Note di Matematica
Note Mat. 28 (2008), n. 2, 187-202
ISSN 1123-2536, e-ISSN 1590-0932
DOI 10.1285/i15900932v28n2p187

Canonical decompositions induced by A-contractions

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Received: 07/09/2006; accepted: 17/03/2008.

Abstract. The classical Nagy-Foiaş-Langer decomposition of an ordinary contraction is generalized in the context of the operators T on a complex Hilbert space $\mathcal{H}$ which, relative to a positive operator A on $\mathcal{H}$, satisfy the inequality $T^*AT \leq A$. As a consequence, a version of the classical von Neumann-Wold decomposition for isometries is derived in this context. Also one shows that, if $T^*AT = A$ and $AT = A^{1/2}TA^{1/2}$, then the decomposition of $\mathcal{H}$ in normal part and pure part relative to $A^{1/2}T$ is just a von Neumann-Wold type decomposition for $A^{1/2}T$, which can be completely described. As applications, some facts on the quasi-isometries recently studied in [4], [5], are obtained.

Keywords: A-contraction, A-isometry, quasi-isometry, von Neumann-Wold decomposition.

MSC 2000 classification: primary 47A15, 47A63, secondary 47B20.

1 Introduction and preliminaries

Let $\mathcal{H}$ be a complex Hilbert space and $\mathcal{B}(\mathcal{H})$ the Banach algebra of all bounded linear operators on $\mathcal{H}$. The range and the null-space of $T \in \mathcal{B}(\mathcal{H})$ are denoted by $\mathcal{R}(T)$ and $\mathcal{N}(T)$, respectively.

Let $A \in \mathcal{B}(\mathcal{H})$ be a fixed positive operator, $A \neq 0$. An operator $T \in \mathcal{B}(\mathcal{H})$ is called an A-*contraction* if it satisfies the inequality

$$T^*AT \leq A, \tag{1}$$

where T^* stands for the adjoint of T. Also, T is called an A-*isometry* if the equality occurs in (1). According to [8] we say that T is a pure A-contraction if T is an A-contraction and there exists no non zero subspace in $\mathcal{H}$ which reduces A and T on which T is an A-isometry. Such operators appear in many papers, for instance [1, 2, 4, 5, 7–9].

Clearly, an ordinary contraction means an I-contraction, where $I = I_{\mathcal{H}}$ is the identity operator in $\mathcal{B}(\mathcal{H})$. A contraction T is also a T^*T-contraction and a S_T-isometry, where S_T is the strong limit of the sequence $\{T^{*n}T^n : \ n \geq 1\}$.

[i]This research was supported by the Contract CEEX 957/28.06.2005.

According to [4], [5], an operator $T \in \mathcal{B}(\mathcal{H})$ which is a T^*T-isometry is called a *quasi-isometry*. A quasi-isometry T is a partial isometry if and only if T is quasinormal, which means that T commutes with T^*T ([4], [7]).

If T is a quasinormal contraction then T and T^* are T^*T-contractions such that T and T^* commute with T^*T, these being a particular case of A-contractions S satisfying $AS = SA$.

In general, for an A-contraction T on $\mathcal{H}$ one has $AT \neq TA$, and furthermore, T^* is not an A-contraction (see [7]). This shows that the properties of A-contractions are quite different from the ones of ordinary contractions. However, an A-contraction T is partially related to the contraction $\widehat{T}$ on $\overline{\mathcal{R}(A)}$ defined (using (1)) by

$$\widehat{T}A^{1/2}h = A^{1/2}Th \quad (h \in \mathcal{H}), \tag{2}$$

where $A^{1/2}$ is the square root of A. Recall that $\overline{\mathcal{R}(A)} = \overline{\mathcal{R}(A^{1/2})}$.

If T is a regular A-contraction, that is it satisfies the condition $AT = A^{1/2}TA^{1/2}$, then it is easy to see that T is a lifting of $\widehat{T}$, or equivalently, T^* is an extension of $\widehat{T}^*$. Even in this case $\mathcal{N}(A)$ is not invariant for T^*, in general, (see [7]) but it is immediate from (1) that $\mathcal{N}(A)$ is invariant for T.

This paper deals with some decompositions of $\mathcal{H}$ induced by A-contractions and particularly, A-isometries.

Thus, in Section 2 we find natural generalizations of Nagy-Foiaş-Langer decomposition and of von Neumann-Wold decomposition, in the context of A-contractions T with $AT = TA$, that is in the commutative case. As consequences, we recover the normal part and the pure (completely non normal) part, as well as the normal partial isometric part, of a quasinormal contraction.

In Section 3 we completely describe the normal-pure decomposition of $\mathcal{H}$ relative to the operator $A^{1/2}T$, when T is a regular A-isometry on $\mathcal{H}$. In fact, this decomposition is a von-Neumann-Wold type decomposition for $A^{1/2}T$, by analogy with the case $A = I$ (when T is an isometry). We give this decomposition in terms of A and T, also using the polar decomposition of $A^{1/2}T$.

As applications, we recover and we complete some facts from Section 2, and we also obtain some results concerning the quasi-isometries, recently studied in [4], [5]. More precisely, our characterizations of normal quasi-isometries are related to a problem posed by Patel in Remark 2.1 [4].

2 Decompositions in the commutative case

It is known [8] that for any A-contraction on $\mathcal{H}$ the subspace

$$\mathcal{N}_\infty(A,T) = \bigcap_{n=1}^{\infty} \mathcal{N}(A - T^{*n}AT^n) \tag{3}$$

is invariant for T, but it is not invariant for A, in general. However, this subspace reduces A if T is a regular A-contraction (Theorem 4.6 [8]), but even in this case it is not invariant for T^*, as happens when T is an ordinary contraction. When the subspace $\mathcal{N}_\infty(A,T)$ reduces A, it is the maximum invariant subspace for A and T on which T is an A-isometry (Proposition 2.1 [8]).

Using this fact, we can now generalize the classical Nagy-Foiaş-Langer theorem ([2], [10]) for ordinary contractions, in the context of A-contractions T with $AT = TA$. First we give the following

1 Lemma. *For an A-contraction T on $\mathcal{H}$ the following assertions are equivalent:*

(i) $AT = TA$;

(ii) $\mathcal{N}(A)$ *reduces* T, *and* T *is a regular* A*-contraction;*

(iii) T^* *is a regular* A*-contraction;*

(iv) T^* *is an* A*-contraction and either* T, *or* T^* *is regular.*

Proof. Clearly, the implications $(i) \Rightarrow (ii)$ and $(iii) \Rightarrow (iv)$ are trivial. Now, the assumption (ii) means that $AT = A^{1/2}TA^{1/2}$ and $\overline{\mathcal{R}(A)} = \overline{\mathcal{R}(A^{1/2})}$ reduces T, whence we obtain $A^{1/2}T = TA^{1/2}$ because $A^{1/2}$ is injective on $\overline{\mathcal{R}(A)}$. This gives

$$\widehat{T}A^{1/2} = A^{1/2}T = TA^{1/2}$$

so that $\widehat{T} = T|_{\overline{\mathcal{R}(A)}}$, and later one obtains for $h \in \mathcal{H}$

$$TAT^*h = A^{1/2}TT^*A^{1/2}h = A^{1/2}T\widehat{T}^*A^{1/2}h = A^{1/2}\widehat{T}\widehat{T}^*A^{1/2}h.$$

Next, since $\widehat{T}$ is a contraction on $\overline{\mathcal{R}(A)}$ it follows that $TAT^* \leq A$, that is T^* is an A-contraction on $\mathcal{H}$. Also one has $A^{1/2}T^* = T^*A^{1/2}$, or equivalently $AT^* = A^{1/2}T^*A^{1/2}$, which means that T^* is a regular A-contraction. Hence (ii) implies (iii).

Finally, from the hypothesis on T and the assumption (iv) we infer that $\mathcal{N}(A)$ reduces T and also that $AT = A^{1/2}TA^{1/2}$, or $AT^* = A^{1/2}T^*A^{1/2}$. But these imply $A^{1/2}T = TA^{1/2}$, or equivalently $AT = TA$. Consequently (iv) implies (v), which ends the proof. QED

We remark from the above proof that under the conditions $(i)-(iv)$ we have $T|_{\overline{\mathcal{R}(A)}} = \widehat{T}$, hence T is a contraction on $\overline{\mathcal{R}(A)}$.

2 Theorem. *Let T be an A-contraction on $\mathcal{H}$ such that $AT = TA$. Then we have*

$$\begin{aligned} \mathcal{N}_\infty^* : &= \mathcal{N}_\infty(A,T) \cap \mathcal{N}_\infty(A,T^*) \\ &= \mathcal{N}(A) \oplus \mathcal{N}(I - S_{\widehat{T}}) \cap \mathcal{N}(I - S_{\widehat{T}^*}) \end{aligned} \tag{4}$$

and it is the maximum reducing subspace for A and T on which T and T^* are A-isometries. Moreover,

$$\mathcal{N}_u := \mathcal{N}_\infty^* \ominus \mathcal{N}(A) \tag{5}$$

is the maximum subspace contained in $\overline{R(A)}$ which reduces T to a unitary operator.

Proof. Let $\mathcal{N}_\infty = \mathcal{N}_\infty(A,T)$ and $\mathcal{N}_{\infty *} = \mathcal{N}_\infty(A,T^*)$. Since $AT = TA$ the subspaces $\mathcal{N}_\infty$ and $\mathcal{N}_{\infty *}$ reduce A. Now if $h \in \mathcal{N}_\infty \cap \mathcal{N}_{\infty *}$ then for every integer $j \geq 1$ we have $Ah = T^{*j}AT^j h = T^j AT^{*j} h$, and for $n \geq 1$ we obtain

$$\begin{aligned} T^{*n}AT^nT^*h &= T^{*n}T^nAT^*h = T^{*n}T^{n-1}Ah \\ &= T^{*n}AT^{n-1}h = T^*Ah = AT^*h. \end{aligned}$$

Hence $T^*h \in \mathcal{N}_\infty$, and similarly one has $Th \in \mathcal{N}_{\infty *}$. Having in view that $\mathcal{N}_\infty$ and $\mathcal{N}_{\infty *}$ are also invariant for T and T^* respectively, it follows that $\mathcal{N}_\infty^* = \mathcal{N}_\infty \cap \mathcal{N}_{\infty *}$ reduces T, and obviously T and T^* are A-isometries on $\mathcal{N}_\infty^*$. In addition, $\mathcal{N}_\infty^*$ is the maximum reducing subspace for A and T on which T and T^* are A-isometries, because $\mathcal{N}_\infty$ and $\mathcal{N}_{\infty *}$ have similar properties relative to T and T^* respectively, as invariant subspaces.

Now since $\mathcal{N}(A)$ reduces A and T, while T, T^* are A-isometries on $\mathcal{N}(A)$, it follows that $\mathcal{N}(A) \subset \mathcal{N}_\infty^*$. Therefore $\mathcal{G} = \mathcal{N}_\infty^* \ominus \mathcal{N}(A)$ also reduces A and T, and T, T^* are A-isometries on $\mathcal{G}$, hence we have for $h \in \mathcal{G}$

$$AT^*Th = T^*ATh = Ah = TAT^*h = ATT^*h.$$

As $\mathcal{G} \subset \overline{\mathcal{R}(A)}$ and A is injective on $\overline{\mathcal{R}(A)}$, we infer from these relations that T is a unitary operator on $\mathcal{G}$. Next, let $\mathcal{M} \subset \overline{\mathcal{R}(A)}$ be another subspace which reduces T to a unitary operator. Then for $h \in \mathcal{M}$ and $n \geq 1$ we have

$$Ah = AT^{*n}T^nh = AT^nT^{*n}h = T^nAT^{*n}h = T^{*n}AT^nh,$$

which provides that $\mathcal{M} \subset \mathcal{N}_\infty \cap \mathcal{N}_{\infty *}$, having in view (3). Hence $\mathcal{M} \subset \mathcal{G}$, what proves the required maximality property of $\mathcal{G}$.

Finally, it is easy to see from (3) that the subspace $\mathcal{N}_\infty$ can be expressed as following

$$\begin{aligned} \mathcal{N}_\infty &= \{h \in \mathcal{H} : Ah = T^{*n}T^nAh,\ n \geq 1\} \\ &= \mathcal{N}(A) \oplus \mathcal{N}(A_0 - S_{\widehat{T}}A_0) = \mathcal{N}(A) \oplus \mathcal{N}(I - S_{\widehat{T}}), \end{aligned}$$

where $A_0 = A|_{\overline{\mathcal{R}(A)}}$, $\widehat{T} = T|_{\overline{\mathcal{R}(A)}}$. Clearly, we used here that $AT = TA$ and that A_0 is injective. Analogously (by Lemma 1) one has

$$\mathcal{N}_{\infty *} = \mathcal{N}(A) \oplus \mathcal{N}(I - S_{\widehat{T}^*})$$

and thus one obtains the second equality in (4). QED

In what follows we say that an operator $T \in \mathcal{B}(\mathcal{H})$ is A-*unitary* if T and T^* are A-isometries. Obviously, if $AT = TA$ then T is A-unitary if and only if T is an A-isometry and T is normal on $\overline{\mathcal{R}(A)}$, or equivalently (by Theorem 2) T is unitary on $\overline{\mathcal{R}(A)}$.

Using this concept, we can generalize in the context of A-contractions the Nagy-Foiaş-Langer decomposition for contractions.

3 Corollary. *Let T be an A-contraction on $\mathcal{H}$ such that $AT = TA$. Then there exists a unique orthogonal decomposition for $\mathcal{H}$ of the form*

$$\mathcal{H} = \mathcal{H}_u \oplus \mathcal{H}_c \tag{6}$$

where the two subspaces reduce A and T, such that $\mathcal{N}(A) \subset \mathcal{H}_u$ and T is A-unitary on $\mathcal{H}_u$, while T is a completely non unitary contraction on $\mathcal{H}_c$. In addition one has $\mathcal{H}_u = \mathcal{N}_\infty^*$.

Proof. By Theorem 2 the subspace $\mathcal{H}_u = \mathcal{N}_\infty^*$ has the required properties. Also, since $\mathcal{H}_c = \mathcal{H} \ominus \mathcal{H}_u \subset \overline{\mathcal{R}(A)} \ominus \mathcal{N}_\infty^* \cap \overline{\mathcal{R}(A)}$ and $T|_{\overline{\mathcal{R}(A)}} = \widehat{T}$, we infer also from Theorem 2 that T is a completely non unitary contraction on $\mathcal{H}_c$. Thus T has the above quoted properties relative to the decomposition (6). Let now $\mathcal{H} = \mathcal{H}'_u \oplus \mathcal{H}'_c$ be another decomposition with $\mathcal{N}(A) \subset \mathcal{H}'_u$ and $\mathcal{H}'_u$ be a reducing subspace for A and T, such that T is A-unitary on $\mathcal{H}'_u$ and T is a completely non unitary contraction on $\mathcal{H}'_c$. Then since $\mathcal{N}(A) \subset \mathcal{H}_u \cap \mathcal{H}'_u$, one has

$$\mathcal{H}_u \ominus \mathcal{H}'_u = \mathcal{H}_u \cap \overline{\mathcal{R}(A)} \ominus \mathcal{H}'_u \cap \overline{\mathcal{R}(A)},$$

and so $\mathcal{H}_u \ominus \mathcal{H}'_u$ reduces T to a unitary operator (by Theorem 2). But $\mathcal{H}_u \ominus \mathcal{H}'_u \subset \mathcal{H}'_c$, hence T is also completely non unitary on $\mathcal{H}_u \ominus \mathcal{H}'_u$. Thus, $\mathcal{H}_u \ominus \mathcal{H}'_u = \{0\}$ that is $\mathcal{H}_u = \mathcal{H}'_u$, and consequently $\mathcal{H}_c = \mathcal{H}'_c$. This shows that the decomposition (6) is unique with respect to the quoted properties. QED

4 Corollary. *If T is a regular A-contraction on $\mathcal{H}$ and A is injective, then T is a contraction on $\mathcal{H}$ and the maximum subspace which reduces T to a unitary operator is*

$$\mathcal{H}_u = \mathcal{N}(I - S_T) \cap \mathcal{N}(I - S_{T^*}). \tag{7}$$

Proof. Since $AT = A^{1/2}TA^{1/2}$ and $A^{1/2}$ is injective it follows that $TA^{1/2} = A^{1/2}T = \widehat{T}A^{1/2}$, hence $TA = AT$ and $T = \widehat{T}$, that is T is a contraction on $\mathcal{H}$. In this case, $\mathcal{H}_u = \mathcal{N}_\infty^*$ has the form (7), having in view (4) and that $\mathcal{N}(A) = \{0\}$. QED

Clearly, in the case $A = I$ every of the above corollaries just give the Nagy-Foiaş-Langer theorem concerning the unitary and the completely non unitary part of a contraction.

5 Corollary. *Let T be an A-isometry such that T^* is a regular pure A-contraction on $\mathcal{H}$. Then T is a shift on $\mathcal{H}$.*

Proof. By Lemma 1 one has $AT = TA$ and since $T^*AT = A$, one obtains that $AT^*T = A$ on $\mathcal{H}$. Also, since $\mathcal{N}(A)$ reduces T^* to an A-isometry and T^* is a pure A-contraction, it follows that $\mathcal{N}(A) = \{0\}$, that is A is injective. Then the previous equality implies $T^*T = I$ so that T is an isometry on $\mathcal{H}$. On the other hand, from Theorem 2 we have that $\mathcal{N}_\infty^*$ reduces T^* to an A-isometry, hence $\mathcal{N}_\infty^* = \{0\}$ (having in view the hypothesis). This implies $\mathcal{H}_u = \{0\}$ and by Corollary 4 this means that T is completely non unitary, that is a shift on $\mathcal{H}$. QED

As a consequence one obtains a version for A-isometries of the von Neumann–Wold decomposition [2, 10] for isometries.

6 Corollary. *Let T be an A-isometry such that $AT = TA$. Then there exists a unique orthogonal decomposition for $\mathcal{H}$ of the form*

$$\mathcal{H} = \mathcal{H}_u \oplus \mathcal{H}_s \tag{8}$$

where the two subspaces reduce A and T, such that $\mathcal{N}(A) \subset \mathcal{H}_u$ and T is A-unitary on $\mathcal{H}_u$, while T is a shift on $\mathcal{H}_s$. Moreover, $\mathcal{H}_u$ is the normal part for $A^{1/2}T$ and we have

$$\mathcal{H}_u = \mathcal{N}(A) \oplus \mathcal{N}(I - S_{\widehat{T}^*}), \quad \mathcal{H}_s = \mathcal{N}(I - S_{\widehat{T}}) \ominus \mathcal{N}(I - S_{\widehat{T}^*}). \tag{9}$$

Proof. Since T is an A-isometry one has $\mathcal{N}_\infty(A,T) = \mathcal{H}$, and so $\mathcal{H}_u = \mathcal{N}_\infty(A,T^*)$ is the subspace from (6) in this case. Also, $\mathcal{H}_u$ is the maximum subspace which reduces A and T on which T^* is an A-isometry (by Theorem 2). Hence T^* is a pure A-contraction on $\mathcal{H}_s = \mathcal{H} \ominus \mathcal{H}_u$, therefore T is a shift on $\mathcal{H}_s$ (by Corollary 5). This gives the decomposition (8) with the required properties relative to T.

Now since T and T^* are A-isometries on $\mathcal{H}_u$, $\mathcal{H}_u$ will reduces $A^{1/2}T$ to a normal operator. Then applying Proposition 2.2 [9] for the regular A-contraction T^*, we obtain that $\mathcal{H}_u$ is the maximum subspace which reduces $A^{1/2}T^* = T^*A^{1/2}$ on which we have $TAT^* = A = T^*AT$. This means that $\mathcal{H}_u$ is the normal part for $T^*A^{1/2}$, or equivalently for $A^{1/2}T$.

Clearly, $\mathcal{H}_u = \mathcal{N}_\infty^*$ has the form from (9) obtained in the proof of Theorem 2. On the other hand, by the same theorem T is unitary on $\mathcal{N}(I - S_{\widehat{T}^*})$, hence T is an isometry on $\overline{\mathcal{R}(A)} = \mathcal{N}(I - S_{\widehat{T}^*}) \oplus \mathcal{H}_s$. This means that $\overline{\mathcal{R}(A)} = \mathcal{N}(I - S_{\widehat{T}})$, and thus we find the form of $\mathcal{H}_s$ from (9). The proof is finished. QED

7 Remark. Let T be as in Corollary 6. Since $A = T^*TA$ one has $\overline{\mathcal{R}(A)} \subset \mathcal{N}(I - T^*T)$, hence

$$\mathcal{H} = \mathcal{N}(A) \vee \mathcal{N}(I - T^*T)$$

but the two subspaces are not orthogonal, in general. In fact, it is easy to see that $\overline{\mathcal{R}(A)} = \mathcal{N}(I - T^*T)$ if and only if $\mathcal{N}(I - T^*T)$ is invariant for T and T is completely non isometric on $\mathcal{N}(A)$.

We also remark that if $A = A^2$ then $A^{1/2}T = AT$ is an A-isometry and AT commutes with A. In this case is not difficult to see that the corresponding decompositions (8) for the A-isometries T and AT coincide, hence AT is A-unitary on $\mathcal{H}_u$ and a shift on $\mathcal{H}_s$.

As an application of Theorem 2 we obtain the following

8 Corollary. *Let T be a quasinormal contraction on $\mathcal{H}$. Then the maximum subspace which reduces T to a T^*T-unitary operator is $\mathcal{N}(T) \oplus \mathcal{N}(I - S_{T^*})$, and $\mathcal{N}(I - S_{T^*})$ is the maximum subspace which reduces T to a unitary operator. Hence T is T^*T-unitary on $\mathcal{H}$ if and only if T is a normal partial isometry.*

Proof. The hypothesis on T gives that T is a T^*T-contraction and T commutes with T^*T. Since $TT^* \leq T^*T$ and $(T^*T)^n = T^{*n}T^n$ for $n \geq 1$, it follows that $T^nT^{*n} \leq T^{*n}T^n$ and also $I - T^{*n}T^n \leq I - T^nT^{*n}$ for $n \geq 1$. This implies that $I - S_T \leq I - S_{T^*}$, whence one obtains

$$\mathcal{N}(I - S_{T^*}) \subset \mathcal{N}(I - S_T) \subset \overline{\mathcal{R}(T^*)}.$$

But $\overline{\mathcal{R}(T^*)}$ reduces T and $\mathcal{N}(I - S_{T^*}) = \mathcal{N}(I - S_{T_0^*})$, $\mathcal{N}(I - S_T) = \mathcal{N}(I - S_{T_0})$, where $T_0 = T|_{\overline{\mathcal{R}(T^*)}}$. Thus, from Theorem 2 we infer in this case that $\mathcal{N}_\infty^* = \mathcal{N}(T) \oplus \mathcal{N}(I - S_{T^*})$, and this subspace and $\mathcal{N}(I - S_{T^*})$ have the required properties. Clearly, T is a normal partial isometry on $\mathcal{N}_\infty^*$, and it is easy to see that $\mathcal{N}_\infty^*$ is also the maximum subspace with this property. This fact ensures the last assertion of the corollary. QED

In the sequel we denote as usually $|T| = (T^*T)^{1/2}$, that is the module of T.

9 Corollary. *Let T be a quasinormal contraction on $\mathcal{H}$ with the polar decomposition $T = W|T|$. Then the normal part in $\mathcal{H}$ for T is*

$$\mathcal{H}_n = \mathcal{N}(T) \oplus \mathcal{N}(I - S_{W^*}),$$

where $\mathcal{N}(I - S_{W^*})$ is the unitary part in $\mathcal{H}$ for W. Also, the pure part in $\mathcal{H}$ for T is

$$\mathcal{H}_p = \mathcal{N}(S_{W^*}) \ominus \mathcal{N}(T)$$

that is the shift part in $\overline{\mathcal{R}(T^*)}$ for W.

Proof. Since T is quasinormal, W is a quasinormal partial isometry with $\mathcal{N}(W) = \mathcal{N}(T)$ satisfying $WT^*T = T^*TW$, hence W is also a T^*T-isometry. Then by Corollary 8 the maximum reducing subspace for W and T^*T on which W is T^*T-unitary is $\mathcal{H}_n = \mathcal{N}(T) \oplus \mathcal{N}(I - S_{W^*})$, and by Corollary 6, $\mathcal{H}_n$ is also the normal part for $|T|W = T$. Since $S_{W^*} = S_{W^*}^2$ (W being quasinormal; see [2], [8]) one has

$$\mathcal{H} = \mathcal{N}(S_{W^*} - S_{W^*}^2) = \mathcal{N}(S_{W^*}) \oplus \mathcal{N}(I - S_{W^*}),$$

hence the pure part in $\mathcal{H}$ for T is the subspace $\mathcal{H}_p = \mathcal{H} \ominus \mathcal{H}_n = \mathcal{N}(S_{W^*}) \ominus \mathcal{N}(T)$. But $\mathcal{N}(I - S_{W^*})$ is the unitary part of W, and so it follows that $\mathcal{H}_p$ is the shift part in $\overline{\mathcal{R}(T^*)}$ for the isometry $W|_{\overline{\mathcal{R}(T^*)}}$. QED

3 Von Neumann-Wold type decomposition for $A^{1/2}T$

As we remarked, the decomposition (8) gives the normal and pure subspaces for the operator $A^{1/2}T$ in the special case when the A-isometry T satisfies the condition $AT = TA$, these subspaces being expressed in the terms of the operators $S_{\widehat{T}}$ and $S_{\widehat{T}^*}$ where $\widehat{T} = T|_{\overline{\mathcal{R}(A)}}$. More general, if instead of condition $AT = TA$ we ask $A^{1/2}T$ to be quasinormal, then Corollary 9 gives the above quoted subspaces in the terms of the partial isometry from the polar decomposition of $A^{1/2}T$. But in this last case, these subspaces can be intrinsic described in the terms of A and T, and thus we obtain a von Neumann-Wold type decomposition for $A^{1/2}T$, as below. Recall that a subspace $\mathcal{G} \subset \mathcal{H}$ is wandering for a sequence $\{S_n : n \geq 1\} \subset \mathcal{B}(\mathcal{H})$ if $S_n\mathcal{G} \perp S_m\mathcal{G}$, $n \neq m$.

10 Theorem. *Let T be a regular A-isometry on $\mathcal{H}$. Then $\mathcal{L} = \mathcal{N}(T^*A^{1/2})$ is a wandering subspace for the operators $A^{1/2}T^n$ $(n \geq 0)$, and the maximum subspace which reduces $A^{1/2}T$ to a normal operator is*

$$\mathcal{H}_n = \bigcap_{n=0}^{\infty} (T^{*n}A^{1/2})^{-1}\mathcal{L}^{\perp}. \tag{10}$$

Moreover, $\mathcal{H}_n$ is invariant for A and T, and $A^{1/2}T$ is a pure injective quasinormal operator on the subspace

$$\mathcal{H} \ominus \mathcal{H}_n = \bigoplus_{n=0}^{\infty} \overline{A^{1/2}T^n\mathcal{L}} = \bigvee_{n=0}^{\infty} A^{1/2}T^n(\mathcal{L} \ominus \mathcal{N}(A)). \tag{11}$$

Proof. Let A and T be as above. It is easy to see that, because $A = T^*AT$, the regularity condition $AT = A^{1/2}TA^{1/2}$ is equivalent to the fact that $A^{1/2}T$ is quasinormal. Also we have $|A^{1/2}T| = A^{1/2}$, $\mathcal{N}(A) = \mathcal{N}(A^{1/2}T)$ and $\overline{\mathcal{R}(A)} = \overline{\mathcal{R}(T^*A^{1/2})}$.

Let $\mathcal{L} := \mathcal{N}(T^*A^{1/2})$. Clearly, $\mathcal{N}(A) \subset \mathcal{L}$ and $\mathcal{L}$ reduces A because $T^*A^{1/2}A = AT^*A^{1/2}$. In fact, one has

$$A^{1/2}\mathcal{L} = \mathcal{N}(T^*) \cap \mathcal{R}(A^{1/2}) = \mathcal{L} \cap \mathcal{R}(A^{1/2}).$$

Let us prove that $\mathcal{L}$ is a wandering subspace for the operators $A^{1/2}T^n$, $n \geq 0$, that is $A^{1/2}T^n\mathcal{L} \perp A^{1/2}T^m\mathcal{L}$ for $n \neq m$. Indeed, for $l, l' \in \mathcal{L}$ we have if $n \geq 1$ and $m = 0$,

$$\langle A^{1/2}T^n l, A^{1/2}l'\rangle = \langle l, T^{*n}Al'\rangle = \langle l, T^{*(n-1)}A^{1/2}T^*A^{1/2}l'\rangle = 0,$$

and if $n, m \geq 1$, $m < n$, then

$$\begin{aligned}\langle A^{1/2}T^n l, A^{1/2}T^m l'\rangle &= \langle l, T^{*n}AT^m l'\rangle = \langle l, T^{*(n-m)}T^{*m}AT^m l'\rangle \\ &= \langle l, T^{*(n-m)}Al'\rangle = \langle l, T^{*(n-m-1)}A^{1/2}T^*A^{1/2}l'\rangle \\ &= 0.\end{aligned}$$

Here we used the fact that T^m is also a regular A-isometry for $m \geq 1$.

Now we define the subspace

$$\mathcal{H}_p := \bigoplus_{n=0}^{\infty} \overline{A^{1/2}T^n\mathcal{L}} = \bigvee_{n=0}^{\infty} A^{1/2}T^n\mathcal{L} = \bigvee_{n=0}^{\infty} A^{1/2}T^n(\mathcal{L} \ominus \mathcal{N}(A)),$$

which is invariant for $A^{1/2}T^m$ $(m \geq 0)$ because using the regularity condition one obtains for $n, m \geq 0$,

$$A^{1/2}T^m A^{1/2}T^n\mathcal{L} = AT^{m+n}\mathcal{L} = A^{1/2}T^{m+n}A^{1/2}\mathcal{L} \subset A^{1/2}T^{m+n}\mathcal{L} \subset \mathcal{H}_p.$$

In particular, $\mathcal{H}_p$ reduces A. Also, $\mathcal{H}_p$ is invariant for $T^{*m}A^{1/2}$, $m \geq 1$. For this, firstly we remark that $T^*A\mathcal{L} = \{0\}$ since $A^{1/2}\mathcal{L} \subset \mathcal{L}$. So, if $m \geq n \geq 0$ then

$$T^{*m}A^{1/2}A^{1/2}T^n\mathcal{L} = T^{*m-n}A\mathcal{L} = \{0\},$$

and in the case $m < n$ we get

$$T^{*m}A^{1/2}A^{1/2}T^n\mathcal{L} = T^{*m}AT^mT^{n-m}\mathcal{L} = T^{*m}A^{1/2}T^mA^{1/2}T^{n-m}\mathcal{L} =$$

$$AT^{n-m}\mathcal{L} \subset \mathcal{H}_p,$$

because $T^{*m}A^{1/2}T^m = A^{1/2}$, T being also a regular $A^{1/2}$-contraction (by Theorem 2.6 [8]). Thus it follows that $\mathcal{H}_p$ reduce $A^{1/2}T^n$ for any n. Now we remark that $\mathcal{H}_p$ is invariant for T^* because

$$T^*A^{1/2}T^n\mathcal{L} = T^*A^{1/2}TT^{n-1}\mathcal{L} = A^{1/2}T^{n-1}\mathcal{L} \subset \mathcal{H}_p$$

if $n \geq 1$, and $T^*A^{1/2}\mathcal{L} = \{0\}$ (the case $n = 0$).

Next, we prove that

$$\mathcal{H}_q := \mathcal{H} \ominus \mathcal{H}_p = \bigcap_{n=0}^{\infty} (A^{1/2}T^n\mathcal{L})^{\perp}$$

is the maximum subspace which reduces $A^{1/2}T$ to a normal operator. First, it is easy to see that

$$\mathcal{H}_q = \{h \in \mathcal{H} : \ T^{*n}A^{1/2}h \in \overline{\mathcal{R}(A^{1/2}T)}, \ n \geq 0\} = \bigcap_{n=0}^{\infty} (T^{*n}A^{1/2})^{-1}\mathcal{L}^{\perp}.$$

Let D be the self-commutator of $A^{1/2}T$, that is

$$D = T^*AT - A^{1/2}TT^*A^{1/2} = A^{1/2}(I - TT^*)A^{1/2}.$$

Clearly $D\mathcal{L} \subset A\mathcal{L} \subset \mathcal{L}$, hence $\mathcal{L}$ is a reducing subspace for D. It is also known from Theorem 1.4 [3] that the maximum subspace which reduces $A^{1/2}T$ to a normal operator is

$$\mathcal{H}_n = \{h \in \mathcal{H} : \ DT^{*n}A^{1/2}h = 0, \ n \geq 0\}.$$

We will show that $\mathcal{H}_q = \mathcal{H}_n$.

Let $h \in \mathcal{H}_q$, $h = l + k$ where $l \in \mathcal{L}$ and $k \in \overline{\mathcal{R}(A^{1/2}T)}$. Let $\{h_n\} \subset \mathcal{H}$ such that $k = \lim_n A^{1/2}Th_n$. Then $A^{1/2}(h-k) \in \overline{\mathcal{R}(A^{1/2}T)}$ and $A^{1/2}l \in \mathcal{L}$, therefore $A^{1/2}l = 0$ and $A^{1/2}h = A^{1/2}k$. Thus we obtain

$$\begin{aligned} A^{1/2}TT^*A^{1/2}h = A^{1/2}TT^*A^{1/2}k = \lim_n A^{1/2}TT^*A^{1/2}A^{1/2}Th_n \\ = \lim_n A^{1/2}TAh_n = \lim_n AA^{1/2}Th_n = Ak = Ah, \end{aligned}$$

which means $Dh = 0$. Hence $D\mathcal{H}_q = \{0\}$, that is the operator $A^{1/2}T$ is normal on $\mathcal{H}_q$, which gives the inclusion $\mathcal{H}_q \subset \mathcal{H}_n$.

Now let $h \in \mathcal{H}_n$. Since $(A^{1/2}T)^*h \in \mathcal{H}_n$ one has $DT^*A^{1/2}h = 0$, hence using the regularity condition on A and T we obtain

$$\begin{aligned} AT^*A^{1/2}h =& A^{1/2}TT^*A^{1/2}T^*A^{1/2}h = A^{1/2}TA^{1/4}T^*A^{1/2}T^*A^{1/4}h \\ =& A^{1/2}TA^{1/2}T^{*2}A^{1/2}h = ATT^{*2}A^{1/2}h. \end{aligned}$$

This implies by the injectivity of $A^{1/2}$ on his range that

$$T^*Ah = A^{1/2}T^*A^{1/2}h = A^{1/2}TT^{*2}A^{1/2}h \in \mathcal{R}(A^{1/2}T).$$

Now using an approximation polynomial for the square root $A^{1/2}$ (as in [6], pg. 261), one infers that $T^*A^{1/2}h \in \overline{\mathcal{R}(A^{1/2}T)}$. This yields to $T^{*2}Ah = (T^*A^{1/2})^2h \in \overline{\mathcal{R}(A^{1/2}T)}$, and as above $T^{*2}A^{1/2}h \in \overline{\mathcal{R}(A^{1/2}T)}$. Then by induction one obtains $T^{*n}A^{1/2}h \in \overline{\mathcal{R}(A^{1/2}T)}$ for any $n \geq 1$, which gives $h \in \mathcal{H}_q$. Therefore we have $\mathcal{H}_n \subset \mathcal{H}_q$ and finally $\mathcal{H}_n = \mathcal{H}_q$.

Consequently, $\mathcal{H}_n$ has the form (11), and $\mathcal{N}(A) \subset \mathcal{H}_n$ because $\mathcal{N}(A) \subset \mathcal{L}$, which implies that $\mathcal{H}_p = \mathcal{H} \ominus \mathcal{H}_n$ reduces $A^{1/2}T$ to a pure injective quasinormal operator. The proof is finished. QED

Theorem 10 can be completed by the following

11 Theorem. *Let T be a regular A-isometry on $\mathcal{H}$ and V be the unique partial isometry on $\mathcal{H}$ satisfying $VA^{1/2} = A^{1/2}T$ and $\mathcal{N}(V) = \mathcal{N}(A)$. Then the subspaces from (10) and (11) have the form*

$$\mathcal{H}_n = \bigcap_{n=0}^{\infty} V^n\mathcal{H} \oplus \mathcal{N}(A) = \bigcap_{n=0}^{\infty} V_0^n\overline{\mathcal{R}(A)} \oplus \mathcal{N}(A), \tag{12}$$

and respectively

$$\mathcal{H} \ominus \mathcal{H}_n = \bigoplus_{n=0}^{\infty} V^n(\mathcal{N}(V^*) \ominus \mathcal{N}(A)) = \bigoplus_{n=0}^{\infty} V_0^n\mathcal{N}(V_0^*), \tag{13}$$

where $V_0 = V|_{\overline{\mathcal{R}(A)}}$ is an isometry on $\overline{\mathcal{R}(A)}$. Furthermore, we have

$$\mathcal{L} = \mathcal{N}(V^*) = \mathcal{N}(V_0^*) \oplus \mathcal{N}(A) = (A^{1/2})^{-1}(\mathcal{N}(V_0^*)), \tag{14}$$

and

$$\overline{A^{1/2}\mathcal{L}} = \mathcal{L} \cap \overline{\mathcal{R}(A)} = \mathcal{N}(V_0^*). \tag{15}$$

In particular, one has $\mathcal{L} = \mathcal{N}(V_0^*)$ if and only if A is injective.

Proof. Let A, T, V as above. Then $A^{1/2}T$ is quasinormal and $A^{1/2}T = VA^{1/2}$ is just the polar decomposition of $A^{1/2}T$ because $|A^{1/2}T| = A^{1/2}$ and $\mathcal{N}(V) = \mathcal{N}(A^{1/2}T) = \mathcal{N}(A)$. Also, $\mathcal{N}(V^*) = \mathcal{N}(T^*A^{1/2}) = \mathcal{L}$ and V commutes with $A^{1/2}$, hence $\mathcal{N}(A)$ reduces V. Thus for $h \in \mathcal{H}$ we have

$$VA^{1/2}h = A^{1/2}Th = V_0A^{1/2}h,$$

therefore $V|_{\overline{\mathcal{R}(A)}} = V_0$ and V_0 is an isometry on $\overline{\mathcal{R}(A)}$ because V is a partial isometry with $\mathcal{N}(V) = \mathcal{N}(A)$. In addition one has

$$\mathcal{N}(V_0^*) = \mathcal{N}(V^*) \cap \overline{\mathcal{R}(A)} = \mathcal{L} \cap \overline{\mathcal{R}(A)},$$

or equivalently $\mathcal{L} = \mathcal{N}(V_0^*) \oplus \mathcal{N}(A)$. Also, for $h \in \mathcal{H}$ we have (T being a regular $A^{1/2}$-contraction)

$$T^*A^{1/2}h = A^{1/4}V_0^*A^{1/4}h = V_0^*A^{1/2}h,$$

because V_0 commutes with $A^{1/2}|_{\overline{\mathcal{R}(A)}}$. Hence $h \in \mathcal{L}$ if and only if $A^{1/2}h \in \mathcal{N}(V_0^*)$, which gives that $\mathcal{L} = (A^{1/2})^{-1}\mathcal{N}(V_0^*)$. Thus, all relations (14) and the second relation from (15) are proved.

Next, obviously one has $\overline{A^{1/2}\mathcal{L}} \subset \mathcal{L} \cap \overline{\mathcal{R}(A)}$. Conversely, let $h \in \mathcal{L} \cap \overline{\mathcal{R}(A)}$ such that $h \perp A^{1/2}\mathcal{L}$. Then $Ah \in A^{1/2}\mathcal{L}$, so $h \perp Ah$ which gives $A^{1/2}h = 0$. Hence $h \in \overline{\mathcal{R}(A)} \cap \mathcal{N}(A)$, that is $h = 0$. Thus we infer that $\overline{A^{1/2}\mathcal{L}} = \mathcal{L} \cap \overline{\mathcal{R}(A)}$, this being the first relation from (15).

Now, from (11) we obtain

$$\mathcal{H} \ominus \mathcal{H}_n = \bigvee_{n=0}^{\infty} A^{1/2}T^n\mathcal{L} = \bigvee_{n=0}^{\infty} V^n\overline{A^{1/2}\mathcal{L}}$$
$$= \bigoplus_{n=0}^{\infty} V^n(\mathcal{N}(V^*) \ominus \mathcal{N}(A)) = \bigoplus_{n=0}^{\infty} V_0^n\mathcal{N}(V_0^*),$$

which give the relations (12). This shows that $\mathcal{H} \ominus \mathcal{H}_n$ is the shift part in $\overline{\mathcal{R}(A)}$ for the isometry V_0, hence we have

$$\overline{\mathcal{R}(A)} \ominus (\mathcal{H} \ominus \mathcal{H}_n) = \bigcap_{n=0}^{\infty} V_0^n\overline{\mathcal{R}(A)} = \bigcap_{n=0}^{\infty} V^n\mathcal{H},$$

and finally we obtain the relations (12). It is clear from (14) that $\mathcal{L} = \mathcal{N}(V_0^*)$ if and only if A is injective. This ends the proof. QED

According to [9], an operator $T \in \mathcal{B}(\mathcal{H})$ is called an *A-weighted isometry* if $T^*T = A$. Then we can also describe the above subspace $\mathcal{H}_n$ using this concept, as follows.

12 Proposition. *Let T be a regular A-isometry on $\mathcal{H}$ and $\mathcal{H}_n$ be as above. Then $\mathcal{H}_n$ is the maximum subspace which reduces A and $A^{1/2}T$ on which $(A^{1/2}T)^*$ is an A-weighted isometry. Moreover, one has $\mathcal{H}_n = \mathcal{R}_u \oplus \mathcal{N}(A)$, where $\mathcal{R}_u$ is the unitary part in $\overline{\mathcal{R}(A)}$ for V_0, V_0 being as in Theorem 11. In addition, $(T|_{\mathcal{H}_n})^*$ is an A-isometry on $\mathcal{R}_u$.*

Proof. From (12) we infer $\mathcal{H}_n = \mathcal{R}_u \oplus \mathcal{N}(A)$ and as $A^{1/2}T$ is normal on $\mathcal{H}_n$ we obtain $A^{1/2}TT^*A^{1/2} = A$ on $\mathcal{H}_n$, and this means that $(A^{1/2}T)^*$ is an A-weighted isometry on $\mathcal{H}_n$. Conversely, both the previous relation and the hypothesis $T^*AT = A$ imply that $A^{1/2}T$ is normal, hence any reducing subspace for A and $A^{1/2}T$ on which $T^*A^{1/2}$ is an A-weighted isometry is contained in $\mathcal{H}_n$. In conclusion, $\mathcal{H}_n$ is the maximum subspace with the above quoted property.

Now since $\mathcal{H}_n$ is invariant for T and A, $\mathcal{R}_u$ will be invariant for A and $(T|_{\mathcal{H}_n})^*$, and we prove that $(T|_{\mathcal{H}_n})^*$ is an A-isometry on $\mathcal{R}_u$. Let $h \in \mathcal{R}_u$. As $\mathcal{R}_u \subset \overline{\mathcal{R}(A)}$ we have $h = \lim_n A^{1/2}h_n$ for some sequence $\{h_n\} \subset \mathcal{H}$. Then if P_n is the orthogonal projection onto $\mathcal{H}_n$, we have

$$A^{1/2}(T|_{\mathcal{H}_n})^*h = A^{1/2}P_nT^*h = P_nA^{1/2}T^*h = P_n(\lim_n A^{1/2}T^*A^{1/2}h_n)$$
$$= P_n \lim_n T^*Ah_n = P_nT^*A^{1/2}h = T^*A^{1/2}h,$$

because $\mathcal{H}_n$ reduces A and $A^{1/2}T$. Next we obtain

$$||A^{1/2}(T|_{\mathcal{H}_n})^*h||^2 = ||T^*A^{1/2}h||^2 = \langle A^{1/2}TT^*A^{1/2}h, h\rangle = \langle Ah, h\rangle = ||A^{1/2}h||^2,$$

because $A^{1/2}T$ is normal on $\mathcal{R}_u$. This relation just shows that the operator $(T|_{\mathcal{H}_n})^*|_{\mathcal{R}_u}$ is an $A|_{\mathcal{R}_u}$-isometry on $\mathcal{R}_u$. This ends the proof. QED

Remark from the above proof that in fact we have

$$A^{1/2}(T|_{\mathcal{H}_n})^*h = (T|_{\mathcal{H}_n})^*A^{1/2}h \quad (h \in \mathcal{R}_u),$$

that is $(T|_{\mathcal{H}_n})^*|_{\mathcal{R}_u}$ commutes with $A^{1/2}|_{\mathcal{R}_u}$, but $(T|_{\mathcal{H}_n})^*$ and $A^{1/2}|_{\mathcal{H}_n}$ are not commutative on all $\mathcal{H}_n$, in general. Concerning the commutative case we have the following proposition, where by (i) we recover the fact that the above subspace $\mathcal{H}_n$ coincides with the subspace $\mathcal{H}_u$ from (8), and by (ii) and (iii) we characterize the subspace $\mathcal{H}_n \ominus \mathcal{N}(A)$ and $\mathcal{H} \ominus \mathcal{H}_n$ respectively, as reducing subspaces for A and T, in $\mathcal{H}$.

13 Proposition. *Let T be an A-isometry on $\mathcal{H}$ such that $AT = TA$. Then the following assertions hold:*

(i) $\mathcal{H}_n$ is the maximum reducing subspace for A and T, on which T^* is an A-isometry.

(ii) $\mathcal{R}_u = \mathcal{H}_n \ominus \mathcal{N}(A)$ is the maximum subspace which reduces T to a unitary operator such that $\mathcal{R}_u = \overline{A\mathcal{R}_u}$.

(iii) $\mathcal{H}_p = \mathcal{H} \ominus \mathcal{H}_n$ is the maximum subspace which reduces T to a shift such that $\mathcal{H}_p = \overline{A\mathcal{H}_p}$.

In particular, if A is injective then T is an isometry and $\mathcal{H} = \mathcal{H}_n \oplus \mathcal{H}_p$ is the von Neumann-Wold decomposition for T.

Proof. Let V be the isometry from Theorem 11. Under the assumption $AT = TA$ we have $VA^{1/2} = A^{1/2}T = TA^{1/2}$, and we infer that $T|_{\overline{\mathcal{R}(A)}} = V|_{\overline{\mathcal{R}(A)}} = V_0$ so that T is an isometry on $\overline{\mathcal{R}(A)}$. Hence, from Theorem 11 we have that $\mathcal{R}_u$ reduces A and T such that T is unitary on $\mathcal{R}_u$, which implies that T^* is an A-isometry on $\mathcal{H}_n$. So, $\mathcal{H}_n \subset \mathcal{H}_u$ (the subspace from (8)) and trivially $\mathcal{H}_u \subset \mathcal{H}_n$ because $\mathcal{H}_u \ominus \mathcal{N}(A)$ reduces T to a normal operator. This gives the assertion (i).

Now one has $\overline{A\mathcal{R}_u} \subset \mathcal{R}_u$, and if $h \in \mathcal{R}_u \ominus \overline{A\mathcal{R}_u}$ then $Ah = 0$ that is $h \in \mathcal{N}(A)$, and since $\mathcal{R}_u \subset \overline{\mathcal{R}(A)}$ we have $h = 0$. Hence $\mathcal{R}_u = \overline{A\mathcal{R}_u}$, and T is unitary on $\mathcal{R}_u$. Let $\mathcal{M} \subset \mathcal{H}$ be another subspace having the above properties of $\mathcal{R}_u$. Since

$T|_{\mathcal{M}}$ is unitary and $T = V_0$ is completely non unitary on $\mathcal{H} \ominus \mathcal{H}_n$, it follows that $\mathcal{M} \subset \mathcal{H}_n$. Thus we obtain

$$\mathcal{M} = \overline{A\mathcal{M}} \subset \overline{A\mathcal{H}_n} = \overline{A\mathcal{R}_u} = \mathcal{R}_u$$

and consequently $\mathcal{R}_u$ has the required properties in (ii).

Next, from Theorem 11 we have that $\mathcal{H}_p$ reduces T to a shift because $T = V_0$ on $\mathcal{H}_p$. As $\mathcal{H}_p$ also reduces A and $\mathcal{H}_p \subset \overline{\mathcal{R}(A)}$, one obtains (as for $\mathcal{R}_u$) that $\mathcal{H}_p = \overline{A\mathcal{H}_p}$. If $\mathcal{M} \subset \mathcal{H}$ is another subspace which reduces T to a shift such that $\mathcal{M} = \overline{A\mathcal{M}}$, then $\mathcal{M} \subset \overline{\mathcal{R}(A)}$ and from the assertion (ii) it follows that $\mathcal{M} \subset \overline{\mathcal{R}(A)} \ominus \mathcal{R}_u = \mathcal{H}_p$. So $\mathcal{H}_p$ has the required properties in (iii).

Clearly, if $\mathcal{N}(A) = \{0\}$ one has $T = V$, therefore T is an isometry on $\mathcal{H}$, while $\mathcal{H}_n = \mathcal{R}_u$ and $\mathcal{H}_p$ are the unitary and shift parts in $\mathcal{H}$ for T, respectively. The proof is finished. QED

As an application to quasi-isometries we have the following

14 Corollary. *Let T be a quasi-isometry on $\mathcal{H}$ such that $|T|T$ is a quasi-normal operator. Then $|T|T$ is normal if and only if*

$$\mathcal{N}(T^{*2}T) = \mathcal{N}(T).$$

Proof. From the hypothesis we infer that T is a T^*T-isometry which is also regular because $S = |T|T$ is quasinormal. Let $T = W|T|$ be the polar decomposition of T. Then Theorem 2.1 [4] ensures that $|T|W$ is a partial isometry with $\mathcal{N}(|T|W) = \mathcal{N}(|T|) = \mathcal{N}(|S|)$. Hence $S = |T|W|T|$ is the polar decomposition of S. Now the corresponding subspace from (13) which reduce S to a pure operator is

$$\mathcal{H}_p = \bigoplus_{n=0}^{\infty} S^n(\mathcal{N}(W^*|T|) \ominus \mathcal{N}(T)).$$

But we have

$$\mathcal{N}(W^*|T|) = \mathcal{N}(S^*) = \mathcal{N}(T^*|T|) = \mathcal{N}(T^*|T|^2) = \mathcal{N}(T^{*2}T)$$

where we used the fact that $T^*|T|^2 = |T|T^*|T|$ (T being a regular T^*T-contraction) and that $\mathcal{N}(T) = \mathcal{N}(|T|)$, $\mathcal{N}(T^*) = \mathcal{N}(TT^*)$. Thus we conclude that S is normal if and only if $\mathcal{H}_p = \{0\}$, or equivalent $\mathcal{N}(T^{*2}T) = \mathcal{N}(T)$. QED

15 Remark. In general one has $T^{*2}T \neq T^*$ even if T is a quasi-isometry and $|T|T$ is quasinormal, for instance if T is the operator on $\mathbb{C}^2$ given by

$$T = \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix}.$$

But any quasi-isometry T with $||T|| = 1$ satisfies $T^{*2}T = T^*$ (see [4], [9]). In this last case, the assumption that $|T|T$ is quasinormal leads to the fact that $|T|T = T$ and that $T^*T = (T^*T)^2$, that is T is a quasinormal partial isometry. Indeed, supposing that $|T|T$ is quasinormal, one has $T^*T^2 = |T|T|T|$ because $||T|T| = |T|$. Then with the above remark one obtains $T = |T|T|T|$, whence one infers

$$T^*T = |T|T^*|T|^2T|T| = |T|T^{*2}T^2|T| = |T|T^*T|T| = (T^*T)^2.$$

So T^*T is an orthogonal projection, or equivalently T is a partial isometry, and hence $T^*T = |T|$. Finally, it follows

$$|T|T = T^*T^2 = T,$$

therefore T is a quasinormal partial isometry.

Clearly, any quasinormal partial isometry $T \neq 0$ is a quasi-isometry with $||T|| = 1$. Having in view this fact, we obtain from Corollary 14 the following

16 Corollary. *Let T be a quasinormal partial isometry. Then T is normal if and only if $\mathcal{N}(T) = \mathcal{N}(T^*)$.*

Proof. Since T is a quasi-isometry and $||T|| = 1$ (supposing $T \neq 0$), we have $T^* = T^{*2}T$ by Remark 15. Thus, if $\mathcal{N}(T) = \mathcal{N}(T^*)$ then $|T|T$ is normal by Corollary 14, and from above remark we find $T = |T|T$, hence T is normal. The converse assertion is trivial. QED

This corollary can be also obtained from Theorem 2.6 [4].

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