quadratic polynomial of the curvature tensor.

3 The first k-th order valued reduction theorem for general linear and classical connections

Let us introduce the following notations.

Let $\mathcal{W}_0\boldsymbol{M} \overset{\text{def}}{=} \mathcal{W}\boldsymbol{M} = T^*\boldsymbol{M}\otimes T\boldsymbol{M}\otimes\bigwedge^2 T^*\boldsymbol{M}$, $\mathcal{W}_i\boldsymbol{M} = \mathcal{W}\boldsymbol{M}\otimes\otimes^i T^*\boldsymbol{M}$, $i \geq 0$. Let us put $\mathcal{W}^{(k,r)}\boldsymbol{M} = \mathcal{W}_k\boldsymbol{M} \underset{\boldsymbol{M}}{\times} \dots \underset{\boldsymbol{M}}{\times} \mathcal{W}_r\boldsymbol{M}$. We set $\mathcal{W}^{(r)}\boldsymbol{M} \overset{\text{def}}{=} \mathcal{W}^{(0,r)}\boldsymbol{M}$. Then $\mathcal{W}_i\boldsymbol{M}$ and $\mathcal{W}^{(k,r)}\boldsymbol{M}$ are natural bundles of order one and the corresponding standard fibers will be denoted by $\mathcal{W}_i$ and $\mathcal{W}^{(k,r)}$, where $\mathcal{W}_0 \overset{\text{def}}{=} \mathcal{W} = I\!R^{m*} \otimes I\!R^m \otimes \bigwedge^2 I\!R^{m*}$, $\mathcal{W}_i = \mathcal{W} \otimes \otimes^i I\!R^{m*}$, $i \geq 0$, and $\mathcal{W}^{(k,r)} = \mathcal{W}_k \times \dots \times \mathcal{W}_r$. Let us denote by $(w_\nu{}^\rho{}_{\lambda\mu\sigma_1\dots\sigma_i})$ the coordinates on $\mathcal{W}_i$.

We denote by

$$\mathcal{R}_{C,i} : T^{i+1}_m Q \to \mathcal{W}_i$$

the G^{i+3}_m-equivariant map associated with the i-th covariant differential of the curvature tensors of classical connections

$$\nabla^i R[\Lambda] : C^\infty(\mathrm{Cla}\,\boldsymbol{M}) \to C^\infty(\mathcal{W}_i\boldsymbol{M}) .$$

The map $\mathcal{R}_{C,i}$ is said to be the *formal curvature map of order* i of classical connections.

Let $C_{C,i} \subset \mathcal{W}_i$ be a subset given by identities of the i-th covariant differentials of the curvature tensors of classical connections, i.e., by covariant differentials of the Bianchi identities and the antisymmetrization of the second order covariant differentials, see Remark 14. So $C_{C,i}$ is given by the following system of equations

$$w_{(\nu}{}^\rho{}_{\lambda\mu)\sigma_1\dots\sigma_i} = 0 , \tag{4}$$

$$w_\nu{}^\rho{}_{(\lambda\mu\sigma_1)\sigma_2\dots\sigma_i} = 0 , \tag{5}$$

$$w_\nu{}^\rho{}_{\lambda\mu\sigma_1\dots[\sigma_{j-1}\sigma_j]\dots\sigma_i} + \mathrm{pol}(\mathcal{W}^{(i-2)}) = 0 , \tag{6}$$

where $j = 2,\dots,i$ and [..] denotes the antisymmetrization.

Let us put $C^{(r)}_C = C_{C,0} \times \dots \times C_{C,r}$ and denote by $C^{(k,r)}_{C,r^{(k-1)}_C}$, $k \leq r$, the fiber in $r^{(k-1)}_C \in C^{(k-1)}_C$ of the canonical projection $\mathrm{pr}^r_{k-1} : C^{(r)}_C \to C^{(k-1)}_C$. For $r < k$ we put $C^{(k,r)}_{C,r^{(k-1)}_C} = \emptyset$. Let us note that there is an affine structure on the fibres of the projection $\mathrm{pr}^r_{r-1} : C^{(r)}_C \to C^{(r-1)}_C$, [6]. Really, $C^{(r)}_C$ is a subbundle in $C^{(r-1)}_C \times \mathcal{W}_r$ given by the solution (for $i = r$) of the system of nonhomogeneous equations (4) – (6).

Then we put

$$\begin{aligned} &\mathcal{R}^{(k,r)}_C \overset{\text{def}}{=} (\mathcal{R}_{C,k},\dots,\mathcal{R}_{C,r}) : T^{r+1}_m Q \to \mathcal{W}^{(k,r)} , \\ &\mathcal{R}^{(r)}_C \overset{\text{def}}{=} \mathcal{R}^{(0,r)}_C , \end{aligned} \tag{7}$$

which has values, for any $j_0^{r+1}\gamma \in T_m^{r+1}Q$, in $C^{(k,r)}_{C,\mathcal{R}^{(k-1)}(j_0^k\gamma)}$. In [6] it was proved that $C_C^{(r)}$ is a submanifold in $\mathcal{W}^{(r)}$ and the restriction of $\mathcal{R}_C^{(r)}$ to $C_C^{(r)}$ is a surjective submersion. Then we can consider the fiber product $T_m^kQ \times_{C_C^{(k-1)}} C_C^{(r)}$ which will be denoted by $T_m^kQ \times C_C^{(k,r)}$. In [5] it was proved that the mapping

$$(\pi_k^{r+1}, \mathcal{R}_C^{(k,r)}) : T_m^{r+1}Q \to T_m^kQ \times C_C^{(k,r)}$$

is a surjective submersion.

Similarly let $\mathcal{U}_0\boldsymbol{E} \overset{\text{def}}{=} \mathcal{U}\boldsymbol{E} = \boldsymbol{E}^* \otimes \boldsymbol{E} \otimes \bigwedge^2 T^*\boldsymbol{M}$, $\mathcal{U}_i\boldsymbol{E} = \mathcal{U}\boldsymbol{E} \otimes \otimes^i T^*\boldsymbol{M}$, $i \geq 0$, $\mathcal{U}^{(k,r)}\boldsymbol{E} = \mathcal{U}_k\boldsymbol{E} \underset{\boldsymbol{M}}{\times} \dots \underset{\boldsymbol{M}}{\times} \mathcal{U}_r\boldsymbol{E}$. Especially, $\mathcal{U}^{(r)}\boldsymbol{E} \overset{\text{def}}{=} \mathcal{U}^{(0,r)}\boldsymbol{E}$. Then $\mathcal{U}_i\boldsymbol{E}$ and $\mathcal{U}^{(k,r)}\boldsymbol{E}$ are G-gauge-natural bundles of order $(1,0)$ and the corresponding standard fibers will be denoted by $\mathcal{U}_i$ and $\mathcal{U}^{(k,r)}$, where $\mathcal{U}_0 \overset{\text{def}}{=} \mathcal{U} = I\!R^{n*} \otimes I\!R^n \otimes \bigwedge^2 I\!R^{m*}$, $\mathcal{U}_i = \mathcal{U} \otimes \otimes^i I\!R^{m*}$, $i \geq 0$, and $\mathcal{U}^{(k,r)} = \mathcal{U}_k \times \dots \times \mathcal{U}_r$. Let us denote by $(u_j{}^i{}_{\lambda\mu\sigma_1\dots\sigma_i})$ the coordinates on $\mathcal{U}_i$.

We denote by

$$\mathcal{R}_{L,i} : T_m^{i-1}Q \times T_m^{i+1}R \to \mathcal{U}_i$$

the $\mathcal{W}_m^{(i+2,i+2)}G$-equivariant map associated with the i-th covariant differential of the curvature tensors of linear connections

$$\nabla^i R[K] : C^\infty(\mathrm{Cla}\,\boldsymbol{M} \underset{\boldsymbol{M}}{\times} \mathrm{Lin}\,\boldsymbol{E}) \to C^\infty(\mathcal{U}_i\boldsymbol{E})\,.$$

The map $\mathcal{R}_{L,i}$ is said to be the *formal curvature map of order* i of general linear connections.

Let $C_{L,i} \subset \mathcal{U}_i$ be a subset given by identities of the i-th covariant differentials of the curvature tensors of linear connections, i.e., by covariant differentials of the Bianchi identity and the antisymmetrization of the second order covariant differentials, see Theorem 11 and Remark 13. So $C_{L,i}$ is given by the following system of equations

$$u_j{}^i{}_{(\lambda\mu\sigma_1)\sigma_2\dots\sigma_i} = 0\,, \tag{8}$$

$$u_j{}^i{}_{\lambda\mu\sigma_1\dots[\sigma_{j-1}\sigma_j]\dots\sigma_i} + \mathrm{pol}(C_C^{(i-2)} \times \mathcal{U}^{(i-2)}) = 0\,, \tag{9}$$

$j = 2,\dots,i$, where $\mathrm{pol}(C_C^{(i-2)} \times \mathcal{U}^{(i-2)})$ are some polynomials on $C_C^{(i-2)} \times \mathcal{U}^{(i-2)}$.

Let us put $C_L^{(r)} = C_{L,0} \times \dots \times C_{L,r}$ and denote by $C^{(k,r)}_{L,r_L^{(k-1)}}$, $k \leq r$, the fiber in $r_L^{(k-1)} \in C_L^{(k-1)}$ of the canonical projection $\mathrm{pr}^r_{k-1} : C_L^{(r)} \to C_L^{(k-1)}$. For $r < k$ we put $C^{(k,r)}_{L,r_L^{(k-1)}} = \emptyset$. Let us note that there is an affine structure on the projection $\mathrm{pr}^r_{r-1} : C^{(r)}_L \to C^{(r-1)}_L$, [4]. Really, $C^{(r)}_L$ is a subbundle in $C^{(r-1)}_L \times \mathcal{U}_r$ given as the solution (for $i = r$) of the system of nonhomogeneous equations (8) – (9).

Then we set

$$\begin{aligned}\mathcal{R}^{(k,r)}_L &\overset{\mathrm{def}}{=} (\mathcal{R}_{L,k}, \dots, \mathcal{R}_{L,r}) : T^{r-1}_m Q \times T^{r+1}_m R \to \mathcal{U}^{(k,r)}\,,\\ \mathcal{R}^{(r)}_L &\overset{\mathrm{def}}{=} \mathcal{R}^{(0,r)}_L\,,\end{aligned}$$

which has values in $C^{(k,r)}_{L,\mathcal{R}^{(k-1)}_L(j^{k-2}_0\lambda, j^k_0\gamma)}$ for any $(j^{r-1}_0\lambda, j^{r+1}_0\gamma) \in T^{r-1}_m Q \times T^{r+1}_m R$.

In [4] it was proved that $C^{(s)}_C \times C^{(r)}_L$, $s \geq r-2$, $r \geq 0$, is a submanifold of $\mathcal{W}^{(s)} \times \mathcal{U}^{(r)}$ and the restriction

$$(\mathcal{R}^{(s)}_C, \mathcal{R}^{(r)}_L) : T^{s+1}_m Q \times T^{r+1}_m R \to C^{(s)}_C \times C^{(r)}_L$$

is a surjective submersion. Then we can consider the fiber product

$$(T^{k_1}_m Q \times T^{k_2}_m R) \underset{C^{(k_1-1)}_C \times C^{(k_2-1)}_L}{\times} (C^{(s)}_C \times C^{(r)}_L)\,,$$

$k_1 \geq k_2 - 2$, and denote it by $T^{k_1}_m Q \times T^{k_2}_m R \times C^{(k_1,s)}_C \times C^{(k_2,r)}_L$.

Now we shall prove the technical

15 Lemma. *If $s \geq r-2$, $k_1 \geq k_2 - 2$, $s+1 \geq k_1$, $r+1 \geq k_2$, then the restricted map*

$$(\pi^{s+1}_{k_1} \times \pi^{r+1}_{k_2}, \mathcal{R}^{(k_1,s)}_C, \mathcal{R}^{(k_2,r)}_L) : \\ T^{s+1}_m Q \times T^{r+1}_m R \to T^{k_1}_m Q \times T^{k_2}_m R \times C^{(k_1,s)}_C \times C^{(k_2,r)}_L$$

is a surjective submersion.

Proof. In [5] it was proved that

$$(\pi^{s+1}_{k_1}, \mathcal{R}^{(k_1,s)}_C) : T^{s+1}_m Q \to T^{k_1}_m Q \times C^{(k_1,s)}_C$$

is a surjective submersion. The mapping of Lemma 15 is then a surjective submersion if and only if the mapping $(\pi^{r+1}_{k_2}, \mathcal{R}^{(k_2,r)}(j^{s+1}_0\lambda, -)) : T^{r+1}_m R \to T^{k_2}_m R \times C^{(k_2,r)}_L$ is a surjective submersion for any $j^{s+1}_0\lambda \in T^{s+1}_m Q$. Let us assume $i = k_2, \dots, r$. By [4] the mapping $\mathcal{R}^{(i)}_L(j^{s+1}_0\lambda, -) : T^{i+1}_m R \to C^{(i)}_L$ is a surjective submersion and we have the commutative diagram

$$\begin{array}{ccc} T^{i+1}_m R & \xrightarrow{\mathcal{R}^{(i)}_L(j^{s+1}_0\lambda, -)} & C^{(i)}_L \\ {\scriptstyle \pi^{i+1}_i}\big\downarrow & & \big\downarrow{\scriptstyle \mathrm{pr}^i_{i-1}} \\ T^i_m R & \xrightarrow{\mathcal{R}^{(i-1)}_L(j^{s+1}_0\lambda, -)} & C^{(i-1)}_L \end{array}$$

All morphisms in the above diagram are surjective submersions which implies that the mapping $(\pi^{i+1}_i, \mathcal{R}^{(i)}_L(j^{s+1}_0\lambda, -)) : T^{i+1}_m R \to T^i_m R \underset{C^{(i-1)}_L}{\times} C^{(i)}_L$ is a surjection over $\mathcal{R}^{(i-1)}_L(j^{s+1}_0\lambda, -)$ given by $(\pi^{i+1}_i, \mathcal{R}_{L,i}(j^{s+1}_0\lambda, -))$. But the mapping $\mathcal{R}_{L,i}(j^{s+1}_0\lambda, -)$ is affine morphisms over $\mathcal{R}^{(i-1)}_L(j^{s+1}_0\lambda, -)$ (with respect to the affine structures on $\pi^{i+1}_i : T^{i+1}_m R \to T^i_m R$ and $\mathrm{pr}^i_{i-1} : C^{(i)}_L \to C^{(i-1)}_L$) which has a constant rank. So the surjective morphism $(\pi^{i+1}_i, \mathcal{R}_{L,i}(j^{s+1}_0\lambda, -))$ has a constant rank and hence is a submersion. $(\pi^{r+1}_{k_2}, \mathcal{R}^{(k_2,r)}_L(j^{s+1}_0\lambda, -))$ is then a composition of surjective submersions

$$(\pi^{k_2+1}_{k_2}, \mathcal{R}_{L,k_2}(j^{s+1}_0\lambda, -), \mathrm{id}_{C^{(k_2+1,r)}_L}) \circ \dots$$
$$\dots \circ (\pi^r_{r-1}, \mathcal{R}_{L,r-1}(j^{s+1}_0\lambda, -), \mathrm{id}_{C^{(r,r)}_L}) \circ (\pi^{r+1}_r, \mathcal{R}_{L,r}(j^{s+1}_0\lambda, -)) . \qquad \boxed{QED}$$

Let F be a G-gauge-natural bundle of order k, i.e., S_F is a $W^{(k,k)}_m G$-manifold.

16 Theorem. *Let $s \geq r-2$, $r+1, s+2 \geq k \geq 1$. For every $W^{(s+2,r+1)}_m G$-equivariant map*

$$f : T^s_m Q \times T^r_m R \to S_F$$

there exists a unique $W^{(k,k)}_m G$-equivariant map

$$g : T^{k-2}_m Q \times T^{k-1}_m R \times C^{(k-2,s-1)}_C \times C^{(k-1,r-1)}_L \to S_F$$

satisfying

$$f = g \circ (\pi^s_{k-2} \times \pi^r_{k-1}, \mathcal{R}^{(k-2,s-1)}_C, \mathcal{R}^{(k-1,r-1)}_L) .$$

Proof. Let us consider the space

$$S_{C,s} \overset{\text{def}}{=} I\!R^m \otimes S^s I\!R^{m*} \quad \text{or} \quad S_{L,r} \overset{\text{def}}{=} I\!R^{n*} \otimes I\!R^n \otimes S^r I\!R^{m*}$$

with coordinates $(s^\lambda{}_{\mu_1\mu_2\dots\mu_s})$ or $(s_j{}^i{}_{\mu_1\dots\mu_r})$, respectively. Let us consider the action of G^s_m on $S_{C,s}$ and the action of $W^{(r,r)}_m G$ on $S_{L,r}$ given by

$$\bar{s}^\lambda{}_{\mu_1\mu_2\dots\mu_s} = s^\lambda{}_{\mu_1\mu_2\dots\mu_s} - \tilde{a}^\lambda_{\mu_1\dots\mu_s} , \quad \bar{s}_j{}^i{}_{\mu_1\dots\mu_r} = s_j{}^i{}_{\mu_1\dots\mu_r} - \tilde{a}^i_{j\mu_1\dots\mu_r} . \qquad (10)$$

From (1), (3) and (10) it is easy to see that the symmetrization maps

$$\sigma_{C,s} : T^s_m Q \to S_{C,s+2} , \quad \sigma_{L,r} : T^r_m R \to S_{L,r+1}$$

given by

$$(s^\lambda{}_{\mu_1\mu_2\dots\mu_{r+1}}) \circ \sigma_{C,s} = \Lambda_{(\mu_1}{}^\lambda{}_{\mu_2,\mu_3\dots\mu_{s+2})} , \quad (s_j{}^i{}_{\mu_1\dots\mu_{r+1}}) \circ \sigma_{L,r} = K_j{}^i{}_{(\mu_1,\mu_2\dots\mu_{r+1})}$$

are equivariant.

We have the G_m^{s+2}-equivariant map

$$\begin{aligned}\varphi_{C,s} &\stackrel{\text{def}}{=} (\sigma_{C,s}, \pi^s_{s-1}, \mathcal{R}_{C,s-1}) \\ &: T^s_m Q \to S_{C,s+2} \times T^{s-1}_m Q \times \mathcal{W}_{s-1}\,.\end{aligned}$$

On the other hand we define the G_m^{s+2}-equivariant map

$$\psi_{C,s} : S_{C,s+2} \times T^{s-1}_m Q \times \mathcal{W}_{s-1} \to T^s_m Q$$

over the identity of $T^{s-1}_m Q$ by the following coordinate expression

$$\Lambda_\mu{}^\lambda{}_{\nu,\rho_1\dots\rho_s} = s^\lambda{}_{\mu\nu\rho_1\dots\rho_s} + \operatorname{lin}(w_\mu{}^\lambda{}_{\nu\rho_1\dots\rho_s} - \operatorname{pol}(T^{s-1}_m Q))\,, \tag{11}$$

where lin denotes the linear combination with real coefficients which arises in the following way. We recall that $\mathcal{R}_{C,s-1}$ gives the coordinate expression

$$\Lambda_\mu{}^\lambda{}_{\nu,\rho_1\dots\rho_s} - \Lambda_\mu{}^\lambda{}_{\rho_1,\nu\rho_2\dots\rho_s} = w_\mu{}^\lambda{}_{\nu\rho_1\dots\rho_s} - \operatorname{pol}(T^{s-1}_m Q))\,. \tag{12}$$

We can write

$$\Lambda_\mu{}^\lambda{}_{\nu,\rho_1\dots\rho_s} = s^\lambda{}_{\mu\nu\rho_1\dots\rho_s} + (\Lambda_\mu{}^\lambda{}_{\nu,\rho_1\dots\rho_s} - \Lambda_{(\mu}{}^\lambda{}_{\nu,\rho_1\dots\rho_s)})\,.$$

Then the term in brackets can be written as a linear combination of terms of the type

$$\Lambda_\mu{}^\lambda{}_{\nu,\rho_i\rho_1\dots\rho_{i-1}\rho_{i+1}\dots\rho_s} - \Lambda_\mu{}^\lambda{}_{\rho_i,\nu\rho_1\dots\rho_{i-1}\rho_{i+1}\dots\rho_s}\,,$$

$i = 1, \dots, s$, and from (12) we get (11).

Moreover,

$$\psi_{C,s} \circ \varphi_{C,s} = \mathrm{id}_{T^s_m Q}\,.$$

Similarly we have the $W_m^{(r+1,r+1)}G$-equivariant map

$$\begin{aligned}\varphi_{L,r} &\stackrel{\text{def}}{=} (\sigma_{L,r}, \mathrm{id}_{T^{r-2}_m Q} \times \pi^r_{r-1}, \mathcal{R}_{L,r-1}) \\ &: T^{r-2}_m Q \times T^r_m R \to S_{L,r+1} \times T^{r-2}_m Q \times T^{r-1}_m R \times \mathcal{U}_{r-1}\end{aligned}$$

and we define the $W_m^{(r+1,r+1)}G$-equivariant map

$$\psi_{L,r} : S_{L,r+1} \times T^{r-2}_m Q \times T^{r-1}_m R \times \mathcal{U}_{r-1} \to T^{r-2}_m Q \times T^r_m R$$

over the identity of $T^{r-2}_m Q \times T^{r-1}_m R$ by the following coordinate expression

$$K_j{}^i{}_{\lambda,\rho_1\dots\rho_r} = s_j{}^i{}_{\lambda\rho_1\dots\rho_r} + \operatorname{lin}(u_j{}^i{}_{\lambda\rho_1\dots\rho_r} - \operatorname{pol}(T^{r-2}_m Q \times T^{r-1}_m R))\,, \tag{13}$$

where lin denotes the linear combination with real coefficients which arises in the following way. We recall that $\mathcal{R}_{L,r-1}$ gives the coordinate expression

$$K_j{}^i{}_{\lambda,\rho_1\dots\rho_r} - K_j{}^i{}_{\rho_1,\lambda\rho_2\dots\rho_r} = u_j{}^i{}_{\lambda\rho_1\dots\rho_r} - \mathrm{pol}(T^{r-2}_m Q \times T^{r-1}_m R)) \,. \tag{14}$$

We can write

$$K_j{}^i{}_{\lambda,\rho_1\dots\rho_r} = s_j{}^i{}_{\lambda\rho_1\dots\rho_r} + (K_j{}^i{}_{\lambda,\rho_1\dots\rho_r} - K_j{}^i{}_{(\lambda,\rho_1\dots\rho_r)}) \,.$$

Then the term in brackets can be written as a linear combination of terms of the type

$$K_j{}^i{}_{\lambda,\rho_i\rho_1\dots\rho_{i-1}\rho_{i+1}\dots\rho_r} - K_j{}^i{}_{\rho_i,\lambda\rho_1\dots\rho_{i-1}\rho_{i+1}\dots\rho_r} \,,$$

$i = 1,\dots,r$, and from (14) we get (13).

Moreover,

$$\psi_{L,r} \circ \varphi_{L,r} = \mathrm{id}_{T^{r-2}_m Q \times T^r_m R} \,.$$

Now we have to distinguish three possibilities.

A) Let $s = r-1$. We have the same orders of groups G^{r+1}_m and $W^{(r+1,r+1)}_m G$ acting on $T^{r-1}_m Q$ and $T^r_m R$.

Let us denote by

$$A^r \stackrel{\text{def}}{=} T^{r-2}_m Q \times T^{r-1}_m R \times \mathcal{W}_{r-2} \times \mathcal{U}_{r-1} \,.$$

Then the map $f \circ (\psi_{C,r-1}, \psi_{L,r}) : S_{C,r+1} \times S_{L,r+1} \times A^r \to S_F$ satisfies the conditions of the orbit reduction Theorem 5 for the group epimorphism $\pi^{r+1,r+1}_{r,r} : W^{(r+1,r+1)}_m G \to W^{(r,r)}_m G$ and the surjective submersion $\mathrm{pr}_3 : S_{C,r+1} \times S_{L,r+1} \times A^r \to A^r$. Indeed, the space $S_{C,r+1} \times S_{L,r+1}$ is a $B^{r+1,r+1}_{r,r} G$-orbit. Moreover, (10) implies that the action of $B^{r+1,r+1}_{r,r} G$ on $S_{C,r+1} \times S_{L,r+1}$ is simply transitive. Hence there exists a unique $W^{(r,r)}_m G$-equivariant map

$$g_r : A^r = T^{r-2}_m Q \times T^{r-1}_m R \times \mathcal{W}_{r-2} \times \mathcal{U}_{r-1} \to S_F$$

such that the following diagram

$$\begin{array}{ccccc}
S_{C,r+1} \times S_{L,r+1} \times A^r & \xrightarrow{(\psi_{C,r-1},\psi_{L,r})} & T^{r-1}_m Q \times T^r_m R & \xrightarrow{f} & S_F \\
\mathrm{pr}_3 \big\downarrow & & (\pi^{r-1}_{r-2} \times \pi^r_{r-1}, \mathcal{R}_{C,r-2}, \mathcal{R}_{L,r-1}) \big\downarrow & & \mathrm{id}_{S_F} \big\downarrow \\
A^r & \xrightarrow{\mathrm{id}_{A^r}} & A^r & \xrightarrow{g_r} & S_F
\end{array}$$

commutes. So $f \circ (\psi_{C,r-1}, \psi_{L,r}) = g_r \circ \mathrm{pr}_3$ and if we compose both sides with $(\varphi_{C,r-1}, \varphi_{L,r})$, by considering

$$\mathrm{pr}_3 \circ (\varphi_{C,r-1}, \varphi_{L,r}) = (\pi^{r-1}_{r-2} \times \pi^r_{r-1}, \mathcal{R}_{C,r-2}, \mathcal{R}_{L,r-1}),$$

we obtain

$$f = g_r \circ (\pi^{r-1}_{r-2} \times \pi^r_{r-1}, \mathcal{R}_{C,r-2}, \mathcal{R}_{L,r-1})\,.$$

In the second step we consider the same construction for the map g_r and obtain the commutative diagram

$$\begin{tikzcd}
S_{C,r} \times S_{L,r} \times A^{r-1} \times \mathcal{W}_{r-2} \times \mathcal{U}_{r-1} \arrow[r, "(\psi_{C,r-2}, \psi_{L,r-1}, \mathrm{id}_{\mathcal{W}_{r-2}\times\mathcal{U}_{r-1}})"] \arrow[d, "\mathrm{pr}_{3,4,5}"'] & A^r \arrow[r, "g_r"] \arrow[d, "(\pi^{r-2}_{r-3}\times\pi^{r-1}_{r-2}, \mathcal{R}_{C,r-3}, \mathcal{R}_{L,r-2}, \mathrm{id}_{\mathcal{W}_{r-2}\times\mathcal{U}_{r-1}})"'] & S_F \arrow[d, "\mathrm{id}_{S_F}"] \\
A^{r-1} \times \mathcal{W}_{r-2} \times \mathcal{U}_{r-1} \arrow[r, "\mathrm{id}_{A^{r-1}\times\mathcal{W}_{r-2}\times\mathcal{U}_{r-1}}"] & A^{r-1} \times \mathcal{W}_{r-2} \times \mathcal{U}_{r-1} \arrow[r, "g_{r-1}"] & S_F
\end{tikzcd}$$

So that there exists a unique $W^{(r-1,r-1)}_m G$-equivariant map $g_{r-1} : A^{r-1} \times \mathcal{W}_{r-2} \times \mathcal{U}_{r-1} \to S_F$ such that

$$g_r = g_{r-1} \circ (\pi^{r-2}_{r-3} \times \pi^{r-1}_{r-2}, \mathcal{R}_{C,r-3}, \mathcal{R}_{L,r-2}, \mathrm{id}_{\mathcal{W}_{r-2}\times\mathcal{U}_{r-1}}),$$

i.e.

$$f = g_{r-1} \circ (\pi^{r-1}_{r-3} \times \pi^r_{r-2}, \mathcal{R}_{C,r-3}, \mathcal{R}_{C,r-2}, \mathcal{R}_{L,r-2}, \mathcal{R}_{L,r-1})\,.$$

Proceeding in this way we get in the last step a unique $W^{(k,k)}_m G$-equivariant map

$$g_k : T^{k-2}_m Q \times T^{k-1}_m R \times \mathcal{W}^{(k-2,r-2)} \times \mathcal{U}^{(k-1,r-1)} \to S_F$$

such that

$$f = g_k \circ (\pi^{r-1}_{k-2} \times \pi^r_{k-1}, \mathcal{R}^{(k-2,r-2)}_C, \mathcal{R}^{(k-1,r-1)}_L)\,.$$

B) Let $s = r - 2$. We have the action of the group G^r_m on $T^{r-2}_m Q$ and the action of the group $W^{(r+1,r+1)}_m G$ on $T^r_m R$.

Then the map $f \circ (\mathrm{id}_{T_m^{r-2}Q}, \psi_{L,r}) : S_{L,r+1} \times T_m^{r-2}Q \times T_m^{r-1}R \times \mathcal{U}_{r-1} \to S_F$ satisfies the conditions of the orbit reduction theorem 5 for the group epimorphism $\pi^{r+1,r+1}_{r,r} : W_m^{(r+1,r+1)}G \to W_m^{(r,r)}G$ and the surjective submersion $\mathrm{pr}_{2,3,4} : S_{L,r+1} \times T_m^{r-2}Q \times T_m^{r-1}R \times \mathcal{U}_{r-1} \to T_m^{r-2}Q \times T_m^{r-1}R \times \mathcal{U}_{r-1}$. Indeed, the space $S_{L,r+1}$ is a $B^{r+1,r+1}_{r,r}G$-orbit. Let us note that the action of $B^{r+1,r+1}_{r,r}G$ on $S_{L,r+1}$ is transitive, but not simple transitive. Hence there exists a unique $W_m^{(r,r)}G$-equivariant map $g_r : T_m^{r-2}Q \times T_m^{r-1}R \times \mathcal{U}_{r-1} \to S_F$ such that the following diagram

$$\begin{tikzcd}
S_{L,r+1} \times T_m^{r-2}Q \times T_m^{r-1}R \times \mathcal{U}_{r-1} \arrow[r, "(\mathrm{id}_{T_m^{r-2}Q}, \psi_{L,r})"] \arrow[d, "\mathrm{pr}_{2,3,4}"'] & T_m^{r-2}Q \times T_m^{r}R \arrow[r, "f"] \arrow[d, "(\mathrm{id}_{T_m^{r-2}Q} \times \pi^r_{r-1}, \mathcal{R}_{L,r-1})"'] & S_F \arrow[d, "\mathrm{id}_{S_F}"'] \\
T_m^{r-2}Q \times T_m^{r-1}R \times \mathcal{U}_{r-1} \arrow[r, "\mathrm{id}_{T_m^{r-2}Q \times T_m^{r-1}R \times \mathcal{U}_{r-1}}"] & T_m^{r-2}Q \times T_m^{r-1}R \times \mathcal{U}_{r-1} \arrow[r, "g_r"] & S_F
\end{tikzcd}$$

commutes. So $f \circ (\mathrm{id}_{T_m^{r-2}Q}, \psi_{L,r}) = g_r \circ \mathrm{pr}_{2,3,4}$ and if we compose both sides with $(\mathrm{id}_{T_m^{r-2}Q}, \varphi_{L,r})$, by considering

$$\mathrm{pr}_{2,3,4} \circ (\mathrm{id}_{T_m^{r-2}Q}, \varphi_{L,r}) = (\mathrm{id}_{T_m^{r-2}Q} \times \pi^r_{r-1}, \mathcal{R}_{L,r-1}),$$

we obtain

$$f = g_r \circ (\mathrm{id}_{T_m^{r-2}Q} \times \pi^r_{r-1}, \mathcal{R}_{L,r-1})\,.$$

Further we proceed as in the second step in A) and we get a unique $W_m^{(k,k)}G$-equivariant map

$$g_k : T_m^{k-2}Q \times T_m^{k-1}R \times \mathcal{W}^{(k-2,r-3)} \times \mathcal{U}^{(k-1,r-1)} \to S_F$$

such that

$$f = g_k \circ (\pi^{r-2}_{k-2} \times \pi^r_{k-1}, \mathcal{R}_C^{(k-2,r-3)}, \mathcal{R}_L^{(k-1,r-1)})\,.$$

C) Let $s > r-1$. We have the action of the group $W_m^{(s+2,r+1)}G$ on $T_m^sQ \times T_m^rR$.

By [5] there exists a $W_m^{(r+1,r+1)}G$-equivariant mapping

$$g_{r+1} : T_m^{r-1}Q \times T_m^rR \times \mathcal{W}^{(r-1,s-1)} \to S_F$$

such that

$$f = g_{r+1} \circ (\pi^s_{r-1} \times \mathrm{id}_{T^r_m R}, \mathcal{R}^{(r-1,s-1)}_C)\,.$$

g_{r+1} is then the mapping satisfying the condition A), i.e. there is a unique $W^{(k,k)}_m G$-equivariant map

$$g_k : T^{k-2}_m Q \times T^{k-1}_m R \times \mathcal{W}^{(k-2,r-2)} \times \mathcal{U}^{(k-1,r-1)} \to S_F$$

such that

$$g_{r+1} = g_k \circ (\pi^{r-2}_{k-2} \times \pi^r_{k-1}, \mathcal{R}^{(k-2,r-2)}_C, \mathcal{R}^{(k-1,r-1)}_L)\,,$$

i.e.,

$$f = g_k \circ (\pi^s_{k-2} \times \pi^r_{k-1}, \mathcal{R}^{(k-2,s-1)}_C, \mathcal{R}^{(k-1,r-1)}_L)\,.$$

Summarizing all cases we have

$$f = g_k \circ (\pi^s_{k-2} \times \pi^r_{k-1}, \mathcal{R}^{(k-2,s-1)}_C, \mathcal{R}^{(k-1,r-1)}_L)$$

for any $s \geq r-2$ and the restriction of g_k to $T^{r-2}_m Q \times T^{r-1}_m R \times C^{(k-2,s-1)}_C \times C^{(k-1,r-1)}_L$ is uniquely determined map g we wished to find. $\boxed{QED}$

In the above Theorem 16 we have found a map g which factorizes f, but we did not prove, that

$$(\pi^s_{k-2} \times \pi^r_{k-1}, \mathcal{R}^{(k-2,s-1)}_C, \mathcal{R}^{(k-1,r-1)}_L) : \\ T^s_m Q \times T^r_m R \to T^{k-2}_m Q \times T^{k-1}_m R \times C^{(k-2,s-1)}_C \times C^{(k-1,r-1)}_L$$

satisfy the orbit conditions, namely we did not prove that

$$(\pi^s_{k-2} \times \pi^r_{k-1}, \mathcal{R}^{(k-2,s-1)}_C, \mathcal{R}^{(k-1,r-1)}_L)^{-1}(j^{k-2}_0\lambda, j^{k-1}_0\gamma, r^{(k-2,s-1)}_C, r^{(k-1,r-1)}_L)$$

is a $B^{s+2,r+1}_{k,k}G$-orbit for any $(j^{k-2}_0\lambda, j^{k-1}_0\gamma, r^{(k-2,s-1)}_C, r^{(k-1,r-1)}_L) \in T^{k-2}_m Q \times T^{k-1}_m R \times C^{(k-2,s-1)}_C \times C^{(k-1,r-1)}_L$. Now we shall prove it.

17 Lemma. *If $(j^s_0\lambda, j^r_0\gamma), (j^s_0\acute{\lambda}, j^r_0\acute{\gamma}) \in T^s_m Q \times T^r_m R$ satisfy*

$$(\pi^s_{k-2} \times \pi^r_{k-1}, \mathcal{R}^{(k-2,s-1)}_C, \mathcal{R}^{(k-1,r-1)}_L)(j^s_0\lambda, j^r_0\gamma) = \\ (\pi^s_{k-2} \times \pi^r_{k-1}, \mathcal{R}^{(k-2,s-1)}_C, \mathcal{R}^{(k-1,r-1)}_L)(j^s_0\acute{\lambda}, j^r_0\acute{\gamma})\,,$$

then there is an element $h \in B^{s+2,r+1}_{k,k}G$ such that $h\,.\,(j^s_0\acute{\lambda}, j^r_0\acute{\gamma}) = (j^s_0\lambda, j^r_0\gamma)$.

Proof. Consider the orbit set $(T^s_m Q \times T^r_m R)/B^{s+2,r+1}_{k,k} G$. This is a $W^{(k,k)}_m G$-set. Clearly the factor projection

$$p : T^s_m Q \times T^r_m R \to (T^s_m Q \times T^r_m R)/B^{s+2,r+1}_{k,k} G$$

is a $W^{(s+2,r+1)}_m G$-map. By Theorem 16 there is a $W^{(k,k)}_m G$-equivariant map

$$g : T^{k-2}_m Q \times T^{k-1}_m R \times C^{(k-2,s-1)}_C \times C^{(k-1,r-1)}_L \to (T^s_m Q \times T^r_m R)/B^{s+2,r+1}_{k,k} G$$

satisfying $p = g \circ (\pi^s_{k-2} \times \pi^r_{k-1}, \mathcal{R}^{(k-2,s-1)}_C, \mathcal{R}^{(k-1,r-1)}_L)$. If

$$\begin{aligned}(\pi^s_{k-2} \times \pi^r_{k-1}, \mathcal{R}^{(k-2,s-1)}_C, \mathcal{R}^{(k-1,r-1)}_L)(j^s_0\lambda, j^r_0\gamma)\\ = (\pi^s_{k-2} \times \pi^r_{k-1}, \mathcal{R}^{(k-2,s-1)}_C, \mathcal{R}^{(k-1,r-1)}_L)(j^s_0\acute{\lambda}, j^r_0\acute{\gamma})\\ = (j^{k-2}_0\lambda, j^{k-1}_0\gamma, r^{(k-2,s-1)}_C, r^{(k-1,r-1)}_L)\,,\end{aligned}$$

then

$$p(j^s_0\lambda, j^r_0\gamma) = g(j^{k-2}_0\lambda, j^{k-1}_0\gamma, r^{(k-2,s-1)}_C, r^{(k-1,r-1)}_L) = p(j^s_0\acute{\lambda}, j^r_0\acute{\gamma})\,,$$

i.e. $(j^s_0\lambda, j^r_0\gamma), (j^s_0\acute{\lambda}, j^r_0\acute{\gamma})$ are in the same $B^{s+2,r+1}_{k,k} G$-orbit, proving Lemma 17. $\boxed{QED}$

The space $T^{k-2}_m Q \times T^{k-1}_m R \times C^{(k-2,s-1)}_C \times C^{(k-1,r-1)}_L$ is a left $W^{(k,k)}_M G$-space corresponding to the G-gauge-natural bundle $J^{k-2}\operatorname{Cla}\boldsymbol{M} \underset{\boldsymbol{M}}{\times} J^{k-1}\operatorname{Lin}\boldsymbol{E} \underset{\boldsymbol{M}}{\times} C^{(k-2,s-1)}_C\boldsymbol{M} \underset{\boldsymbol{M}}{\times} C^{(k-1,r-1)}_L\boldsymbol{E}$. Setting $\nabla^{(k,s)} = (\nabla^k, \dots, \nabla^s)$, then, as a direct consequence of Theorem 16, we obtain the *first k-th order valued reduction theorem for linear and classical connections* in the form.

18 Theorem. *Let $s \geq r-2$, $r+1, s+2 \geq k \geq 1$. Let F be a G-gauge-natural bundle of order k. All natural differential operators*

$$f : C^\infty(\operatorname{Cla}\boldsymbol{M} \underset{\boldsymbol{M}}{\times} \operatorname{Lin}\boldsymbol{E}) \to C^\infty(F\boldsymbol{E})$$

which are of order s with respect to classical connections and of order r with respect to linear connections are of the form

$$f(j^s\Lambda, j^r K) = g(j^{k-2}\Lambda, j^{k-1}K, \nabla^{(k-2,s-1)}R[\Lambda], \nabla^{(k-1,r-1)}R[K])$$

where g is a unique natural operator

$$g : J^{k-2}\operatorname{Cla}\boldsymbol{M} \underset{\boldsymbol{M}}{\times} J^{k-1}\operatorname{Lin}\boldsymbol{E} \underset{\boldsymbol{M}}{\times} C^{(k-2,s-1)}_C\boldsymbol{M} \underset{\boldsymbol{M}}{\times} C^{(k-1,r-1)}_L\boldsymbol{E} \to F\boldsymbol{E}\,.$$

19 Remark. From the proof of Theorem 16 it follows that the operator g is the restriction of a zero order operator defined on the k-th order G-gauge-natural bundle $J^{k-2}\operatorname{Cla}\boldsymbol{M} \underset{\boldsymbol{M}}{\times} J^{k-1}\operatorname{Lin}\boldsymbol{E} \underset{\boldsymbol{M}}{\times} \mathcal{W}^{(k-2,s-1)}\boldsymbol{M} \underset{\boldsymbol{M}}{\times} \mathcal{U}^{(k-1,r-1)}\boldsymbol{E}$.

4 The second k-th order valued reduction theorem for linear and classical connections

Write $(\boldsymbol{E}^{p_1,p_2}_{q_1,q_2})_i \overset{\text{def}}{=} \boldsymbol{E}^{p_1,p_2}_{q_1,q_2} \otimes \otimes^i T^*\boldsymbol{M}$, $i \geq 0$, and set

$$(\boldsymbol{E}^{p_1,p_2}_{q_1,q_2})^{(k,r)} \overset{\text{def}}{=} (\boldsymbol{E}^{p_1,p_2}_{q_1,q_2})_k \underset{\boldsymbol{M}}{\times} \dots \underset{\boldsymbol{M}}{\times} (\boldsymbol{E}^{p_1,p_2}_{q_1,q_2})_r \,, \qquad (\boldsymbol{E}^{p_1,p_2}_{q_1,q_2})^{(r)} \overset{\text{def}}{=} (\boldsymbol{E}^{p_1,p_2}_{q_1,q_2})^{(0,r)} \,.$$

The i-th order covariant differential of sections of $\boldsymbol{E}^{p_1,p_2}_{q_1,q_2}$ with respect to (Λ, K) is a natural operator

$$\nabla^i : C^\infty(\operatorname{Cla}\boldsymbol{M} \underset{\boldsymbol{M}}{\times} \operatorname{Lin}\boldsymbol{E} \underset{\boldsymbol{M}}{\times} \boldsymbol{E}^{p_1,p_2}_{q_1,q_2}) \to C^\infty((\boldsymbol{E}^{p_1,p_2}_{q_1,q_2})_i)$$

which is of order $(i-1)$ with respect to classical and linear connections and of order i with respect to sections of $\boldsymbol{E}^{p_1,p_2}_{q_1,q_2}$. Let us note that $\boldsymbol{E}^{p_1,p_2}_{q_1,q_2}$ is a $(1,0)$-order G-gauge-natural bundle and let us denote by $V \overset{\text{def}}{=} \otimes^{p_1} I\!R^n \otimes \otimes^{q_1} I\!R^{n*} \otimes^{p_2} I\!R^m \otimes \otimes^{q_2} I\!R^{m*}$ its standard fiber with coordinates $(v^A) = (v^{i_1\dots i_{p_1}\lambda_1\dots\lambda_{p_2}}_{j_1\dots j_{q_1}\mu_1\dots\mu_{q_2}})$. By V_i or $V^{(k,r)} \overset{\text{def}}{=} V_k \times \dots \times V_r$, $V^{(r)} \overset{\text{def}}{=} V^{(0,r)}$, we denote the standard fibers of $(\boldsymbol{E}^{p_1,p_2}_{q_1,q_2})_i$ or $(\boldsymbol{E}^{p_1,p_2}_{q_1,q_2})^{(k,r)}$, respectively.

Hence we have the associated $W^{(i+1,i+1)}_m G$-equivariant map, denoted by the same symbol,

$$\nabla^i : T^{i-1}_m Q \times T^{i-1}_m R \times T^i_m V \to V_i \,.$$

If $(v^A, v^A{}_\lambda, \dots, v^A{}_{\lambda_1\dots\lambda_i})$ are the induced jet coordinates on $T^i_m V$ (symmetric in all subscripts) and $(V^A{}_{\lambda_1\dots\lambda_i})$ are the canonical coordinates on V_i, then ∇^i is of the form

$$\begin{aligned}&(V^A{}_{\lambda_1\dots\lambda_i}) \circ \nabla^i \\ &\quad = v^A{}_{\lambda_1\dots\lambda_i} + \operatorname{pol}(T^{i-1}_m Q \times T^{i-1}_m R \times T^{i-1}_m V)\,,\end{aligned} \tag{15}$$

where pol is a polynomial on $T^{i-1}_m Q \times T^{i-1}_m R \times T^{i-1}_m V$.

We define the k-th order *formal Ricci equations*, $k \geq 2$, as follows. For $k = 2$ we have by Remark 13

$$V^A{}_{[\lambda\mu]} - \operatorname{pol}(C^{(0)}_C \times C^{(0)}_L \times V) = 0\,. \tag{E_2}$$

For $k > 2$, (E_k) is obtained by the formal covariant differentiating of (E_2) – (E_{k-1}) and antisymmetrization of the last two formal covariant differentials. They are of the form

$$V^A{}_{\lambda_1\dots[\lambda_i\lambda_{i+1}]\dots\lambda_k} - \operatorname{pol}(C^{(k-2)}_C \times C^{(k-2)}_L \times V^{(k-2)}) = 0\,, \tag{E_k}$$

$i = 1, \dots, k-1$.

20 Definition. The k-th order *formal Ricci subspace* $Z^{(k)} \subset C_C^{(k-2)} \times C_L^{(k-2)} \times V^{(k)}$ is defined by equations $(E_2), \dots, (E_k)$, $k \geq 2$. For $k = 0, 1$ we set $Z^{(0)} = V$ and $Z^{(1)} = V^{(1)}$.

In [4] it was proved that $Z^{(k)}$ is a submanifold of $C_C^{(k-2)} \times C_L^{(k-2)} \times V^{(k)}$ and the restricted morphism

$$(\mathcal{R}_C^{(k-2)}, \mathcal{R}_L^{(k-2)}, \nabla^{(k)}) : T_m^{k-1}Q \times T_m^{k-1}R \times T_m^k V \to Z^{(k)}$$

is a surjective submersion. Let us consider the projection $\mathrm{pr}_k^r : Z^{(r)} \to Z^{(k)}$. We have an affine structure on fibres of the projection $\mathrm{pr}_{r-1}^r : Z^{(r)} \to Z^{(r-1)}$. It follows from the fact that $Z^{(r)}$ is a subbundle in $Z^{(r-1)} \times (C_{C,r-2} \times C_{L,r-2} \times V_r)$ given as the space of solutions of the system of nonhomogeneous equations (E_r). Let us denote by $Z_{z^{(k-1)}}^{(k,r)}$ the fiber in $z^{(k-1)} \in Z^{(k-1)}$ of the projection $\mathrm{pr}_{k-1}^r : Z^{(r)} \to Z^{(k-1)}$. Then we can consider the fiber product over $Z^{(k-1)}$

$$(T_m^{k-2}Q \times T_m^{k-2}R \times T_m^{k-1}V) \underset{Z^{(k-1)}}{\times} Z^{(r)}$$

and denote it by

$$T_m^{k-2}Q \times T_m^{k-2}R \times T_m^{k-1}V \times Z^{(k,r)} .$$

21 Lemma. *If $r + 1 \geq k \geq 1$, then the restricted morphism*

$$(\pi_{k-2}^{r-1} \times \pi_{k-2}^{r-1} \times \pi_{k-1}^r) \times (\mathcal{R}_C^{(k-2,r-2)}, \mathcal{R}_L^{(k-2,r-2)}, \nabla^{(k,r)}) :$$
$$: T_m^{r-1}Q \times T_m^{r-1}R \times T_m^r V \to T_m^{k-2}Q \times T_m^{k-2}R \times T_m^{k-1}V \times Z^{(k,r)}$$

is a surjective submersion.

Proof. The proof of Lemma 21 follows from the commutative diagram

$$\begin{array}{ccc} T_m^{r-1}Q \times T_m^{r-1}R \times T_m^r V & \xrightarrow{(\mathcal{R}_C^{(r-2)}, \mathcal{R}_L^{(r-2)}, \nabla^{(r)})} & Z^{(r)} \\ {\scriptstyle \pi_{k-2}^{r-1} \times \pi_{k-2}^{r-1} \times \pi_{k-1}^r} \big\downarrow & & \big\downarrow {\scriptstyle \mathrm{pr}_{k-1}^r} \\ T_m^{k-2}Q \times T_m^{k-2}R \times T_m^{k-1}V & \xrightarrow{(\mathcal{R}_C^{(k-3)}, \mathcal{R}_L^{(k-3)}, \nabla^{(k-1)})} & Z^{(k-1)} \end{array}$$

where all morphisms are surjective submersions. Hence

$$(\pi_{k-2}^{r-1} \times \pi_{k-2}^{r-1} \times \pi_{k-1}^r) \times (\mathcal{R}_C^{(k-2,r-2)}, \mathcal{R}_L^{(k-2,r-2)}, \nabla^{(k,r)}) \tag{16}$$

is surjective. For $k = r$ the map

$$(\mathcal{R}_C^{(r-2,r-2)} = \mathcal{R}_{C,r-2}, \mathcal{R}_L^{(r-2,r-2)} = \mathcal{R}_{L,r-2}, \nabla^{(r,r)} = \nabla^r)$$

is an affine morphism over $(\mathcal{R}_C^{(r-3)}, \mathcal{R}_L^{(r-3)}, \nabla^{(r-1)})$ with constant rank, i.e. $(\pi_{r-2}^{r-1} \times \pi_{r-2}^{r-1} \times \pi_{r-1}^r) \times (\mathcal{R}_{C,r-2}, \mathcal{R}_{L,r-2}, \nabla^r)$ is a submersion. The mapping (16) is then a composition of surjective submersions. $\boxed{QED}$

22 Theorem. *Let S_F be a left $W^{(k,k)}_m G$-manifold. For every $W^{(r+1,r+1)}_m G$-equivariant map $f : T^{r-1}_m Q \times T^{r-1}_m R \times T^r_m V \to S_F$ there exists a unique $W^{(k,k)}_m G$-equivariant map $g : T^{k-2}_m Q \times T^{k-2}_m R \times T^{k-1}_m V \times Z^{(k,r)} \to S_F$ such that*

$$f = g \circ (\pi^{r-1}_{k-2} \times \pi^{r-1}_{k-2} \times \pi^r_{k-1}, \mathcal{R}^{(k-2,r-2)}_C, \mathcal{R}^{(k-2,r-2)}_L, \nabla^{(k,r)})\,.$$

Proof. Consider the map

$$(\mathrm{id}_{T^{r-1}_m Q} \times \mathrm{id}_{T^{r-1}_m R} \times \pi^r_{k-1}, \nabla^{(k,r)}) : T^{r-1}_m Q \times T^{r-1}_m R \times T^r_m V \to \\ \to T^{r-1}_m Q \times T^{r-1}_m R \times T^{k-1}_m V \times V^{(k,r)}$$

and denote by $\widetilde{V}^{(k,r)} \subset T^{r-1}_m Q \times T^{r-1}_m R \times T^{k-1}_m V \times V^{(k,r)}$ its image. By (15), the restricted morphism

$$\widetilde{\nabla}^{(k,r)} : T^{r-1}_m Q \times T^{r-1}_m R \times T^r_m V \to \widetilde{V}^{(k,r)}$$

is bijective for every $(j^{r-1}_0 \lambda, j^{r-1}_0 \gamma) \in T^{r-1}_m Q \times T^{r-1}_m R$, so that $\widetilde{\nabla}^{(k,r)}$ is an equivariant diffeomorphism. Define

$$(\widetilde{\mathcal{R}}^{(k-2,r-2)}_C, \widetilde{\mathcal{R}}^{(k-2,r-2)}_L) : \widetilde{V}^{(k,r)} \to T^{k-2}_m Q \times T^{k-2}_m R \times T^{k-1}_m V \times Z^{(k,r)}$$

by

$$(\widetilde{\mathcal{R}}^{(k-2,r-2)}_C, \widetilde{\mathcal{R}}^{(k-2,r-2)}_L)(j^{r-1}_0 \lambda, j^{r-1}_0 \gamma, j^{k-1}_0 \mu, v) = \\ = (j^{k-2}_0 \lambda, j^{k-2}_0 \gamma, j^{k-1}_0 \mu, \mathcal{R}^{(k-2,r-2)}_C(j^{r-1}_0 \lambda), \mathcal{R}^{(k-2,r-2)}_L(j^{r-1}_0 \lambda, j^{r-1}_0 \gamma), v)\,,$$

$(j^{r-1}_0 \lambda, j^{r-1}_0 \gamma, j^{k-1}_0 \mu, v) \in \widetilde{V}^{(k,r)}$. By Lemma 15 $(\widetilde{\mathcal{R}}^{(k-2,r-2)}_C, \widetilde{\mathcal{R}}^{(k-2,r-2)}_L)$ is a surjective submersion.

Thus, Lemma 15 and Lemma 17 imply that $(\widetilde{\mathcal{R}}^{(k-2,r-2)}_C, \widetilde{\mathcal{R}}^{(k-2,r-2)}_L)$ satisfies the orbit conditions for the group epimorphism $\pi^{r+1,r+1}_{k,k} : W^{(r+1,r+1)}_m G \to W^{(k,k)}_m G$ and there exists a unique $W^{(k,k)}_m G$-equivariant map $g : T^{k-2}_m Q \times T^{k-2}_m R \times T^{k-1}_m V \times Z^{(k,r)} \to S_F$ such that the diagram

$$\begin{tikzcd}
\widetilde{V}^{(k,r)} \arrow[r, "(\widetilde{\nabla}^{(k,r)})^{-1}"] \arrow[d, "(\widetilde{\mathcal{R}}^{(k-2,r-2)}_C, \widetilde{\mathcal{R}}^{(k-2,r-2)}_L)"'] & T^{r-1}_m Q \times T^{r-1}_m R \times T^r_m V \arrow[r, "f"] \arrow[d, "(\pi^{r-1}_{k-2} \times \pi^{r-1}_{k-2} \times \pi^r_{k-1}, \mathcal{R}^{(k-2,r-2)}_C, \mathcal{R}^{(k-2,r-2)}_L, \nabla^{(k,r)})"'] & S_F \arrow[d, "\mathrm{id}_{S_F}"'] \\
T^{k-2}_m Q \times T^{k-2}_m R \times T^{k-1}_m V \times Z^{(k,r)} \arrow[r, "\mathrm{id}"] & T^{k-2}_m Q \times T^{k-2}_m R \times T^{k-1}_m V \times Z^{(k,r)} \arrow[r, "g"] & S_F
\end{tikzcd}$$

commutes. Hence $f \circ (\widetilde{\nabla}^{(k,r)})^{-1} = g \circ (\widetilde{\mathcal{R}}_C^{(k-2,r-2)}, \widetilde{\mathcal{R}}_L^{(k-2,r-2)})$. Composing both sides with $\widetilde{\nabla}^{(k,r)}$, by considering

$$(\widetilde{\mathcal{R}}_C^{(k-2,r-2)}, \widetilde{\mathcal{R}}_L^{(k-2,r-2)}) \circ \widetilde{\nabla}^{(k,r)} = \\ (\pi^{r-1}_{k-2} \times \pi^{r-1}_{k-2} \times \pi^{r}_{k-1}, \mathcal{R}_C^{(k-2,r-2)}, \mathcal{R}_L^{(k-2,r-2)}, \nabla^{(k,r)}),$$

we get

$$f = g \circ (\pi^{r-1}_{k-2} \times \pi^{r-1}_{k-2} \times \pi^{r}_{k-1}, \mathcal{R}_C^{(k-2,r-2)}, \mathcal{R}_L^{(k-2,r-2)}, \nabla^{(k,r)})\,. \qquad \boxed{QED}$$

$T^{k-2}_m Q \times T^{k-2}_m R \times T^{k-1}_m V \times Z^{(k,r)}$ is closed with respect to the action of the group $W^{(k,k)}_m G$. The corresponding natural bundle is $J^{k-2}\,\mathrm{Cla}\,\boldsymbol{M} \underset{\boldsymbol{M}}{\times} J^{k-2}\,\mathrm{Lin}\,\boldsymbol{E} \underset{\boldsymbol{M}}{\times} J^{k-1}\boldsymbol{E}^{p_1,p_2}_{q_1,q_2} \underset{\boldsymbol{M}}{\times} Z^{(k,r)}\boldsymbol{E}$. Then the *second k-th order valued reduction theorem for linear and classical connections* can be formulated as follows.

23 Theorem. *Let F be a G-gauge-natural bundle of order $k \geq 1$ and let $r+1 \geq k$. All natural differential operators $f : C^\infty(\mathrm{Cla}\,\boldsymbol{M} \underset{\boldsymbol{M}}{\times} \mathrm{Lin}\,\boldsymbol{E} \underset{\boldsymbol{M}}{\times} \boldsymbol{E}^{p_1,p_2}_{q_1,q_2}) \to C^\infty(F\boldsymbol{E})$ of order r with respect sections of $\boldsymbol{E}^{p_1,p_2}_{q_1,q_2}$ are of the form*

$$f(j^{r-1}\Lambda, j^{r-1}K, j^r\Phi) = \\ g(j^{k-2}\Lambda, j^{k-2}K, j^{k-1}\Phi, \nabla^{(k-2,r-2)}R[\Lambda], \nabla^{(k-2,r-2)}R[K], \nabla^{(k,r)}\Phi)$$

where g is a unique natural operator

$$g : J^{k-2}\,\mathrm{Cla}\,\boldsymbol{M} \underset{\boldsymbol{M}}{\times} J^{k-2}\,\mathrm{Lin}\,\boldsymbol{E} \underset{\boldsymbol{M}}{\times} J^{k-1}\boldsymbol{E}^{p_1,p_2}_{q_1,q_2} \underset{\boldsymbol{M}}{\times} Z^{(k,r)}\boldsymbol{E} \to F\boldsymbol{E}\,.$$

24 Remark. The order $(r-1)$ of the above operators with respect to linear and classical connections is the minimal order we have to use. The second reduction theorem can be easily generalized for any operators of orders s_1 or s_2 with respect to connections Λ or K, respectively, where $s_1 \geq s_2 - 2$, $s_1, s_2 \geq r-1$. Then

$$f(j^{s_1}\Lambda, j^{s_2}K, j^r\Phi) = \\ g(j^{k-2}\Lambda, j^{k-2}K, j^{k-1}\Phi, \nabla^{(k-2,s_1-1)}R[\Lambda], \nabla^{(k-2,s_2-1)}R[K], \nabla^{(k,r)}\Phi)\,.$$

25 Remark. It is easy to see that the second reduction theorem can be generalized for any number of fields $\overset{i}{\Phi}$, $i = 1, \dots, m$, of order $(1,0)$ and that any finite order operator

$$f(j^{s_1}\Lambda, j^{s_2}K, j^{r_i}\overset{i}{\Phi})\,, \quad s_1, s_2 \geq \max(r_i) - 1\,, s_1 \geq s_2 - 2\,,$$

factorizes through $j^{k-2}\Lambda, j^{k-2}K, j^{k-1}\overset{i}{\Phi}$ and sufficiently high covariant differentials of $R[\Lambda]$, $R[K]$, $\overset{i}{\Phi}$.

References

[1] D. E. Eck: *Gauge-natural bundles and generalized gauge theories*, Mem. Amer. Math. Soc. **33** No. 247 (1981).

[2] J. Janyška: *Natural and gauge-natural operators on the space of linear connections on a vector bundle*, in: Diff. Geom. and Its Appl., Proc. Conf. Brno 1989, World Scientific, Singapore, 1990, 58–68.

[3] J. Janyška: *On the curvature of tensor product connections and covariant differentials*, in: the Proc. of the 23rd Winter School Geometry and Physics, Srní (Czech Republic) 2003, Supplemento ai Rendiconti del Circolo Matematico di Palermo, Serie II **72** (2004) 135–143.

[4] J. Janyška: *Reduction theorems for general linear connections*, Diff. Geom. and its Appl. **20** (2004) 177–196.

[5] J. Janyška: *Higher order valued reduction theorems for classical connections*, to appear in Central European Journal of Mathematics.

[6] I. Kolář, P. W. Michor, J. Slovák: Natural Operations in Differential Geometry Springer–Verlag 1993.

[7] D. Krupka, J. Janyška: Lectures on Differential Invariants, Folia Fac. Sci. Nat. Univ. Purkynianae Brunensis, Brno, 1990.

[8] G. Lubczonok: *On reduction theorems*, Ann. Polon. Math. **26** (1972) 125–133.

[9] A. Nijenhuis: *Natural bundles and their general properties*, Diff. Geom., in honour of K. Yano, Kinokuniya, Tokyo 1972, 317–334.

[10] J. A. Schouten: Ricci calculus, Berlin-Göttingen, 1954.

[11] C. L. Terng: *Natural vector bundles and natural differential operators*, Am. J. Math. **100** (1978) 775–828.

[12] T. Y. Thomas: *The replacement theorem and related questions in the projective geometry of paths*, Ann. Math. **28** (1927) 549–561.

[13] T. Y. Thomas, A. D. Michal: *Differential invariants of affinely connected manifolds*, Ann. Math. **28** (1927) 196–236.