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Invariance of time varying order intervals for linear systems and delay equations

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Abstract. In this paper, we give a characterization of the invariance of time varying order intervals on a Banach lattice with respect to linear systems and linear delay equations. By using the notion of modulus semigroup, the criteria is expressed in terms of a differential inequality.

Keywords: invariant sets, Banach lattice, modulus semigroup, delay equations

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1 Introduction

In this paper, we discuss a comparison between the solution of a linear system and two given trajectories. Namely, consider a linear system described by the

$$\begin{cases} \frac{d}{dt}x(t) & = Ax(t), \quad t \geq t_0, \\ x(t_0) & = x_0, \end{cases} \tag{1}$$

where $x(t)$ is in a Banach lattice E, A is the generator of a strongly continuous semigroup and u, v two time varying functions with values in E^+. We propose to find necessary and sufficient conditions so that the solution of (1) satisfies

$$-v(t_0) \leq x_0 \leq u(t_0) \Longrightarrow -v(t) \leq x(t) \leq u(t) \quad \text{for all } t \geq t_0. \tag{2}$$

In this case we say that the family of order intervals $[-v(t), u(t)]$, $t \geq 0$, is invariant with respect to the linear system (1). By taking the trajectories u and

v vanishing at infinity, the condition (2) ensures the asymptotic stability (in the sense of order convergence) of the solution to the linear system (1).

This extend, to infinite dimensional linear systems, the results obtained by Voicu [16] and Benzaouia-Hmamed [10], and the results obtained in [5, 6, 8] in the case of constant trajectories u, v.

As an application, we use the characterization obtained for the linear system (1) to study the same problem for linear delay equations of the form

$$\begin{cases} \frac{d}{dt}x(t) & = Ax(t) + Lx_t, \quad t \geq t_0, \\ x(t_0) & = x^0, \ x_{t_0}(\theta) = \varphi(\theta), \ \text{a.e. } \theta \in (-h, 0)\,, \end{cases} \tag{3}$$

where A is a linear operator on a Banach lattice X, $L : C([-h, 0], X) \longrightarrow X$ is linear and bounded and the function $x_t(\theta) = x(t+\theta)$, a.e. $\theta \in (-h, 0)$. Namely, we characterize the invariance of the family of order intervals $[-v(t), u(t)]$, $t \geq -h$, with respect to the delay equation (3), in the sense that for every $x^0 \in [-v(t_0), u(t_0)]$ and $\varphi(\theta) \in [-v(t_0+\theta), u(t_0+\theta)]$, $\theta \in (-h, 0)$, $x(t) \in [-v(t), u(t)]$ for all $t \geq t_0$. The case of matrices and one delay was studied by Hmamed in [11] and the case of constant trajectories u, v was considered in [5, 6, 8].

2 Invariant order intervals with respect to linear systems in Banach lattice spaces

In a Banach lattice E we consider the family of order intervals

$$I_t := \{z \in E : -v(t) \leq z \leq u(t)\}\,, \quad t \geq 0, \tag{4}$$

where $u, v : [0, +\infty) \to E^+$, and a linear system

$$\begin{cases} \frac{d}{dt}x(t) & = Ax(t), \quad t \geq t_0, \\ x(t_0) & = x_0, \end{cases} \tag{5}$$

where A generates a C_0-semigroup $(T(t))_{t\geq 0}$ in E. The mild solution of this system is given by

$$x(t) = T(t - t_0)x_0, \quad t \geq t_0.$$

1 Definition. The family $(I_t)_{t\geq 0}$ is said to be invariant with respect to the system (5) if for all $t_0 \geq 0$ and $x_0 \in I_{t_0}$, the solution of system (5) satisfies $x(t) \in I_t$ for all $t \geq t_0$.

2 Remark. $(I_t)_{t\geq 0}$ is invariant with respect to system (5) is equivalent to the following

$$x_0 \in I_{t_0} \Longrightarrow T(\tau)x_0 \in I_{t_0+\tau} \quad \text{for all } \tau \geq 0$$

for all $t_0 \geq 0$.

To prepare the necessary ingredients for our main result, we begin by the following fundamental proposition.

3 Proposition. *Let G be the generator of a positive C_0-semigroup $(S(t))_{t\geq 0}$ on a Banach lattice X and a function $\varphi \in C^1(\mathbb{R}^+, X) \cap C(\mathbb{R}^+, D(G))$, then the following assertions are equivalent.*

(a) $S(\tau)\varphi(t) \leq \varphi(t+\tau)$ *for all* $t, \tau \geq 0$.

(b) $G\varphi(t) \leq \frac{d}{dt}^+\varphi(t)$ *for all* $t \geq 0$.

Proof. $(b) \Rightarrow (a)$. Let us set

$$\psi(\tau) := S(\tau)\varphi(t) - \varphi(t+\tau).$$

We have

$$\begin{aligned} \frac{d}{d\tau}^+\psi(\tau) &= GS(\tau)\varphi(t) - \frac{d}{dt}^+\varphi(t+\tau) \\ &= G\psi(\tau) + G\varphi(t+\tau) - \frac{d}{dt}^+\varphi(t+\tau), \end{aligned}$$

and $\psi(0) = 0$. Then, we obtain

$$\psi(\tau) = \int_0^\tau S(\tau-\sigma)[G\varphi(t+\sigma) - \frac{d}{dt}^+\varphi(t+\sigma)]d\sigma,$$

which implies, by assumption, that $\psi(\tau) \leq 0$. The implication $(a) \Rightarrow (b)$ is obvious. $\boxed{QED}$

Next, we recall the notion of dominating and modulus semigroups of $(T(t))_{t\geq 0}$. We say that a positive C_0-semigroup $(\mathcal{S}(t))_{t\geq 0}$ dominates $(T(t))_{t\geq 0}$ if

$$|T(t)x| \leq \mathcal{S}(t)|x| \quad \text{for all } x \in E. \tag{6}$$

The smallest positive semigroup on E dominating $(T(t))_{t\geq 0}$ is called the modulus semigroup of $(T(t))_{t\geq 0}$, noted by $(T_\#(t))_{t\geq 0}$, see [13, Page 278]. We recall that for the case of matrices the modulus semigroup always exists. More precisely for $A := (a_{ij})_{1\leq i,j\leq n} \in \mathbb{R}^{n\times n}$ the generator of modulus semigroup is given by

$$A_\# = \begin{cases} a_{ii} & \text{if } i = j \\ |a_{ij}| & \text{if } i \neq j. \end{cases} \tag{7}$$

Unfortunately, in infinite dimensional spaces the modulus semigroup not need to be exist in general. However, it is shown in [3], that in a Banach lattice with order continuous norm (i.e. all positive increasing and upper bounded sequences are convergent) every dominated C_0-semigroup $(T(t))_{t\geq 0}$ possesses a modulus semigroup $(T_{\#}(t))_{t\geq 0}$. This latter semigroup is given in [15] by

$$T_{\#}(t) = s - \lim_{n\to\infty} |T(\frac{t}{n})|^n \qquad \text{for all } t \geq 0. \tag{8}$$

The modulus $|T|$ of an operator T is defined by

$$|T|\, x = \sup\{|Ty| : |y| \leq x\} \qquad \text{for all } \ x \geq 0. \tag{9}$$

From now on, E will be a Banach lattice with order continuous norm and that $(T(t))_{t\geq 0}$ has a modulus semigroup $(T_{\#}(t))_{t\geq 0}$.

First, it is easy to check that $(I_t)_{t\geq 0}$ is invariant with respect to the system (5) if and only if for all $t_0 \geq 0$

$$\binom{x^+}{x^-} \leq \binom{u(t_0)}{v(t_0)} \Rightarrow \binom{(T(\tau)x)^+}{(T(\tau)x)^-} \leq \binom{u(t_0+\tau)}{v(t_0+\tau)} \quad \text{for all} \quad \tau \geq 0. \tag{10}$$

On the other hand, we have

$$\begin{pmatrix} (T(t)x)^+ \\ (T(t)x)^- \end{pmatrix} \leq \mathcal{T}_{\Diamond}(t) \begin{pmatrix} x^+ \\ x^- \end{pmatrix} \quad \text{for all } x \in E \ \text{ and } \ t \geq 0, \tag{11}$$

where $(\mathcal{T}_{\Diamond}(t))_{t\geq 0}$ is the positive C_0-semigroup on $E \times E$ given by

$$\mathcal{T}_{\Diamond}(t) := \frac{1}{2} \begin{pmatrix} T_{\#}(t) + T(t) & T_{\#}(t) - T(t) \\ T_{\#}(t) - T(t) & T_{\#}(t) + T(t) \end{pmatrix}, \qquad t \geq 0. \tag{12}$$

It is difficult to determine its generator $\mathcal{A}_{\Diamond}$. However if A and $A_{\#}$ have the same domain, which is the case for example if A is bounded, we can obtain $\mathcal{A}_{\Diamond}$ explicitly on a core, i.e. a dense subspace of $D(\mathcal{A}_{\Diamond})$ endowed with the graph norm.

4 Proposition. *Assume that $D(A_{\#}) = D(A)$, then we have $D(A) \times D(A)$ is a core of $\mathcal{A}_{\Diamond}$ and*

$$\mathcal{A}_{\Diamond|D(A)\times D(A)} = \frac{1}{2} \begin{pmatrix} A_{\#} + A & A_{\#} - A \\ A_{\#} - A & A_{\#} + A \end{pmatrix}. \tag{13}$$

Proof. It is easy to verify that $D(A)\times D(A) \subset D(\mathcal{A}_{\Diamond})$ and that (13) holds. Since $D(A) \times D(A)$ is invariant under the semigroup $(\mathcal{T}_{\Diamond}(t))_{t\geq 0}$ and dense in $E \times E$, then it is a core of $\mathcal{A}_{\Diamond}$, see [9, Prop. 1.7, p. 53]. $\boxed{QED}$

Next, combining Formula (8) with [8, Proposition 2.8], we give a new expression of the semigroup $(\mathcal{T}_{\Diamond}(t))_{t\geq 0}$. Recall first that the positive and the negative part of a bounded operator T having a modulus $|T|$ are

$$T^+ = \frac{1}{2}(|T| + T), \qquad T^- = \frac{1}{2}(|T| - T). \tag{14}$$

5 Proposition. *For all $t \geq 0$,*

$$\mathcal{T}_{\Diamond}(t) = s - \lim_{n\to\infty} \widetilde{\mathcal{T}}(\frac{t}{n})^n \tag{15}$$

where

$$\widetilde{\mathcal{T}}(s) := \begin{pmatrix} T(s)^+ & T(s)^- \\ T(s)^- & T(s)^+ \end{pmatrix}, \qquad s \geq 0.$$

We also have

$$\widetilde{\mathcal{T}}(t) \left(\begin{smallmatrix} x \\ y \end{smallmatrix}\right) = \sup \left\{ \left(\begin{smallmatrix} (T(t)z)^+ \\ (T(t)z)^- \end{smallmatrix}\right) : \left(\begin{smallmatrix} z^+ \\ z^- \end{smallmatrix}\right) \leq \left(\begin{smallmatrix} x \\ y \end{smallmatrix}\right) \right\}, \quad t \geq 0,\ x, y \in E^+. \tag{16}$$

Using the above results, we give a characterization of the invariance of the family of intervals $(I_t)_{t\geq 0}$.

6 Lemma. *The following assertions are equivalent.*

(a) $(I_t)_{t\geq 0}$ *is invariant with respect to the system* (5)*.*

(b) For all $t_0, \tau \geq 0$*,*

$$\mathcal{T}_{\Diamond}(\tau) \left(\begin{smallmatrix} u(t_0) \\ v(t_0) \end{smallmatrix}\right) \leq \left(\begin{smallmatrix} u(t_0+\tau) \\ v(t_0+\tau) \end{smallmatrix}\right).$$

Proof. The implication $(b) \Rightarrow (a)$ follows from the inequality (11) and the positivity of C_0-semigroup $(\mathcal{T}_{\Diamond}(t))_{t\geq 0}$.
To show $(a) \Rightarrow (b)$, let $t_0, \tau \geq 0$. From (16), we have

$$\widetilde{\mathcal{T}}(\tau) \left(\begin{smallmatrix} u(t_0) \\ v(t_0) \end{smallmatrix}\right) = \sup \left\{ \left(\begin{smallmatrix} (T(\tau)z)^+ \\ (T(\tau)z)^- \end{smallmatrix}\right) : \left(\begin{smallmatrix} z^+ \\ z^- \end{smallmatrix}\right) \leq \left(\begin{smallmatrix} u(t_0) \\ v(t_0) \end{smallmatrix}\right) \right\}. \tag{17}$$

Since $(I_t)_{t\geq 0}$ is invariant with respect to system (5), we obtain $-v(t_0+\tau) \leq T(\tau)z \leq u(t_0+\tau)$ for all $-v(t_0) \leq z \leq u(t_0)$. Hence, (10) and (17) imply that

$$\widetilde{\mathcal{T}}(\tau) \left(\begin{smallmatrix} u(t_0) \\ v(t_0) \end{smallmatrix}\right) \leq \left(\begin{smallmatrix} u(t_0+\tau) \\ v(t_0+\tau) \end{smallmatrix}\right).$$

For all $n \in \mathbb{N}^*$, one then has

$$\widetilde{\mathcal{T}}(\frac{\tau}{n}) \left(\begin{smallmatrix} u(t_0) \\ v(t_0) \end{smallmatrix}\right) \leq \left(\begin{smallmatrix} u(t_0+\frac{\tau}{n}) \\ v(t_0+\frac{\tau}{n}) \end{smallmatrix}\right).$$

By reiterating this inequality, we obtain

$$\widetilde{\mathcal{T}}(\frac{\tau}{n})^n \left(\begin{smallmatrix} u(t_0) \\ v(t_0) \end{smallmatrix}\right) \leq \left(\begin{smallmatrix} u(t_0+\tau) \\ v(t_0+\tau) \end{smallmatrix}\right).$$

By passing to the limit, the result follows by (15). $\boxed{QED}$

Now we are ready to state our main result. The proof is straightforward using Propositions 3, 4 and Lemma 6.

7 Theorem. *Let* $\binom{u}{v} \in C^1(\mathbb{R}^+, E^+ \times E^+) \cap C(\mathbb{R}^+, D(\mathcal{A}_\Diamond))$ *with* $u(t), v(t) \in D(A) \cap D(A_\#)$ *for all* $t \geq 0$*. The following assertions are equivalent.*

(a) $(I_t)_{t\geq 0}$ *is invariant with respect to the system* (5)*,*

(b) the differential inequality holds

$$\frac{1}{2}\begin{pmatrix} A_\# + A & A_\# - A \\ A_\# - A & A_\# + A \end{pmatrix}\begin{pmatrix} u(t) \\ v(t) \end{pmatrix} \leq \frac{d}{dt}^+ \begin{pmatrix} u(t) \\ v(t) \end{pmatrix} \quad \text{for all } t \geq 0. \tag{18}$$

For positive semigroups we have the following more explicit characterization.

8 Corollary. *Assume that* A *generates a positive semigroup and* $u, v \in C^1(\mathbb{R}^+, E^+) \cap C(\mathbb{R}^+, D(A))$*. Then* $(I_t)_{t\geq 0}$ *is invariant with respect to system* (5) *if and only if*

$$\begin{cases} Au(t) \leq \frac{d}{dt}^+ u(t), & t \geq 0, \\ Av(t) \leq \frac{d}{dt}^+ v(t), & t \geq 0. \end{cases}$$

3 Invariance of time varying order intervals for delay equations

Consider the linear delay equation

$$\begin{cases} \frac{d}{dt}x(t) & = Ax(t) + Lx_t, \quad t \geq t_0, \\ x(t_0) & = x^0, \; x_{t_0}(\theta) = \varphi(\theta), \; \text{a.e. } \theta \in (-h, 0)\,, \end{cases} \tag{19}$$

where A is the generator of a C_0-semigroup $(T(t))_{t\geq 0}$ on a Banach lattice X with order continuous norm, $x^0 \in X$, $L : C([-h, 0], X) \longrightarrow X$ is a bounded linear operator given by

$$L\varphi = \int_{-h}^{0} d\eta(\theta)\varphi(\theta), \quad \varphi \in C([-h, 0], X), \tag{20}$$

where $\eta : [-h, 0] \to \mathcal{L}(X)$ is a function of bounded variation with no mass at zero. One denotes by x_t the function defined as $x_t(\theta) = x(t+\theta)$, a.e. $\theta \in (-h, 0)$.

For the definition and the existence of solutions of 19, see [2] and [12]. In [7], Bernier and Manitius studied this equation in finite dimension for the following particular operator

$$Lf := \sum_{k=1}^{N} A_k f(-h_k) + \int_{-h}^{0} B(\theta)f(\theta)d\theta \quad \text{for } f \in C([-h, 0], \mathbb{R}^n),$$

where, for $k = 1, \cdots, N$, $A_k \in \mathcal{L}(\mathbb{R}^n)$, $0 < h_1 < \cdots < h_N < h$ and the function $B(\cdot)$ is bounded and measurable. Consider the family of sets

$$J_t := \{x \in E : -v(t) \le x \le u(t)\}, \quad t \ge -h,$$

with $u, v : [-h, \infty) \to X^+$. We give the following definition.

9 Definition. The family $(J_t)_{t \ge -h}$ is invariant with respect to the delay equation (19) if for every $t_0 \ge 0$, one has: for every $x^0 \in J_{t_0}$ and $\varphi(\theta) \in J_{t_0+\theta}$, $\theta \in (-h, 0)$, the solution x of (19) satisfies $x(t) \in J_t$ for all $t \ge t_0$.

In order to characterize the invariance of J_t, $t \ge -h$, with respect to the delay equation (19), we consider its equivalent Cauchy problem

$$\begin{cases} \frac{d}{dt} Z(t) & = \mathcal{A} Z(t), \quad t \ge t_0, \\ Z(t_0) & = \begin{pmatrix} x^0 \\ \varphi \end{pmatrix}, \end{cases} \tag{21}$$

in the Banach lattice $E := X \times L^p((-h, 0), X)$, where

$$\mathcal{A} := \begin{pmatrix} A & L \\ 0 & \frac{d}{d\theta} \end{pmatrix} \text{ and } D(\mathcal{A}) = \left\{ \begin{pmatrix} z \\ f \end{pmatrix} \in D(A) \times W_p^1((-h, 0), X) : f(0) = z \right\}.$$

In [2, 12], it is shown that $\mathcal{A}$ generates a C_0-semigroup $(T(t))_{t \ge 0}$ on E and

$$T(t) \begin{pmatrix} x^0 \\ \varphi \end{pmatrix} = \begin{pmatrix} x(t, \varphi, x^0) \\ x_t(\cdot, \varphi, x^0) \end{pmatrix}, \quad t \ge 0,$$

with $x(\cdot, \varphi, x^0)$ is the mild solution of (19).

Consider in E the sets

$$I_t := \left\{ \begin{pmatrix} z \\ f \end{pmatrix} \in E : -\begin{pmatrix} v(t) \\ v_t \end{pmatrix} \le \begin{pmatrix} z \\ f \end{pmatrix} \le \begin{pmatrix} u(t) \\ u_t \end{pmatrix} \right\}, \quad t \ge 0.$$

In the following lemma we relate the invariance with respect to delay equation (19) and the one with respect to the Cauchy problem (21).

10 Lemma. *The following assertions are equivalent.*

(i) $(J_t)_{t \ge -h}$ *is invariant with respect to the delay equation* (19)*,*

(ii) $(I_t)_{t \ge 0}$ *is invariant with respect to the system* (21)*.*

Proof. *(i)* $\Rightarrow$ *(ii)*. Let $t_0 \ge 0$ and $Z_0 = \begin{pmatrix} x^0 \\ \varphi \end{pmatrix} \in I_{t_0}$. So, we have $-v(t_0) \le x^0 \le u(t_0)$ and $-v(t_0 + \theta) \le \varphi(\theta) \le u(t_0 + \theta)$, a.e. $\theta \in (-h, 0)$.
Since, for all $\tau \ge 0$,

$$x(t_0 + \tau, \varphi, x^0) = P_1 T(\tau) \begin{pmatrix} x^0 \\ \varphi \end{pmatrix},$$

where P_1 is the projection onto the first component of E, we obtain

$$-v(t_0+\tau) \leq P_1 T(\tau)\binom{x^0}{\varphi} \leq u(t_0+\tau). \tag{22}$$

On the other hand, for all $\theta \in (-h, 0)$, we have

$$-v(t_0+\tau+\theta) \leq x(t_0+\tau+\theta, \varphi, x^0) \leq u(t_0+\tau+\theta).$$

Let P_2 be the projection onto the second component of E. Then,

$$\begin{aligned} P_2 T(\tau)(\theta) &= x_{t_0+\tau}(\theta) \\ &= \begin{cases} x(t_0+\tau+\theta, \varphi, x^0) & \text{if } \tau+\theta \geq 0, \\ \varphi(\tau+\theta) & \text{if not.} \end{cases} \end{aligned}$$

which implies by assumption that

$$-v_{t_0+\tau} \leq P_2 T(\tau)\binom{x^0}{\varphi} \leq u_{t_0+\tau} \tag{23}$$

Finally, (22) together with (23) imply that $T(\tau)\binom{x^0}{\varphi} \in I_{t_0+\tau}$.
Similarly, we obtain the implication $(ii) \Rightarrow (i)$. $\boxed{QED}$

To give the main result of this section, we recall from [4, 14] the following result concerning the modulus semigroup of $(T(t))_{t\geq 0}$.

11 Theorem. *The semigroup $(T(t))_{t\geq 0}$ admits a modulus semigroup $(T_\#(t))_{t\geq 0}$. Moreover the generator of $(T_\#(t))_{t\geq 0}$ is given by*

$$\mathcal{A}_\# := \begin{pmatrix} A_\# & |L| \\ 0 & \frac{d}{d\theta} \end{pmatrix},$$

$$D(\mathcal{A}_\#) = \left\{ \binom{z}{f} \in D(A_\#) \times W_p^1((-h,0), X) : f(0) = z \right\}.$$

In order to give an explicit characterization of the invariance of the family of intervals $(J_t)_{t\geq -h}$, we restrict our self to the case where A is bounded. In this case, we can give explicitly the generator $\mathcal{A}_\Diamond$ and its domain $D(\mathcal{A}_\Diamond)$. More precisely we have the following result.

12 Proposition. *The operator* $\mathcal{A}_\Diamond = \frac{1}{2}\begin{pmatrix} \mathcal{A}_\#+\mathcal{A} & \mathcal{A}_\#-\mathcal{A} \\ \mathcal{A}_\#-\mathcal{A} & \mathcal{A}_\#+\mathcal{A} \end{pmatrix}$ *and* $D(\mathcal{A}_\Diamond) = D(\mathcal{A}) \times D(\mathcal{A})$.

Proof. In view of Proposition 4, it suffices to show that $\widetilde{\mathcal{A}} := \frac{1}{2}\begin{pmatrix} \mathcal{A}_\#+\mathcal{A} & \mathcal{A}_\#-\mathcal{A} \\ \mathcal{A}_\#-\mathcal{A} & \mathcal{A}_\#+\mathcal{A} \end{pmatrix}$ is the generator of a C_0-semigroup on $E \times E$. Let us decompose the operator $\mathcal{A}$ as follows

$$\mathcal{A} := \mathbb{A} + \mathbb{L} + \mathbb{A}_2$$

with

$$\mathbb{A} := \begin{pmatrix} A_1 & 0 & 0 & 0 \\ 0 & \frac{d}{d\theta} & 0 & 0 \\ 0 & 0 & A_1 & 0 \\ 0 & 0 & 0 & \frac{d}{d\theta} \end{pmatrix}, \ \mathbb{L} := \begin{pmatrix} 0 & L^+ & 0 & L^- \\ 0 & 0 & 0 & 0 \\ 0 & L^- & 0 & L^+ \\ 0 & 0 & 0 & 0 \end{pmatrix} \ \text{and}\, \mathbb{A}_2 := \begin{pmatrix} 0 & 0 & A_2 & 0 \\ 0 & 0 & 0 & 0 \\ A_2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

where $A_1 := \frac{1}{2}(A_\# + A), A_2 := \frac{1}{2}(A_\# - A)$. Since $\mathbb{A}_2 \in \mathcal{L}(E \times E)$ ($\mathbb{A}_2 = 0$ in the positive case), we have only to show that $\mathbb{A} + \mathbb{L}$ generates a C_0-semigroup. From [2], we know that the operator $\begin{pmatrix} A_1 & 0 \\ 0 & \frac{d}{d\theta} \end{pmatrix}$, with domain $D(\mathcal{A})$ is the generator of the C_0-semigroup $(\mathcal{T}_0(t))_{t\geq 0}$ given by

$$\mathcal{T}_0(t) = \begin{pmatrix} S(t) & 0 \\ S_t & T_0(t) \end{pmatrix}, \quad t \geq 0,$$

where $(T_0(t))_{t\geq 0}$ is the nilpotent left shift semigroup on $L^p([-h,0],X)$, $(S(t))_{t\geq 0}$ is the uniformly continuous semigroup generated by A_1 and

$$S_t : X \to L^p([-h,0],X)$$

is defined by

$$(S_t x)(\sigma) := \begin{cases} S(t+\sigma)x, & -t < \sigma \leq 0, \\ 0, & -h \leq \sigma \leq -t. \end{cases}$$

Consequently, the matrix operator $\mathbb{A}$, with the diagonal domain $D(\mathcal{A}) \times D(\mathcal{A})$, generates the C_0-semigroup $(\mathbb{T}(t))_{t\geq 0}$ given by

$$\mathbb{T}(t) = \begin{pmatrix} \mathcal{T}_0(t) & 0 \\ 0 & \mathcal{T}_0(t) \end{pmatrix}, \quad t \geq 0.$$

Next, we are going to show that $\mathbb{L}$ is a small Miyadera perturbation of $\mathbb{A}$ in $E \times E$. We know from [17], that $|L|$ the modulus of L is similarly given by (20) with some bounded variation function $\tilde{\eta}$. Hence, both $L^+ = \frac{1}{2}(|L| + L)$ and $L^- = \frac{1}{2}(|L| - L)$ are also given by (20) with some η_+ and η_-. Hence, for $p > 1$, one has from [2],

$$\int_0^t \|L^\pm(S(s)x + T_0(s)f)\| ds \leq t^{\frac{1}{p'}} M |\eta_\pm|([-h,0]) \| \left(\begin{smallmatrix} x \\ f \end{smallmatrix}\right) \|$$

for all $t \in]0,1[$. Let us choose t_0 sufficiently small such that

$$q := t_0^{\frac{1}{p'}} M \max(|\eta_+|([-h,0]), |\eta_-|([-h,0])) < \frac{1}{2}.$$

Now, for $\zeta_1 = \binom{x_1}{\varphi_1}$ and $\zeta_2 = \binom{x_2}{\varphi_2}$, one can obtain

$$\begin{aligned}\int_0^t \|\mathbb{L}\mathbb{T}(s)\tbinom{\zeta_1}{\zeta_2}\|ds \leq & \int_0^t \|L^+(S(s)x_1 + T_0(s))\varphi_1\|ds+ \\ & + \int_0^t \|L^-(S(s)x_1 + T_0(s)\varphi_1)\|ds+ \\ & + \int_0^t \|L^-(S(s)x_1 + T_0(s)\varphi_1)\|ds+ \\ & + \int_0^t \|L^+(S(s)x_2 + T_0(s)\varphi_2)\|ds \leq 2q\|\tbinom{\zeta_1}{\zeta_2}\|.\end{aligned}$$

Consequently, $\mathbb{L}$ is a small Miyadera perturbation of $\mathbb{A}$. Then, by the Miyadera-Voigt perturbation theorem, see [9, Corollary III.3.16], it follows that $\mathbb{A}+\mathbb{L}$ generates a C_0-semigroup. The case $p = 1$ can be treated as in [14]. We endow E by the equivalent norm

$$\|\tbinom{x}{f}\|_c = \|x\| + c\|f\|_{L^p}$$

with $c > 2\max(|\eta_+|([-h,0]), |\eta_-|([-h,0])$ and show that $\mathbb{L}$ is a small Miyadera perturbation of $\mathbb{A}$ on $(E, \|\cdot\|_c) \times (E, \|\cdot\|_c)$. Then the result follows. $\boxed{QED}$

From Theorem 7, Proposition 12 and Theorem 11, by making explicit the condition (18), we obtain the following main result.

13 Theorem. *Assume that $u, v \in C^1([-h,\infty), X^+)$. The family of sets $(J_t)_{t\geq -h}$ is invariant with respect to the delay system* (19) *if and only if*

$$\begin{pmatrix} A_1 & A_2 \\ A_2 & A_1 \end{pmatrix}\begin{pmatrix} u(t) \\ v(t) \end{pmatrix} + \begin{pmatrix} L^+ & L^- \\ L^- & L^+ \end{pmatrix}\begin{pmatrix} u_t \\ v_t \end{pmatrix} \leq \frac{d}{dt}^+ \begin{pmatrix} u(t) \\ v(t) \end{pmatrix}. \tag{24}$$

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