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Low codimension Fano–Enriques threefolds

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Abstract. In this paper we study Fano threefolds with a torsion divisor (Fano–Enriques). Due to this torsion divisor, they can be described as quotients of Fano threefolds by a finite abelian group action. We start from lists of Fano threefolds by Reid, Fletcher and Altınok and check which of them admit such an action with a Fano–Enriques quotient.

Keywords: Fano threefolds, torsion divisors, quotient singularities.

MSC 2000 classification: primary 14J30, secondary 14J45

Introduction

In [9], Reid introduced the graded rings method for the explicit classification of surfaces, which he used to produce a list of 95 K3 quasi-smooth hypersurfaces in weighted projective spaces (which were proved to be the only ones). Later, Fletcher used this method to create more lists of different weighted complete intersections. From the K3 surfaces he developed two lists of anticanonically polarised Fano threefolds that have the K3s as hyperplane sections. These two lists for Fano threefolds of codimension one and two can be found in [6]. Later on, Altınok developed in [1] a formula to compute the Hilbert series of a Fano threefold (which is very important for the graded rings method). Also, Altınok wrote a list of codimension three K3 surfaces (which produces a list of codimension three Fano threefolds). All lists are also in [7].

In this paper we deal with Fano–Enriques threefolds (Fano threefolds with a torsion divisor σ). These varieties can be expressed as quotients of Fano threefolds under an action by a $\mathbb{Z}/(r)$ group, where r is the order of σ. We use the above lists of codimension 1, 2 and 3 to give in this paper all possible Fano–Enriques quotients that can be obtained from these lists.

To find these quotients, one finds restrictions for the covers. Then, one tests all members of the known lists of Fano threefolds and, for all the members satisfying the restrictions, one computes the Hilbet series to apply a slightly modified graded rings method and search for a quotient.

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The distribution of this paper is as follows. In the first section, we recall some preliminaries about Fano–Enriques threefolds. We describe Altınok's method to compute the Hibert series for the anticanonical ring of a Fano threefold and the graded rings method in section 2. In section 3 we introduce our modification for these methods (Altınok's and graded rings) to Fano–Enriques threefolds. Finally, in section 4, we show the complete way to obtain the lists of Fano–Enriques quotients and give these lists.

1 Preliminaries

Throughout this paper, we work over the complex field. We recall that a singularity of type $\frac{1}{r}(a_1,\dots,a_l)$ is a point with an analytic neighbourhood which is isomorphic to a neighbourhood of the origin in $\mathbb{C}^l$ under an action by $\mathbb{Z}/(r)$ consisting on multiplying by $(\epsilon^{a_1},\dots,\epsilon^{a_l})$ where $\epsilon = e^{2\pi\frac{i}{r}}$. Sometimes we just say singularity of type $\frac{1}{r}$ without specifying $a_1,\dots,a_l$. We consider just tridimensional irreducible and reduced schemes over $\mathbb{C}$ with at most isolated cyclic singularities of index $\frac{1}{r}(1,a,-a)$ with r and a coprime. The necessary background about the weighted projective space and quasi-smooth subvarieties can be found in [5] or [6].

1 Definition. A threefold X is *Fano* if it has at most isolated quitient singularities of index $\frac{1}{r}(1,a,-a)$, with r and a coprime, and the anticanonical class $-K_X$ is ample.

2 Definition. A Fano threefold X is *Fano–Enriques* if it has a torsion divisor σ, (i.e. there exists $r\in\mathbb{Z}^+$ so that $r\sigma \overset{lin}{\sim} 0$).

We recall now the standard construction that relates every Fano–Enriques threefold with a Fano threefold. Let X be a Fano–Enriques threefold with torsion divisor σ of order r. It is well known that we can define a covering $\pi : Y \to X$ where

$$Y = \operatorname{Spec}(\mathcal{O}_X \oplus \mathcal{O}_X(\sigma) \oplus \cdots \oplus \mathcal{O}_X((r-1)\sigma)).$$

This covering is $r:1$ at each point in which σ is a Cartier divisor. Moreover, there is a regular action by $\mathbb{Z}/(r)$ on Y such that X is the quotient of Y by this action. This action can be described this way: For $j\in\mathbb{Z}/(r)$, consider $\epsilon = e^{\frac{2\pi i}{r}}$. Then:

$$\mathcal{O}_X \oplus \mathcal{O}_X(\sigma) \oplus \cdots \oplus \mathcal{O}_X((r-1)\sigma) \xrightarrow{j} \mathcal{O}_X \oplus \mathcal{O}_X(\sigma) \oplus \cdots \oplus \mathcal{O}_X((r-1)\sigma))$$

where

$$(a_0,\dots,a_{r-1}) \mapsto (a_0, \epsilon^j a_1, \dots, \epsilon^{(r-1)j} a_{r-1}).$$

Then, for $I \in Y = \operatorname{Spec}(\mathcal{O}_X \oplus \mathcal{O}_X(\sigma) \oplus \cdots \oplus \mathcal{O}_X((r-1)\sigma))$ its image is the prime ideal $j^{-1}(I)$.

3 Remark. If X is quotient of Y by a $\mathbb{Z}/(r)$ action, that action can be extended to the weighted projective space that contains Y. The reason is that the representation of $\mathbb{Z}/(r)$ has a reflection in the anticanonical ring of Y (via the action on the sheaf of structure). The action on the anticanonical ring

$$R(Y, -K_Y) = \bigoplus_{d\in\mathbb{Z}} H^0(Y, -dK_Y)$$

preserves the degrees of the elements. This means that we have an action for every $R(Y, -K_Y)_d = H^0(Y, -dK_X)$. Therefore, each space will be generated by eigenvectors (this is because the automorphism is an r-th root of the identity), which means that the matrix of the automorphism of the ring is diagonal after a suitable change of coordinates. Now we can use Orbifold Riemann-Roch to classify by eigenvalues (r-th roots of 1) the generators of $R(Y, -K_Y)$ (which are eigenvectors after the change of coordinates). This describes the action that, together with Y, determines X.

4 Proposition. *Let X be a Fano–Enriques threefold with torsion divisor σ of order r. Let $Y = \mathrm{Spec}(\mathcal{O}_X \oplus \mathcal{O}_X(\sigma) \oplus \cdots \oplus \mathcal{O}_X((r-1)\sigma))$ as above. Then:*

(1) All the singularities of Y are terminal.

(2) The cover Y is a $\mathbb{Q}$-Fano threefold.

Proof. Let $U \subset X$ be a sufficiently small analytic neighbourhood around a singularity Q of type $\frac{1}{r_Q}(b_Q, 1, -1)$. We know that $\mathrm{Pic}(U)$ is isomorphic to $\mathbb{Z}/(r_Q)$ and is generated by the restriction to U of the canonical class, so the local expression of the torsion divisor $\sigma_{|U}$ is $l_Q K_{X|U}$ for some l_Q in $\mathbb{Z}/(r_Q)$. Then, $\alpha := \frac{r_Q}{\gcd(r_Q, l_Q)}$ is the order of $\sigma_{|U}$ in $\mathrm{Pic}(U)$ (i.e. $\alpha\sigma$ is Cartier in Q) and divides r (the order of σ in the divisor class group of X). Let Y_α be the cover of X associated to $\alpha\sigma$. Clearly, Y is a cover for Y_α (there is an obvious monomorphism of rings which defines an epimorphism of schemes with a commutative diagram). Therefore, Q comes from $\frac{r}{\alpha}$ different points with disjoint analytic neighbourhoods V_i, $i = 1, \ldots, \frac{r}{\alpha}$. Observe now that the canonical cover of a $\frac{1}{r_Q}(b_Q, 1, -1)$ singularity is an analytic neighbourhood of O in $\mathbb{C}^3$ defined as

$$\mathrm{Spec}(\mathcal{O}_U \oplus \mathcal{O}_U(K_{X|U}) \oplus \cdots \oplus \mathcal{O}_U((r-1)K_{X|U})).$$

Hence, working locally, we easily deduce that each V_i is of type $\frac{1}{\frac{r_Q}{\alpha}}(b_Q, 1, -1)$ which proves (1).

Since π is an analytically local isomorphism away from the singularities of X (which are isolated), the anticanonical bundle of Y is $\pi^*(-K_X)$ (i.e.

$\pi^{-1}(-K_X) \oplus \pi^{-1}(-K_X + \sigma) \oplus \cdots \oplus \pi^{-1}(-K_X + (r-1)\sigma))$. Therefore Y is a Fano threefold and we have (2).

QED

5 Example. Consider the complete intersection $Y_{2,2,2} \subset \mathbb{P}^6$ with $\mathbb{Z}/(2)$ acting as the multiplication by $+,+,+,+,-,-,-$ (i.e. the generator of $\mathbb{Z}/(2)$ takes $(x_0, x_1, x_2, x_3, x_4, x_5, x_6)$ to $(x_0, x_1, x_2, x_3, -x_4, -x_5, -x_6)$). Then, since a general Y has eight fixed points (the points in the intersection with $\{x_4 = x_5 = x_6 = 0\}$), X has eight singularities of index $\frac{1}{2}(1,1,1)$.

The aim of this paper is to reverse this process, i.e. to describe the possible X's for a fixed Y with a prescribed $\mathbb{Z}/(r)$ action.

2 The graded rings method for Fano threefolds

In this section we recall Reid's graded rings method from [10] in order to find appropriate ambient spaces and equations for Fano threefolds as shown in [1]. The aim is to manage some numerical data (i.e. the type of the singularities and the selfintersection of the canonical divisor $-K_Y^3 \in \mathbb{Q}$) to search for a Fano threefold embedded in a weighted projective space by the anticanonical morphism. For this purpose we use the Hilbert series of the polarized variety $(Y, -K_Y)$ (which depends only on the above numerical data) to guess a possible set of generators and relations of the ring associated to our threefold embedded in the appropriate weighted projective space. In general, for an ample divisor D (for a Fano threefold, we use $D = -K_X$) we can consider the ring

$$R(Y, D) := \bigoplus_{n \geq 0} H^0(Y, nD).$$

If we can find generators and relations for such a ring, then we have a good description of $Y = \mathrm{Proj}(R(Y, D))$.

We use Hilbert series to find this $R(Y, D)$. The Hilbert series of Y is:

$$P_Y(t) := \sum_{n \geq 0} h^0(Y, nD) t^n.$$

To compute t$P(t)$, we can recall Orbifold-RR formula from [10]:

$$\chi(\mathcal{O}_Y(D)) = \chi(\mathcal{O}_Y) + \frac{1}{12} D(D - K_Y)(2D - K_Y) + \\ + \frac{1}{12} D c_2(Y) + \sum_{Q \in \mathcal{B}} c_Q(D) \quad (1)$$

where $\mathcal{B}$ represents the basket of terminal quotient singularities $\frac{1}{r_Q}(1, a_Q, -a_Q) = \frac{1}{r_Q}(b_Q, 1, -1)$ (where $a_Q b_Q \equiv 1 \bmod r_Q$) and the contributions are:

$$c_Q(D) = -i_Q \frac{r_Q^2 - 1}{12 r_Q} + \sum_{j=1}^{i_Q - 1} \frac{[b_Q j]_{r_Q}(r_Q - [b_Q j]_{r_Q})}{2 r_Q}$$

where:

- i_Q is defined by the condition $\mathcal{O}(D) \simeq \mathcal{O}(i_Q K_Y)$ near the singularity Q.
- $[a]_r$ is the minimal nonnegative remainder of a mod r.

6 Remark. Since $-K_X$ is ample, the equality $\chi(-nK_Y) = h^0(Y, -nK_Y)$ holds for all $n \geq 0$ by Kodaira vanishing theorem.

7 Theorem. *[Altınok [1]] The Hilbert series $\sum_{n \geq 0} h^0(Y, -nK_Y)t^n$ of a Fano threefold Y can be computed as a rational function on t by the formula:*

$$P_Y(t) = \frac{1+t}{(1-t)^2} + \frac{t+t^2}{(1-t)^4} \frac{-K_Y^3}{2} + \\ - \sum_{Q \in \mathcal{B}} \frac{1}{(1-t)(1-t^r)} \sum_{i=1}^{r_Q - 1} \frac{[b_Q i]_{r_Q}(r_Q - [b_i]_{r_Q})}{2r} t^i. \quad (2)$$

We call numerical data of a Fano threefold to $\mathcal{B}$ and the selfintersection of the canonical class $-K_Y^3$.

8 Remark. We recall from [1] that the selfintersection of the canonical class of a Fano threefold X is

$$-K_Y^3 = \sum \frac{b_Q(r_Q - b_Q)}{r_Q} + 2k$$

for some integer k.

9 Example. Now we illustrate the graded rings method. Consider a hypothetical Fano threefold with just a $\frac{1}{2}(1,1,1)$ singularity and $-K_Y^3 = \frac{5}{2}$. Using Alınok's formula (2), we obtain the Hilbert series:

$$P_Y(t) := 1 + 4t + 11t^2 + 24t^3 + 46t^4 + 79t^5 + 126t^6 + 189t^7 + 271t^8 + 374t^9 + \cdots$$

By definition, the coefficient of t^d is the dimension of the $\mathbb{C}$- vector space of all homogeneous elements in $R(Y, -K_Y)$ of degree d. In particular, we have generators x_1, x_2, x_3, x_4 in degree one. Since a generator contributes to this series multiplying by $\frac{1}{1-t^a}$, where a is the degree of the generator, we can multiply by $(1-t)^4$ to simplify the series and discover new generators and relations:

$$(1-t)^4 P_Y(t) = 1 + t^2 + t^4 - t^5 + t^6 - t^7 + t^8 - t^9 + \cdots$$

This shows that there is a new generator y in degree two. Then, we multiply by $(1-t^2)$ and get $1-t^5$. Therefore, one expects to have a relation in degree five, which means having a hypersurface $Y_5 \subset \mathbb{P}(1,1,1,1,2)$. And in fact, any quasi-smooth equation works and give the numerical data we started from (i.e. a Fano threefold with a singularity of type $\frac{1}{2}(1,1,1)$ and $-K_Y^3 = \frac{5}{2}$).

10 Remark. The expression "one expects" means, as in [2], "if there are no extra generators and relations". If there is an extra relation among the monomials (products of the generators), we need a new generator of the same degree to fill the dimension given by Orbifold-RR and the Hilbert series does not change. We avoid these special cases (see also Remark 22).

3 The graded rings method for Fano–Enriques threefolds

This section is devoted to modify the graded rings method in order to apply it to Fano–Enriques threefolds. We start with an analogue of Altınok's formula. We need an useful remark which explains why formula (3) has been motivated.

From Remark 3, one naturally attempts to generalize Altınok's formula by defining a new Hilbert series considering two degrees: the standard one in $\mathbb{Z}$ and other in $\mathbb{Z}/(r) = \{r-\text{th roots of } 1\}$:

Clearly, $H^0(Y, -nK_Y) = \bigoplus_{i=0}^{r-1} H^0(X, -nK_X + i\sigma)$. Therefore, $P_Y(t) = \sum_{i=0}^{r-1} P_X^i(t)$ where $P_X^i(t) = \sum_{n\geq 0} h^0(X, -nK_X + i\sigma)t^n$.

So we can define the new Hilbert series in $\mathbb{Z}[[t,e]]/(e^r - 1)$ as $\sum_{i=0}^{r-1} e^i P_X^i(t)$. The main point of this is that the product in the power series ring agrees with the $\mathbb{Z}/(r)$ action (i.e. a generator in $H^0(X, -nK_X + i\sigma)$ contributes to the Hilbert series multiplying by $\frac{1}{(1-e^i t^n)} = 1 + e^i t^n + e^{[2i]_r} t^{2n} + \cdots$). So we grade $R(Y, -K_Y)$ in a new way:

11 Definition. We call *bidegree* of an element in $H^0(X, -nK_X + i\sigma) \subset R(Y, -K_Y)$ as $(n, i) \in \mathbb{Z} \oplus \mathbb{Z}/(r)$.

Now we need a formula for $\sum h^0(X, -nK_X + \tau)$, where τ is a numerically trivial divisor (in particular, it can be a torsion divisor).

12 Lemma. *Let X be a Fano threefold with a numerically trivial divisor τ and a basket of singularities $\mathcal{B}$. For every singularity $Q \in \mathcal{B}$, define l_Q such that,* *locally in* Q, $\tau \simeq \mathcal{O}(l_Q K_X)$. *Then we have that* $\sum_{n\geq 0} h^0(X, -nK_X+\tau)t^n =$

$$P_X(t) + \sum_{Q\in\mathcal{B}} \Big(\frac{r_Q - l_Q}{1-t}\frac{r_Q^2-1}{12r_Q} + \\ + \frac{1}{1-t^{r_Q}} \sum_{j=0}^{r_Q-1} \big(\sum_{i=j+1}^{j+r_Q-l_Q} \frac{[b_Q i](r_Q-[b_Q i])}{2r_Q} \big) t^j \Big). \quad (3)$$

Proof. By Orbifold Riemann-Roch:

$$\chi(-nK_X) = \chi(\mathcal{O}_X) + \frac{2n^3+3n^2+n}{12}(-K_X^3) + \frac{n}{12}(-K_X)c_2 + \sum_{Q\in\mathcal{B}} c_Q(n)$$

where

$$c_Q(n) = -[-n]\frac{r_Q^2-1}{12r_Q} + \sum_{j=1}^{[-n]-1} \frac{[b_Q j](r_Q-[b_Q j])}{2r_Q}$$

and

$$\chi(-nK_X+\tau) = \chi(\mathcal{O}_X) + \frac{2n^3+3n^2+n}{12}(-K_X) + \\ + \frac{n}{12}(-K_X)c_2 + \sum_{Q\in\mathcal{B}} c_Q(-nK_X+\tau)$$

where

$$c_Q(-nK_X+\tau) = -[l_Q-n]\frac{r_Q^2-1}{12r_Q} + \sum_{j=1}^{[l_Q-n]-1} \frac{[b_Q j](r_Q-[b_Q j])}{2r_Q}.$$

It is clear that $\chi(-nK_X+\tau) - \chi(-nK_X)$ is the sum in $\mathcal{B}$ of $c_Q(-nK_X+\tau) - c_Q(n)$. This takes the value:

$$\sum_{Q\in\mathcal{B}} \Big((r_Q-l_Q)\frac{r_Q^2-1}{12r_Q} - \sum_{i=n+1}^{n+r_Q-l_Q} \frac{[b_Q j](r_Q-[b_Q j])}{2r_Q} \Big).$$

This expression is clearly periodic with period r_Q, so we get (3) QED

13 Remark. Lemma 12 shows that the only numerical data of a Fano–Enriques threefold we need to consider consist of $-K_X^3$, the order of the torsion divisor r, and the basket $\mathcal{B}$ divided in $\mathcal{B}t$ (singularities where the torsion divisor is not trivial) and $\mathcal{B}e$ (rest of the basket). Moreover, for every singularity $Q \in \mathcal{B}t$ we add the number l_Q, which is determined by the local value of the torsion divisor in the singularity Q.

The generalization of the graded rings method to Fano–Enriques threefolds comes by applying Lemmas 7 and 12. We continue by applying the method to a concrete example.

14 Example. Consider a Fano–Enriques variety X whose basket is $\mathcal{B} = \{\frac{1}{10}(1,3,7), 2\times\frac{1}{5}(1,2,3)\}$ with respective $l_Q = 6,1,1$ and $-K_X^3 = \frac{1}{2}$. We look for generators as in Example 9 but paying attention also to the new weight in $\mathbb{Z}/(5)$. From formulas (2) and (3), we get the Hilbert series:

$$\begin{aligned}&(1+t+2t^2+5t^3+9t^4+16t^5+25t^6+38t^7+54t^8+74t^9+\cdots)+\\&\quad+e(t+2t^2+5t^3+9t^4+15t^5+26t^6+38t^7+54t^8+75t^9+\cdots)+\\&\quad+e^2(3t^2+5t^3+9t^4+16t^5+25t^6+38t^7+54t^8+75t^9+\cdots)+\\&\quad+e^3(t+2t^2+5t^3+9t^4+16t^5+25t^6+37t^7+55t^8+75t^9+\cdots)+\\&\quad+e^4(t+2t^2+5t^3+9t^4+15t^5+26t^6+38t^7+54t^8+75t^9+\cdots)\end{aligned}$$

The coefficient of e^it^j is the dimension of the subspace of elements of bidegree (i,j) in $R(X,-K_X,\sigma)$, so we have generators x_1,x_2,x_3,x_4 in the subspaces $H^0(\mathcal{O}_X(-K_X))$, $H^0(\mathcal{O}_X(-K_X+\sigma))$, $H^0(\mathcal{O}_X(-K_X+3\sigma))$, $H^0(\mathcal{O}_X(-K_X+4\sigma))$ respectively. They have respective bidegrees (1,0), (1,1), (1,3), (1,4).

A generator in $H^0(\mathcal{O}_X(-jK_X+i\sigma))$ contributes now to the Hilbert series multiplying by

$$1+e^it^j+e^{[2i]_r}t^{2j}+\cdots$$

This is an inverse for $1-e^it^j$ in $\mathbb{Z}[[e,t]]/(e^r-1)$. Thus we multiply in our case by $1-t, 1-et, 1-e^3t, 1-e^4t$, so we get

$$\begin{aligned}&(1+t^4+t^8+\cdots)+\\&\quad+e(t^6+\cdots)+\\&\quad+e^2(t^2+-t^7+\cdots)+\\&\quad+e^3(t^8+\cdots)+\\&\quad+e^4(t^4-t^9+\cdots)\end{aligned}$$

This shows that we need a generator y in $H^0(\mathcal{O}_X(-2K_X+2\sigma))$ (of bidegree (2,2)). We now multiply by $1-e^2t^2$ and get:

$$1-t^5,$$

so we have a relation in $H^0(\mathcal{O}_X(-5K_X))$ (i.e. a weighted-homogeneous polynomial of degree 5 which is invariant by the action). The expression $x_1^5+x_2^5+x_3^5+x_4^5-x_2y^2$ gives a quasi-smooth relation for $Y_5\subset\mathbb{P}(1,1,1,1,2)$, with the action by $\mathbb{Z}/(5)$ consisting on multiplying each coordinate by $(1,\epsilon,\epsilon^3,\epsilon^4,\epsilon^2)$.

Now we should check that all singularities behave as expected. The only fixed points by the action in $\mathbb{P}(1,1,1,1,2)$ are the five coordinate points and the line $\mathbb{P}(x_2,y)$. So the only possible singular points come from points in $Y \cap \mathbb{P}(x_2,y)$. This intersection consists of three points: $(0:0:0:0:1), (0:1:0:0:1), (0:1:0:0:-1)$. For the two last points, we can use standard coordinates because these two points are not singular in Y (nor in $\mathbb{P}(1,1,1,1,2)$). In fact, we can take $\frac{x_1}{x_2}, \frac{x_3}{x_2}, \frac{x_4}{x_2}$ as affine coordinates. Considering our action as the multiplication by $(\epsilon^4, 1, \epsilon^2, \epsilon^3, 1)$, it is clear that for the second point and the third one we get a quotient singularity $\frac{1}{5}(4,2,3) = \frac{1}{5}(1,2,3)$. Moreover, since $-K_X$ is represented locally by the divisor $\{\frac{x_1}{x_2} = 0\}$ and σ by $\frac{x_4^2}{x_2^2}$, we obtain that σ is, locally, equal to the canonical divisor (i.e. $l_Q = 1$ for both points). For the first point, we immediately see that we can take $\frac{x_1}{y}, \frac{x_3}{y}, \frac{x_4}{y}$ as (analytically) local orbinates. Here, we have to consider the action in $\mathbb{C}^3$ which gives the singularity in Y: it takes (a,b,c) to $(-a,-b,-c)$. Now we also have another action, generated by the morphism that takes (a,b,c) to $(\epsilon^4 a, \epsilon^2 b, \epsilon^3 c)$. This means that we actually have a $\mathbb{Z}/(10)$ action generated by the two morphisms just written (the opposite of a nontrivial 5th root of unit is a nontrivial 10th root of unit). So we get a singularity of type $\frac{1}{10}(1\times 5 + 4\times 2, 1\times 5 + 2\times 2, 1\times 5 + 3\times 2) = \frac{1}{10}(3,9,1) = \frac{1}{10}(1,3,7)$. Working as before, we can see that l_Q is 6 as expected. This means that the example we got does have only terminal singularities, and so it is a Fano–Enriques threefold.

4 The search for Fano–Enriques threefolds

In this section we list all non-special (in the sense of Remark 10) Fano–Enriques threefolds of codimension 1, 2 and 3. To this purpose we give first some restrictions for the numerical data that gives (after a computational search), just 39 possibilities for $\mathcal{B}t$. Then, we combine it with all possible covers (chosen from the lists of [6] and [1] or [7]) to give the quotients. The lists in this section have been found using Magma (see [8]), which was used also to do the computer search for Tables 1.r ($r \in \{2,3,4,5,6,8\}$).

These are some immediate restrictions for the numerical data of a Fano–Enriques threefold:

(1) from Bogomolov's instability, as said in [2], $-K_X c_2 > 0$, so applying Orbifold-RR to the canonical class we get

$$\sum_{Q\in\mathcal{B}_X} \left(r_Q - \frac{1}{r_Q}\right) < 24. \tag{4}$$

(2) the same for the cover Y:

$$\sum_{Q\in\mathcal{B}_Y}\left(r_Q-\frac{1}{r_Q}\right)<24.$$

(3) the Lefschetz number is $L(g,\mathcal{O}_Y)=\sum(-1)^i\mathrm{Trace}(g^*|_{H^i(\mathcal{O}_X)})$. By the Atiyah-Singer-Segal formula, it is a sum of various contributions over $\mathrm{Fix}(g)$. So: $\mathrm{Fix}(g)=\emptyset\Rightarrow L(g,\mathcal{O}_Y)=0$. By Kodaira vanishing, since the covering Y is a Fano threefold, we get that $L(g,\mathcal{O}_Y)=\mathrm{Trace}(g^*|_{H^0(\mathcal{O}_X)})$, and $g^*|_{H^0(\mathcal{O}_X)}$ is not zero. Therefore the action fixes at least a point. This means that, if r is the order of the torsion divisor σ, we have a $\frac{1}{kr}$ singularity for at least one $k\in\mathbb{Z}$ positive. So $r\leq 24$ by

(4).

(4) if we call $\mathcal{B}t$ the subset of $\mathcal{B}$ consisting on the singularities Q where τ is not Cartier (i.e. $l_Q\neq 0$), it is clear that $\chi(-nK_X+\tau)-\chi(-nK_X)$ depends only on $\mathcal{B}t$ and the coefficients l_Q, not on the rest of $\mathcal{B}$ and $-K_X^3$. These numbers must be integer for all $\tau=i\sigma, i\in\{0,\dots,r-1\}$.

(5) Since $-K_X$ is ample and σ is numerically trivial, $-K_X+i\sigma$ is ample for any $i\in\mathbb{Z}/(r)$. Therefore, by Kodaira's vanishing, $\chi(i\sigma)=h^0(i\sigma)\geq 0$.

After doing an exhaustive computer search (testing all baskets with all possible combinations of l_Q), we found that there are 39 possible $\mathcal{B}t$ (with $r=2,3,4,5,6,8$) satisfying the restrictions. The notation we use is

$$\left(\frac{1}{r_Q}(1,a,-a)\right)_{l_Q}.$$

These are the results, which we divide according to the group $\mathbb{Z}/(r)$ acting

Table 1.2. Subsets $\mathcal{B}t$ for order 2 actions:

$$\left(\frac{1}{2}(1,1,1)\right)_1,\left(\frac{1}{14}(1,1,13)\right)_7 \qquad (\mathcal{B}t2.1)$$

$$\left(\frac{1}{2}(1,1,1)\right)_1,\left(\frac{1}{14}(1,3,11)\right)_7 \qquad (\mathcal{B}t2.2)$$

$$\left(\frac{1}{2}(1,1,1)\right)_1,\left(\frac{1}{14}(1,5,9)\right)_7 \qquad (\mathcal{B}t2.3)$$

$$\left(\frac{1}{4}(1,1,3)\right)_2,\left(\frac{1}{12}(1,1,11)\right)_6 \qquad (\mathcal{B}t2.4)$$

$$\left(\frac{1}{4}(1,1,3)\right)_2, \left(\frac{1}{12}(1,5,7)\right)_6 \qquad (\mathcal{B}t2.5)$$

$$\left(\frac{1}{6}(1,1,5)\right)_3, \left(\frac{1}{10}(1,1,9)\right)_5 \qquad (\mathcal{B}t2.6)$$

$$\left(\frac{1}{6}(1,1,5)\right)_3, \left(\frac{1}{10}(1,3,7)\right)_5 \qquad (\mathcal{B}t2.7)$$

$$\left(\frac{1}{8}(1,1,7)\right)_4, \left(\frac{1}{8}(1,1,7)\right)_4 \qquad (\mathcal{B}t2.8)$$

$$\left(\frac{1}{8}(1,1,7)\right)_4, \left(\frac{1}{8}(1,3,5)\right)_4 \qquad (\mathcal{B}t2.9)$$

$$\left(\frac{1}{8}(1,3,5)\right)_4, \left(\frac{1}{8}(1,3,5)\right)_4 \qquad (\mathcal{B}t2.10)$$

$$3 \times \left(\frac{1}{2}(1,1,1)\right)_1, \left(\frac{1}{10}(1,1,9)\right)_5 \qquad (\mathcal{B}t2.11)$$

$$3 \times \left(\frac{1}{2}(1,1,1)\right)_1, \left(\frac{1}{10}(1,3,7)\right)_5 \qquad (\mathcal{B}t2.12)$$

$$2 \times \left(\frac{1}{2}(1,1,1)\right)_1, \left(\frac{1}{4}(1,1,3)\right)_2, \left(\frac{1}{8}(1,1,7)\right)_4 \qquad (\mathcal{B}t2.13)$$

$$2 \times \left(\frac{1}{2}(1,1,1)\right)_1, \left(\frac{1}{4}(1,1,3)\right)_2, \left(\frac{1}{8}(1,3,5)\right)_4 \qquad (\mathcal{B}t2.14)$$

$$2 \times \left(\frac{1}{2}(1,1,1)\right)_1, 2 \times \left(\frac{1}{6}(1,1,5)\right)_3 \qquad (\mathcal{B}t2.15)$$

$$\left(\frac{1}{2}(1,1,1)\right)_1, 2 \times \left(\frac{1}{4}(1,1,3)\right)_2, \left(\frac{1}{6}(1,1,5)\right)_3 \qquad (\mathcal{B}t2.16)$$

$$4 \times \left(\frac{1}{4}(1,1,3)\right)_2 \qquad (\mathcal{B}t2.17)$$

$$5 \times \left(\frac{1}{2}(1,1,1)\right)_1, \left(\frac{1}{6}(1,1,5)\right)_3 \qquad (\mathcal{B}t2.18)$$

$$4 \times \left(\frac{1}{2}(1,1,1)\right)_1, 2 \times \left(\frac{1}{4}(1,1,3)\right)_2 \qquad (\mathcal{B}t2.19)$$

$$8 \times \left(\frac{1}{2}(1,1,1)\right)_1 \qquad (\mathcal{B}t2.20)$$

Table 1.3. Subsets $\mathcal{B}t$ for order 3 actions:

$$\left(\frac{1}{9}(1,1,8)\right)_3, \left(\frac{1}{9}(1,1,8)\right)_6 \qquad (\mathcal{B}t3.1)$$

$$\Big(\frac{1}{9}(1,2,7)\Big)_3, \Big(\frac{1}{9}(1,2,7)\Big)_6 \tag{$\mathcal{B}t3.2$}$$

$$\Big(\frac{1}{9}(1,4,5)\Big)_3, \Big(\frac{1}{9}(1,4,5)\Big)_6 \tag{$\mathcal{B}t3.3$}$$

$$2\times\Big(\frac{1}{3}(1,1,2)\Big)_1, \Big(\frac{1}{12}(1,5,7)\Big)_4 \tag{$\mathcal{B}t3.4$}$$

$$\Big(\frac{1}{3}(1,1,2)\Big)_1, \Big(\frac{1}{3}(1,1,2)\Big)_2, \Big(\frac{1}{6}(1,1,5)\Big)_2, \Big(\frac{1}{6}(1,1,5)\Big)_4 \tag{$\mathcal{B}t3.5$}$$

$$4\times\Big(\frac{1}{3}(1,1,2)\Big)_1, \Big(\frac{1}{6}(1,1,5)\Big)_4 \tag{$\mathcal{B}t3.6$}$$

$$3\times\Big(\frac{1}{3}(1,1,2)\Big)_1, 3\times\Big(\frac{1}{3}(1,1,2)\Big)_2 \tag{$\mathcal{B}t3.7$}$$

Table 1.4. Subsets $\mathcal{B}t$ for order 4 actions:

$$\Big(\frac{1}{4}(1,1,3)\Big)_2, 2\times\Big(\frac{1}{8}(1,3,5)\Big)_2 \tag{$\mathcal{B}t4.1$}$$

$$\times\Big(\frac{1}{2}(1,1,1)\Big)_1, \Big(\frac{1}{4}(1,1,3)\Big)_1, \Big(\frac{1}{12}(1,5,7)\Big)_9 \tag{$\mathcal{B}t4.2$}$$

$$2\times\Big(\frac{1}{2}(1,1,1)\Big)_1, \Big(\frac{1}{8}(1,1,7)\Big)_2, \Big(\frac{1}{8}(1,1,7)\Big)_6 \tag{$\mathcal{B}t4.3$}$$

$$2\times\Big(\frac{1}{2}(1,1,1)\Big)_1, \Big(\frac{1}{8}(1,3,5)\Big)_2, \Big(\frac{1}{8}(1,3,5)\Big)_6 \tag{$\mathcal{B}t4.4$}$$

$$2\times\Big(\frac{1}{2}(1,1,1)\Big)_1, 2\times\Big(\frac{1}{4}(1,1,3)\Big)_1, 2\times\Big(\frac{1}{4}(1,1,3)\Big)_3 \tag{$\mathcal{B}t4.5$}$$

Table 1.5. Subsets $\mathcal{B}t$ for order 5 actions:

$$2\times\Big(\frac{1}{5}(1,2,3)\Big)_1, \Big(\frac{1}{10}(1,3,7)\Big)_6 \tag{$\mathcal{B}t5.1$}$$

$$\Big(\frac{1}{5}(1,1,4)\Big)_1, \Big(\frac{1}{5}(1,1,4)\Big)_2, \Big(\frac{1}{5}(1,1,4)\Big)_3, \Big(\frac{1}{5}(1,1,4)\Big)_4 \tag{$\mathcal{B}t5.2$}$$

$$\Big(\frac{1}{5}(1,1,4)\Big)_1, \Big(\frac{1}{5}(1,1,4)\Big)_4, \Big(\frac{1}{5}(1,2,3)\Big)_1, \Big(\frac{1}{5}(1,2,3)\Big)_4 \tag{$\mathcal{B}t5.3$}$$

$$\Big(\frac{1}{5}(1,2,3)\Big)_1, \Big(\frac{1}{5}(1,2,3)\Big)_2, \Big(\frac{1}{5}(1,2,3)\Big)_3, \Big(\frac{1}{5}(1,2,3)\Big)_4 \tag{$\mathcal{B}t5.4$}$$

Table 1.6. Subsets $\mathcal{B}t$ for order 6 actions:

$$2\times\Big(\frac{1}{3}(1,1,2)\Big)_1,\Big(\frac{1}{4}(1,1,3)\Big)_2,\Big(\frac{1}{12}(1,5,7)\Big)_{10} \qquad (\mathcal{B}t6.1)$$

$$2\times\Big(\frac{1}{2}(1,1,1)\Big)_1,\Big(\frac{1}{3}(1,1,2)\Big)_1,\Big(\frac{1}{3}(1,1,2)\Big)_2,\Big(\frac{1}{6}(1,1,5)\Big)_1,\Big(\frac{1}{6}(1,1,5)\Big)_5 \qquad (\mathcal{B}t6.2)$$

Table 1.8. Subsets $\mathcal{B}t$ for order 8 actions:

$$\Big(\frac{1}{2}(1,1,1)\Big)_1,\Big(\frac{1}{4}(1,1,3)\Big)_1,\Big(\frac{1}{8}(1,3,5)\Big)_3,\Big(\frac{1}{8}(1,3,5)\Big)_7 \qquad (\mathcal{B}t8.1)$$

15 Remark. In the previous tables we omitted the redundant cases that are given by a multiple of the torsion divisor (i.e. we get the same singularities but all the l_Q are multiplied by an integer that is coprime with the order r of σ) because they represent the same $\mathcal{B}t$ but the new torsion divisor is now a multiple of the former one. For instance, $2\times\Big(\frac{1}{5}(1,2,3)\Big)_3,\Big(\frac{1}{10}(1,3,7)\Big)_8$ is ($\mathcal{B}t5.1$) considering 3σ instead of σ.

Now we can use this list of possible subsets $\mathcal{B}t$ to check which among all known Fano threefolds admit a Fano–Enriques quotient. We check three lists, which we do not reproduce here due to their size. The first two, of codimension 1 and 2, due to Reid and Fletcher respectively, can be found in [6]. The other one, of codimension 3, is due to Altınok and is in [1]. For the first list (Reid's codimension 1) we get:

16 Proposition. *Exactly 12 Fano–Enriques threefolds up to deformation can be obtained as quotients of Reid's 95 Fano hypersurfaces. They have torsion 2,3 and 5 and are given in Table 2.*

Proof. It is a case by case proof of the result. As a sample, we retake the case of Example 9 and Example 14: Let us suppose $\mathbb{Z}/(r)$ is acting on $Y_5\subset\mathbb{P}(1,1,1,1,2)$ (we explained in Remark 3 that the action on Y induces a diagonal action on $\mathbb{P}(1,1,1,1,2)$). Necessarily, (0:0:0:0:1) belongs to Y and is a fixed point by the action on $\mathbb{P}(1,1,1,1,2)$, since it is the only singular point on this weighted projective space. Therefore, a singularity of type $\frac{1}{2r}$ appears in the quotient X. In fact, the basket has to be $\mathcal{B}=\mathcal{B}t=\{\frac{1}{r_1},\dots,\frac{1}{r_l},\frac{1}{2r}\}$, with $r_i|r$. No ($\mathcal{B}tr.i$) satisfies this condition for $r=2,4,6,8$. There is a possibility for $\mathbb{Z}/(3)$, with the basket $\{4\times\frac{1}{3},\frac{1}{6}\}$ ($\mathcal{B}t$ 3.6), but it is also impossible because it should be $-K_X^3=\frac{5}{2r}=\frac{5}{6}$. This contradicts Remark 8, which implies that, for this basket, $-K_X^3 = \frac{3}{2} + 2k,\ k \in \mathbb{Z}$. Therefore, $r = 5$ and the only possible quotient is the example we already know (it is No. 2 in Table 2 below). Now we would play the game in Example 14 to observe these invariants lead to no other possibility. QED

17 Remark. For the following tables, we list these data for each element:

- the cover, which is a complete intersection $Y_{d_1,\dots,d_l} \subset \mathbb{P}(a_1,\dots,a_n)$ of hypersurfaces of degrees $d_1,\dots,d_l$
- the action on the cover
- $\mathcal{B}t :=$ {singularities in the basket where the torsion divisor is not trivial with their respective coefficients l_Q }.
- $\mathcal{B}\backslash\mathcal{B}t :=$ rest of the basket

Table 2. Fano–Enriques threefolds from codimension 1 Fano threefolds: No. 1

- **cover:** $Y_4 \subset \mathbb{P}(1,1,1,1,1)$
- **action:** $\mathbb{Z}/(5)$ acts by $(1,\epsilon,\epsilon^2,\epsilon^3,\epsilon^4)$, $\epsilon = e^{\frac{2\pi}{5}}$
- $\mathcal{B}t = \{\left(\frac{1}{5}(1,2,3)\right)_1, \left(\frac{1}{5}(1,2,3)\right)_2, \left(\frac{1}{5}(1,2,3)\right)_3, \left(\frac{1}{5}(1,2,3)\right)_4\} = (\mathcal{B}t5.4)$
- $\mathcal{B}\backslash\mathcal{B}t = \emptyset$

No. 2

- **cover:** $Y_5 \subset \mathbb{P}(1,1,1,1,2)$
- **action:** $\mathbb{Z}/(5)$ acts by $(1,\epsilon,\epsilon^3,\epsilon^4,\epsilon^2)$, $\epsilon = e^{\frac{2\pi}{5}}$
- $\mathcal{B}t = \{2 \times \left(\frac{1}{5}(1,2,3)\right)_1, \left(\frac{1}{10}(1,3,7)\right)_6\} = (\mathcal{B}t5.1)$
- $\mathcal{B}\backslash\mathcal{B}t = \emptyset$

No. 3

- **cover:** $Y_6 \subset \mathbb{P}(1,1,1,2,2)$
- **action:** $\mathbb{Z}/(3)$ acts by $(1,\epsilon,\epsilon^2,\epsilon,\epsilon^2)$, $\epsilon = e^{\frac{2\pi}{3}}$
- $\mathcal{B}t = \{3 \times \left(\frac{1}{3}(1,1,2)\right)_1, 3 \times \left(\frac{1}{3}(1,1,2)\right)_2\} = (\mathcal{B}t3.7)$

- $\mathcal{B}\backslash\mathcal{B}t = \{\frac{1}{2}(1,1,1)\}$

No. 4

- **cover:** $Y_8 \subset \mathbb{P}(1,1,1,2,4)$
- **action:** $\mathbb{Z}/(2)$ acts by $(+,-,-,-,-)$
- $\mathcal{B}t = \{8 \times \left(\frac{1}{2}(1,1,1)\right)_1\} = (\mathcal{B}t2.20)$
- $\mathcal{B}\backslash\mathcal{B}t = \{\frac{1}{2}(1,1,1)\}$

No. 5

- **cover:** $Y_9 \subset \mathbb{P}(1,1,1,3,4)$
- **action:** $\mathbb{Z}/(3)$ acts by $(1,\epsilon,\epsilon^2,\epsilon^2,\epsilon)$, $\epsilon = e^{\frac{2\pi}{3}}$
- $\mathcal{B}t = \{2 \times \left(\frac{1}{3}(1,1,2)\right)_1, \left(\frac{1}{12}(1,5,7)\right)_4\} = (\mathcal{B}t3.4)$
- $\mathcal{B}\backslash\mathcal{B}t = \emptyset$

No. 6

- **cover:** $Y_9 \subset \mathbb{P}(1,1,2,3,3)$
- **action:** $\mathbb{Z}/(3)$ acts by $(1,\epsilon,\epsilon^2,\epsilon,\epsilon^2)$, $\epsilon = e^{\frac{2\pi}{3}}$
- $\mathcal{B}t = \{4 \times \left(\frac{1}{3}(1,1,2)\right)_1, \left(\frac{1}{6}(1,1,5)\right)_4\} = (\mathcal{B}t3.6)$
- $\mathcal{B}\backslash\mathcal{B}t = \{\frac{1}{3}(1,1,2)\}$

No. 7

- **cover:** $Y_{12)} \subset \mathbb{P}(1,1,2,3,6)$
- **action:** $\mathbb{Z}/(2)$ acts by $(+,-,-,-,-)$
- $\mathcal{B}t = \{4 \times \left(\frac{1}{2}(1,1,1)\right)_1, 2 \times \left(\frac{1}{4}(1,1,3)\right)_2\} = (\mathcal{B}t2.19)$
- $\mathcal{B}\backslash\mathcal{B}t = \{\frac{1}{3}(1,1,2)\}$

No. 8

- **cover:** $Y_{14} \subset \mathbb{P}(1,1,2,4,7)$
- **action:** $\mathbb{Z}/(2)$ acts by $(+,-,-,-,-)$
- $\mathcal{B}t = \{2 \times \left(\frac{1}{2}(1,1,1)\right)_1, \left(\frac{1}{4}(1,1,3)\right)_2, \left(\frac{1}{8}(1,3,5)\right)_4\} = (\mathcal{B}t2.14)$
- $\mathcal{B}\backslash\mathcal{B}t = \{\frac{1}{2}(1,1,1)\}$

No. 9

- **cover:** $Y_{16} \subset \mathbb{P}(1,1,2,5,8)$
- **action:** $\mathbb{Z}/(2)$ acts by $(+,-,-,-,-)$
- $\mathcal{B}t = \{3 \times \left(\frac{1}{2}(1,1,1)\right)_1, \left(\frac{1}{10}(1,3,)\right)_5\} = (\mathcal{B}t2.12)$
- $\mathcal{B}\backslash\mathcal{B}t = \{\frac{1}{2}(1,1,1)\}$

No. 10

- **cover:** $Y_{16} \subset \mathbb{P}(1,1,3,4,8)$
- **action:** $\mathbb{Z}/(2)$ acts by $(+,-,-,-,-)$
- $\mathcal{B}t = \{5 \times \left(\frac{1}{2}(1,1,1)\right)_1, \left(\frac{1}{6}(1,1,5)\right)_3\} = (\mathcal{B}t2.18)$
- $\mathcal{B}\backslash\mathcal{B}t = \{\frac{1}{4}(1,1,3)\}$

No. 11

- **cover:** $Y_{20} \subset \mathbb{P}(1,2,3,5,10)$
- **action:** $\mathbb{Z}/(2)$ acts by $(+,-,-,-,-)$
- $\mathcal{B}t = \{\left(\frac{1}{2}(1,1,1)\right)_1, 2 \times \left(\frac{1}{4}(1,1,3)\right)_2, \left(\frac{1}{6}(1,1,5)\right)_3\} = (\mathcal{B}t2.16)$
- $\mathcal{B}\backslash\mathcal{B}t = \{\frac{1}{5}(1,2,3)\}$

No. 12

- **cover:** $Y_{24} \subset \mathbb{P}(1,2,3,7,12)$
- **action:** $\mathbb{Z}/(2)$ acts by $(+,-,-,-,-)$