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Uniqueness of the 2-universality Criterion

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Abstract. Kim, Kim, and Oh gave a minimal criterion for the 2-universality of positive-definite integer-matrix quadratic forms. We show that this 2-universality criterion is unique in the sense of the uniqueness of the Conway-Schneeberger Fifteen Theorem.

Keywords: 2-universality, universality criteria, quadratic forms

MSC 2000 classification: 11E20, 11E25

1 Introduction

By a *quadratic form* (or just *form*) *of rank* n we mean a degree-two homogeneous polynomial in n independent variables. If the quadratic form Q is given by $Q(x_1, \dots, x_n) = \sum_{i,j} a_{ij}x_ix_j$ with $a_{ij} = a_{ji}$, then the matrix given by $L = (a_{ij})$ is the *Gram Matrix* of a $\mathbb{Z}$-lattice L equipped with a symmetric bilinear form $\langle \cdot, \cdot \rangle$ such that $\langle L, L \rangle \subseteq \mathbb{Z}$. We have immediately from these structures that $Q(\mathbf{x}) = \mathbf{x}^T L\mathbf{x} = \langle L\mathbf{x}, \mathbf{x} \rangle$ for $\mathbf{x} \in \mathbb{R}^n$.

For convenience, we use form-theoretic and lattice-theoretic language interchangeably throughout. A complete introduction to both approaches to quadratic form theory can be found in [5].

We say that a rank-n form Q *represents* an integer k if there is an $\mathbf{x} \in \mathbb{Z}^n$ such that $Q(\mathbf{x}) = k$. More generally, we say that a lattice L represents another lattice ℓ if there is a $\mathbb{Z}$-linear, bilinear form-preserving injection $\sigma : \ell \to L$. A form is called *universal* if it represents all positive integers and is similarly called *n-universal* if it represents all positive-definite integer-matrix rank-n quadratic forms. It is clear that a rank-n form Q is universal if and only if it is 1-universal, as for an integer k

$$k = Q(x_1, \dots, x_n) \iff Q(x_1x, \dots, x_nx) = kx^2.$$

In 1993, Conway and Schneeberger announced the *Fifteen Theorem*, giving a criterion characterizing the positive-definite integer-matrix quadratic forms

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which represent all positive integers. Specifically, they showed that any positive-definite integer-matrix form which represents the set of nine critical numbers $\mathcal{S}_1 = \{1, 2, 3, 5, 6, 7, 10, 14, 15\}$ is universal [1, 2]. Kim, Kim, and Oh [4] presented an analogous criterion for 2-universality which we state in Theorem 1 of Section 3.

The set $\mathcal{S}_1$ of the Fifteen Theorem is known to be unique. Indeed, if $\mathcal{S}_1'$ is a set of integers such that a quadratic form is universal if and only if it represents the full set $\mathcal{S}_1'$, then $\mathcal{S}_1 \subseteq \mathcal{S}_1'$. We show an analogous uniqueness result for the 2-universality criterion found by Kim, Kim, and Oh [4].

2 Notations and Terminology

If a $\mathbb{Z}$-lattice L is of the form $L = L_1 \oplus L_2$ for sublattices L_1, L_2 of L and $\langle L_1, L_2 \rangle = 0$ then we write $L \cong L_1 \perp L_2$ and say that L_1 and L_2 are *orthogonal.*

We write $\langle a_1, \ldots, a_n \rangle$ for the rank-n diagonal form

$$a_1x_1^2 + \cdots + a_nx_n^2 \cong \begin{pmatrix} a_1 & & \\ & \ddots & \\ & & a_n \end{pmatrix}$$

and denote by $[a, b, c]$ the rank-2 form

$$ax^2 + 2bxy + cy^2 \cong \begin{pmatrix} a & b \\ b & c \end{pmatrix}.$$

From the classical reduction theory of quadratic forms, we may assume that the form $[a, b, c]$ is always *Minkowski-reduced* so that $0 \leq 2b \leq a \leq c$.

We work with a generalization of the escalation method used by Conway [2] and Bhargava [1]. Extending the definitions of Bhargava [1], we define a *truant* of a lattice L to be a lattice not represented by L. An *escalation* of L by a rank-n truant ℓ is a lattice L' representing ℓ which contains L as a sublattice with codimension at most n.

If $\mathcal{S}$ is a set of rank-n forms such that all escalations by elements in $\mathcal{S}$ eventually produce lattices which are n-universal, then every lattice which represents all of $\mathcal{S}$ must contain an n-universal sublattice and thus is itself n-universal (see [1–3]). We call any such $\mathcal{S}$ an *n-criterion set.* Thus, for example, the set $\mathcal{S}_1$ found by Conway [2] naturally gives the 1-criterion set

$$\{x^2, 2x^2, 3x^2, 5x^2, 6x^2, 7x^2, 10x^2, 14x^2, 15x^2\}.$$

3 Uniqueness of the 2-criterion Set

Kim, Kim, and Oh found the following 2-criterion set in [4]:

1 Theorem (Kim, Kim, and Oh)**.** *A* 2*-criterion set is given by*

$$\mathcal{S}_2 := \{\langle 1,1\rangle\,,\ \langle 2,3\rangle\,,\ \langle 3,3\rangle\,,\ [2,1,2],\ [2,1,3],\ [2,1,4]\}.$$

More can be said about this criterion: the set $\mathcal{S}_2$ is a *minimal* 2-criterion set, in the sense that for every form $\ell \in \mathcal{S}_2$ there is some rank-4 form which represents all of $\mathcal{S}_2$ but ℓ (see [4]). We now strengthen this result, showing that $\mathcal{S}_2$ is the *unique* minimal 2-criterion set.

2 Theorem. *The set of forms $\mathcal{S}_2$ given in Theorem 1 is the unique minimal 2-criterion set—that is, every* 2*-criterion set must contain $\mathcal{S}_2$ as a subset.*

Proof. Throughout, $\mathcal{T}$ denotes a finite set of rank-2 forms not containing some form $\ell \in \mathcal{S}_2$. It suffices to show that for any such $\mathcal{T}$ there is some lattice with truant ℓ which represents all of $\mathcal{T}$, since we know from Theorem 1 that $\mathcal{S}_2$ is a 2-criterion set.

If $\langle 1,1\rangle \notin \mathcal{T}$ then we may write (by Minkowski reduction)

$$\mathcal{T} = \{\langle 1,c_1\rangle\,,\ldots,\langle 1,c_k\rangle\,,L_1,\ldots,L_{k'}\},$$

where $c_i > 1$ for all $1 \le i \le k$ and the first minimum of L_i is also larger than 1 for each $1 \le i \le k'$. Then, the lattice

$$\langle 1,c_1,\ldots,c_k\rangle \perp L_1 \perp \ldots \perp L_{k'}$$

represents all of $\mathcal{T}$ but has truant $\langle 1,1\rangle$. We have therefore shown that any 2-criterion set must contain $\langle 1,1\rangle$.

Now, if $\langle 2,3\rangle \notin \mathcal{T}$ then we may express

$$\begin{aligned}
\{\langle a_1,c_1\rangle\,,\ldots,\langle a_k,c_k\rangle\} &:= \{\langle a,c\rangle \in \mathcal{T} \mid a \in \{1,2,3\}, c > 4\}\,,\\
\{[d_1,1,e_1],\ldots,[d_{k'},1,e_{k'}]\} &:= \{[d,1,e] \in \mathcal{T} \mid d \in \{2,3\}, e > 5\}\,,\\
\{L_1,\ldots,L_{k''}\} &:= \{[p,q,r] \in \mathcal{T} \mid 3 < p \le r\}\,.
\end{aligned}$$

Then, the lattice

$$\langle 1,1,4,c_1,\ldots,c_k\rangle \perp [2,1,2] \perp \langle e_1-2,\ldots,e_{k'}-2\rangle \perp L_1 \perp \ldots \perp L_{k''}$$

represents all of $\mathcal{T}$ but has truant $\langle 2,3\rangle$, whence every 2-criterion set must contain $\langle 2,3\rangle$. An analogous argument shows that every 2-criterion set must also contain $\langle 3,3\rangle$.

Likewise, if $[2,1,e_*] \notin \mathcal{T}$ for some $e_* \in \{2,3,4\}$ then we consider the sets

$$\begin{aligned}\{\langle a_1, c_1\rangle, \ldots, \langle a_k, c_k\rangle\} &:= \{\langle a, c\rangle \in \mathcal{T} \mid a \in \{1,2,3\}, c > e_*\},\\ \{[d_1,1,e_1], \ldots, [d_{k'},1,e_{k'}]\} &:= \{[d,1,e] \in \mathcal{T} \mid d \in \{2,3\}, e > e_*\},\\ \{L_1, \ldots, L_{k''}\} &:= \{[p,q,r] \in \mathcal{T} \mid 3 < p \le r\}.\end{aligned}$$

As the rank-e_* form $\langle 1, \ldots, 1\rangle$ represents $[2,1,e]$ for all $1 < e < e_*$, we observe that the lattice

$$\underbrace{\langle 1, \ldots, 1\rangle}_{e_* \text{ times}} \perp \langle c_1, \ldots, c_k, e_*\rangle \perp [d_1,1,e_1] \perp \ldots \perp [d_{k'},1,e_{k'}] \perp L_1 \perp \ldots \perp L_{k''}$$

represents all of $\mathcal{T}$ but does not represent $[2,1,e_*]$. We therefore see that every 2-criterion set must contain $[2,1,e_*]$ for each $e_* \in \{2,3,4\}$.

Since we shown that every 2-criterion set must include each $\ell \in \mathcal{S}_2$, we have proven the theorem. *QED*

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