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Fano Configurations in Subregular Planes

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Abstract. In this article a wide variety of odd-order translation planes, constructed from Desarguesian planes by multiple derivation, are shown to admit Fano configurations. For example, we show that subregular planes of odd order q^2 obtained by the replacement of at most $(q-1)/4$ disjoint nets admit Fano configurations. Further, for s odd and greater than 1, any subregular plane of order q^2 can be embedded into a subregular plane of order q^{2s} that contains the original plane as a subplane and admits Fano configurations.

Keywords: Fano subplane, derivation, replaceable net

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1 Introduction

This article concerns an old problem in finite geometry: *Given a finite projective plane of order n, for what values k are there subplanes of order k*? Although there are some old results that describe subplanes of Desarguesian projective planes, there is no general theory as to the nature or order of subplanes of an arbitrary projective plane of order n. However, there is something that can be said regarding finite *translation* planes, which are the focus of this paper. It is obvious that any translation plane of order p^r, for p a prime, has a subplane of order p. (For the benefit of readers to whom this is not obvious, we review relevant properties of translation planes in the first few sections.)

More interestingly, in a translation plane of order p^r there are often subplanes of order 2: these are the complete quadrangles ABCD whose diagonal points, $AB \cap CD = P, AC \cap BD = Q$, and $AD \cap BC = R$, are collinear. This configuration, consisting of seven points and seven lines with three points per line and three lines per point, is called a ***Fano configuration*** or ***Fano subplane*** (even though Fano investigated those quadrangles whose diagonal points are *not* collinear). Motivated by a result of H. Lenz, Hanna Neumann [18] investigated the quadrangles of the Hall planes of odd order. (The *Hall planes* form a widely studied class of translation planes.) For each Hall plane, Neumann found a pair of quadrangles, one with collinear diagonal points (that is, a Fano configuration) and one without. Her work led to the conjecture that *a finite projective plane might be non-Desarguesian if and only if it contains both types of quadrangles* (although such a conjecture does not appear in her paper). In this context, it is natural to consider finite projective planes in which every quadrangle generates a Fano configuration. By coincidence, two years after Neumann's article appeared, Andrew Gleason, unaware of that paper, was investigating a question that arose from the work of Ruth Moufang; that investigation produced his important theorem [13]: *A finite projective plane in which every quadrangle generates a Fano configuration is necessarily Desarguesian.* Since all known even-order projective planes contain a Fano subplane as part of their construction, it follows that the conjecture holds in all *known* planes of even order. More than fifty years after Neumann's initial contribution, the odd-order case of the conjecture remains open; in fact, not much has been accomplished in this direction. We propose here a systematic approach to the conjecture for subregular planes. Although our basic idea is much more general, in this article we consider only subregular planes of odd order, and we ask which subset of this class must admit Fano configurations. Even though we are unable to provide a complete description, our main result, the 1/4-Theorem (Theorem 14) found in Section 11, comes close in some respects:

Theorem. Let Σ be a Desarguesian affine plane of odd order q^2 and let Λ be a set of $k+1$ disjoint derivation sets. Let π denote the translation plane obtained by a multiple derivation of the corresponding derivable nets. If $k < (q-1)/4$ then the projective extension of π contains a Fano configuration.

Perhaps our most general and surprising result comes later in that section:

Theorem. Let π_1 be any subregular plane of odd order q^2. Then for any odd integer s, $s > 1$, there exist subregular planes π_s of order q^{2s} that contain π_1 as a subplane and admit Fano configurations.

For André planes in Section 17 we prove

Theorem. An André plane of odd order q^2 either admits Fano configurations, or there is a derivable net $\mathcal{D}$ such that the André plane obtained by replacing $\mathcal{D}$ admits Fano configurations.

In other words, even though we do not yet know whether all non-Desarguesian André planes must have Fano configurations, we know that from any André plane without a Fano configuration we can derive one with a Fano configuration by replacing just one derivable net. We will assess how inclusive our results are and show how they apply to specific projective planes.

2 A Primer on Translation Planes and Derivation

In an affine plane π, a ***translation*** is a collineation that fixes no point of π, but fixes each parallel class and fixes all lines of one parallel class. A ***translation plane*** is an affine plane that admits a group consisting of translations, the ***translation group***, that acts transitively on the points of the plane. Fundamental work of André [3] shows that for each translation plane there is an associated vector space V of dimension $2k$ over $GF(p)$, for p a prime, such that the points of the affine plane may be identified with the vectors of V, while the lines of the plane correspond to translates of k-dimensional subspaces that intersect one another precisely in the zero vector. This implies that the translation group of the affine plane is elementary Abelian of order n^2, and n is a prime power. We shall restrict our attention here to the case where $n = q^2$ for $q = p^r$, so that V is 4-dimensional over $GF(q)$.

In this context, a ***spread*** for a translation plane of order q^2 is a set of $q^2 + 1$ 2-dimensional $GF(q)$-vector subspaces that partition the nonzero vectors of V (that is, they mutually intersect in the zero vector and they cover the vectors of V); the spread elements are called ***components.*** The lines of the translation plane then are the vector translates of the components. Similarly, a ***partial spread*** is a set of $k \leq q^2 + 1$ 2-dimensional vector subspaces that intersect one another exactly the zero vector. A ***net*** of degree k is then a point-line geometry, where the *points* are the vectors of V and the *lines* are the vector translates of the partial spread.

It is clear that nets and partial spreads are equivalent objects.

A standard way to construct new translation planes from a given translation plane of square order is by *derivation*, which we discuss below. However, it is simply a redesignation of certain affine subplanes of order q as lines in a newly constructed translation plane. Given a spread S for a translation plane π of order q^2, with associated 4-dimensional vector space V, every 2-dimensional vector subspace W, which is not a component of π, gives rise to an affine subplane of order q, called a ***Baer*** subplane of π. To see this, we note that the set of $q + 1$ 1-dimensional $GF(q)$-subspaces of W form a spread of W, thereby producing an associated translation plane, which is then a subplane of π of order q. Hence, in the following, we shall identify subplanes of order q with the 2-dimensional $GF(q)$-subspaces that give rise to them.

3 Derivation

An affine plane π of order q^2 has $q^2 + 1$ parallel classes. As is customary, we will sometimes consider each parallel class to be one of the $q^2 + 1$ points on the line at infinity of the projective extension π^+ of π. Since there is a natural correspondence between a set of parallel classes and the net it determines (that is, the set of lines contained in those parallel classes), it will be convenient to employ the same notation to represent a set of parallel classes, the set of points representing those parallel classes, and the set of lines through those points; the intended interpretation will always be clear from the context.

A ***derivation set*** (or, if you prefer, ***derivation net***) in π is a subset $\mathcal{D}$ of $q+1$ parallel classes with the following property: For each pair of distinct points A and B of π such that AB is a line of $\mathcal{D}$ (by which we mean a line in the net determined by $\mathcal{D}$), there is a unique subplane of order q which contains the points A and B, and whose set of parallel classes is precisely $\mathcal{D}$. If π contains a derivation set then it is called ***derivable***.

We note that every derivation set is clearly equivalent to a partial spread of $q+1$ components that produce a corresponding net satisfying the condition on subplanes. The partial spread is said to be a **derivable partial spread** and the corresponding net is said to be a **derivable net**. Thus, in this article, there is an equivalence between *derivation sets*, *derivable partial spreads*, and *derivable nets.*

The importance of this concept is seen in the following theorem due to Ostrom.

1 Proposition. *Let π be an affine plane of order q^2, and let $\mathcal{D}$ be a derivation set. Define a point-line geometry whose points are the points of π and whose lines are of two types:*

- *if the line AB of π belongs to $\mathcal{D}$, then the new line determined by A and B is the unique subplane of order q with parallel classes in $\mathcal{D}$;*
- *if the line AB of π does not belong to $\mathcal{D}$, then it remains a line of the new plane.*

The new geometry is an affine plane of order q^2; moreover, if π is a translation plane, then so is the derived plane.

The new plane is said to be ***$\mathcal{D}$-derived from*** π. If π happens to contain a number of disjoint derivation sets (in the sense that no two of them have a parallel class in common), then we may independently apply the derivation process to each of them. We call this operation ***multiple derivation***, and denote the *replacement set* that consists of the disjoint derivation sets by Λ. We call a translation plane ***subregular*** if it can be obtained by multiple derivation from a Desarguesian affine plane of order q^2.

2 Remark. There are other ways to represent the replacement set Λ. Perhaps the most common: A derivation set corresponds in a natural way to a regulus of a spread in the vector space, mentioned earlier, that is associated with a translation plane. Derivation corresponds to replacing the regulus by its opposite regulus, a process that is the source of the terminology *subregular* for our multiply derived planes. Because of the limited scope of our paper, we will bypass the higher dimensional model, although it will serve as an important tool in the related article [14]. Only in the final section of this paper, where we discuss the literature, will we use the terminology of reguli.

4 Representation of Baer Subplanes

Recall that the affine subplane π_0 of order q in an affine plane π of order q^2 (which appears in the definition of derivation) is called a ***Baer subplane***. Taken projectively, that is extend π and π_0 to the associated projective plane and projective subplane, respectively, say π^+ and π_0^+. Then π_0^+ has the property that every line of π^+ meets π_0^+ in a one point point or in the $q+1$ points of a ***Baer subline***. In the projective interpretation, the line at infinity of any of these affine subplanes is a Baer subline $\mathcal{D}$ of π^+. Thus, we can consider a replacement set Λ to be a subset of points on the line at infinity consisting of disjoint Baer sublines of the associate projective plane π^+.

If the coordinates (x, y) of the points of a Desarguesian affine plane Σ are elements of $GF(q^2)$, then for each $m, n \in GF(q^2),\ m \neq 0$, the set

$$y = x^q m + xn$$

represents one of the $(q^2-1)q^2$ Baer subplanes that meet the line $x = 0$ only at the origin, by our agreement that the spread of 1-dimensional $GF(q)$-subspaces of this particular 2-dimensional $GF(q)$-subspace produces the associated affine subplane of order q. We shall denote this subplane by $B(m, n)$. As an example, consider the lines with slopes in the set $\mathcal{D} = \{(m);\ m^{q+1} = 1\}$: For all m for which $m^{q+1} = 1$, the lines $y = xm$ and their translates are replaced in the derived plane by the Baer subplanes $B(m, 0) = \{(x, y);\ y = xm^q\}$ and their translates.

3 Notation. We shall always denote the parallel classes of lines in Desarguesian planes by field elements enclosed in parentheses. We shall represent the net determined by those parallel classes by the components through the origin.

5 Linear Subsets

In an affine Desarguesian plane Σ, the points of a Baer subplane are fixed by a unique involution, called a ***Baer involution***, that arises from a collineation of Σ. For example, for $\delta \in GF(q)^*$ and any $m \in GF(q^2)$ for which $m^{q+1} = \delta$, the points of the subplane $y = x^q m$ are fixed by the mapping induced by the involution

$$(x, y) \longmapsto \left(\frac{my^q}{\delta}, mx^q \right).$$

By definition, any such involution has orbits of length 2 outside of the fixed-point set. These involutions induce on the line at infinity an involution that fixes the points of a Baer subline $\mathcal{D}$, and interchanges pairs of the points at infinity that are not in $\mathcal{D}$. We call such an orbit of length 2 a ***pair of conjugate points of*** $\mathcal{D}$. Moreover, a set Λ of disjoint derivation sets is defined to be ***linear*** if the derivation sets have a common pair of conjugate points. For example, any union of sets of the form $\mathcal{D}_\delta = \{(m);\ m^{q+1} = \delta\}$ is linear: the slopes (0) and (∞) are interchanged by any involution that for some $\delta \in GF(q)$ fixes the points of $\mathcal{D}_\delta$.

6 The André Planes

The Hall planes were investigated at first using coordinates over an algebraic structure obtained by distorting the multiplication of a finite field. Albert [2] showed that any Hall plane may be constructed from an affine Desarguesian plane by deriving a single derivation set. We now consider a special family of subregular planes that includes the Hall planes.

4 Notation. In an affine Desarguesian plane Σ coordinatized by $GF(q^2)$, for $\delta \in GF(q)^*$ define

$$R_\delta = \left\{ y = xm;\ m^{q+1} = \delta \right\} \quad \text{and} \quad R^*_\delta = \left\{ y = x^q m;\ m^{q+1} = \delta \right\}.$$

Note that both these sets cover the same points of Σ. The slopes of the lines of R_δ determine a Baer subline on the line at infinity, which we also denote by R_δ. Note that one may read R_δ or R^*_δ either as a partial spread or as the net determined by the partial spread.

5 Proposition. *For $\delta \in GF(q)^*$, R_δ determines a derivation set whose associated subplanes are the elements of R^*_δ. Furthermore,*

(1) if $\alpha \neq \beta$ then the nets R_α and R_β have no line in common, whence they determine disjoint derivation sets, and

(2) each derivation set R_δ has (0) and (∞) as a pair of conjugate points.

A translation plane constructed by multiply deriving any union of the sets R_δ is called an ***André plane***. Observe that statement (1) says that as δ runs through the nonzero elements of $GF(q)$, the sets R_δ partition the set of $q^2 - 1$

lines through the origin distinct from $x = 0$ and $y = 0$; statement (2) says that the replacement set for an André plane is linear.

6 Remark. If the replacement net consists of the union of all R_δ (that is, using all $q-1$ values of δ), the resulting subregular plane will be Desarguesian. Conversely, any subregular plane obtained from a Desarguesian plane by the derivation of exactly $q-1$ disjoint derivation sets is Desarguesian by results of Orr [19] for q odd, and of Thas [23] for q even.

7 Lifting Subregular Planes

Although the concepts reviewed above can be substantially generalized, in this article we restrict our attention to the subregular planes of odd order. Because the multiple derivation of $q-1$ disjoint derivation sets leads to another Desarguesian plane, it is not true that all subregular planes admit Fano configurations. However, we show that if we derive fewer than roughly a quarter of the possible $(q-1)$ disjoint derivation sets, then the resulting planes always admit Fano configurations. In one sense this condition is not as restrictive as it might sound. Recently, the second author provided a construction technique by which a subregular plane of order q^2 can be embedded in a subregular plane of order q^{2s}, where s is odd. We will prove that the subregular plane of order q^{2s} will admit Fano configurations, even if the original subregular plane might not.

We have been discussing how a subregular plane is obtained from a Desarguesian plane Σ_{q^2} over the field $GF(q^2)$ by multiply deriving a set of k disjoint derivation sets. We now embed Σ_{q^2} in a Desarguesian affine plane $\Sigma_{q^{2s}}$ over the field $GF(q^{2s})$ and reinterpret those derivation sets in the larger plane.

7 Proposition. (*Johnson [16]*) *If Λ is a set of k disjoint derivation sets in the Desarguesian plane Σ_{q^2} of order q^2 and s is an odd integer, then there is a canonical way to construct from Λ a set Λ' of k disjoint derivation sets in the Desarguesian plane $\Sigma_{q^{2s}}$ of order q^{2s}. Furthermore, if π is the subregular plane multiply derived from Σ_{q^2}, and π_s is the subregular plane multiply derived from $\Sigma_{q^{2s}}$, then π is an affine subplane of π_s.*

8 The Basic Construction and Its Implications

The construction that serves as the foundation for our work is based on a simple idea. As we have seen in the introductory sections, we obtain a subregular plane π (and, indeed, most of the known finite affine planes) from a Desarguesian plane Σ by designating a subset Λ of its parallel classes. The new plane has the same affine points as the old plane; lines with slopes outside Λ will be lines in both planes, while lines of the old plane with slopes in Λ are distorted (by multiple derivation) to become lines in the new plane. Our idea is to start with a complete quadrangle $ABCD$ in the projective extension of Σ, with its six sides and its three diagonal points P, Q, R, that remains a quadrangle in the projective extension of the derived plane. We arrange that the set Λ excludes the slopes of the six sides (so that those six lines will also be lines of the derived plane) while the diagonal points lie in the same component of the partial spread defined by the derivation set. In other words, P, Q, R will lie on a line of the new plane. While we work in the Desarguesian affine plane of order q^2 and restrict consideration in this article to subregular translation planes, the ideas will work just as well in a much more general setting. In fact, the second author continues this line of inquiry in a related article on translation planes with spreads in $PG(2n-1, q)$, for $n > 2$ [14].

9 The Basic Construction

We begin our basic construction by choosing coordinates so that both

(1) (∞) and (0) are parallel classes that are not in Λ, and

(2) $R_1 = \{y = xm;\ m^{q+1} = 1\} \subseteq \Lambda$.

We shall postpone until the next section our discussion of when one can make such a choice. Next, for $c \in GF(q^2),\ c \neq 0, 1$, we define the quadrangle $ABCD$ by

$$A = (\infty),\ \ B = \left(0, \frac{c}{c-1}\right),\ \ C = (c, 0),\ \ D = (1, 0).$$

Because we are working in a Desarguesian plane we easily determine the equations for the six sides:

$$\begin{aligned}
&AD : x = 1, \quad AB : x = 0, \quad AC : x = c, \quad CD : y = 0,\\
&BD : y = -x\left(\frac{1}{1-c^{-1}}\right) + \frac{c}{c-1}, \ \text{ and } BC : y = -x\left(\frac{1}{c-1}\right) + \frac{c}{c-1}.
\end{aligned}$$

It follows at once that

$$P = AB \cap CD = (0,0), \qquad Q = AC \cap BD = (c, -c),$$
$$R = AD \cap BD = (1,1).$$

We have seen that the set of Baer subplanes that cover the same points as R_1 is $R_1^* = \{y = x^q m;\ m^{q+1} = 1\}$ (recall again, the notation is only giving the partial spread; the net lines are the translates of the partial spread components). One easily checks that if c happens to satisfy $c^q = -c$ then P, Q, and R are points of $y = x^q$, which is in R_1^*. As a consequence, these points would lie on a line of our derived plane π, completing our basic construction. In summary,

8 Lemma. *Let π be a subregular translation plane of odd order q^2 obtained from the affine Desarguesian plane Σ of order q^2 by multiple derivation of a set Λ of disjoint derivable sets. When coordinates have been chosen so that (∞) and (0) are parallel classes that are not in Λ while $R_1 = \{y = xm;\ m^{q+1} = 1\} \cap \ell_\infty \subset \Lambda$, if $c \in GF(q^2)$ satisfies $c^q = -c$, while*

$$\left(\frac{1}{c^{-1} - 1}\right) \quad \textit{and} \quad \left(\frac{1}{1 - c}\right)$$

are slopes of Σ that are not in Λ, then the projective extension π^+ of π admits Fano configurations.

Proof. We must verify that the conditions on c are sufficient for the quadrangle $ABCD$ of the basic construction to generate a Fano configuration. Specifically, since $A = (\infty) \notin \Lambda$, A is a point of the derived projective plane π^+ and the lines AD, AB, AC of Σ remain lines of the derived affine plane π. Same with CD, BD, and BC with respective slopes (0), $\left(\frac{1}{c^{-1}-1}\right)$, and $\left(\frac{1}{1-c}\right)$ that are assumed to lie outside Λ. Because we also assume that $c^q = -c$, it follows that P, Q, R (which form a triangle in the Baer subplane $y = x^q$ of Σ) are collinear in π. In other words, A, B, C, D, P, Q, R are the points of a Fano subplane of π^+. $\boxed{QED}$

Here is a simple illustration of how our lemma works; we look at more substantial applications in subsequent sections.

9 Theorem. [*Neumann [18]*] *The projective Hall planes of odd order q^2 admit Fano configurations.*

Proof. The affine Hall plane of order q^2 can be constructed from the Desarguesian plane by the derivation of just one net, which we take to be R_1. Note that (0) and (∞) lie outside R_1. Choose c to be one of the $q-1$ nonzero elements of $GF(q^2)$ for which $c^q = -c$. We need to prove that $\left(\frac{1}{c^{-1}-1}\right)$ and $\left(\frac{1}{1-c}\right)$ are not in the set of parallel classes (m) for which $m^{q+1} = 1$. Note that

$$\left(\frac{1}{1-c}\right)^{q+1} = \left(\frac{1}{1-c^q}\right)\left(\frac{1}{1-c}\right) = \left(\frac{1}{1+c}\right)\left(\frac{1}{1-c}\right) = \frac{1}{1-c^2},$$

which equals 1 if and only if $c^2 = 0$, a contradiction. Similarly,

$$\begin{aligned}\left(\frac{1}{c^{-1}-1}\right)^{q+1} &= \left(\frac{1}{c^{-q}-1}\right)\left(\frac{1}{c^{-1}-1}\right)\\ &= -\left(\frac{1}{c^{-1}+1}\right)\left(\frac{1}{c^{-1}-1}\right) = \frac{1}{1-c^{-2}},\end{aligned}$$

which equals 1 if and only if $c^{-2} = 0$, a contradiction. $\boxed{QED}$

10 Planes of Odd Order Admitting Fano Configurations

We turn now to the question of when it might be possible to choose coordinates as required by the basic construction of Section 9. Let Σ be a Desarguesian affine plane of odd order q^2 and let Λ be a set of disjoint derivation sets of Σ. Whereas any member of Λ can be represented by $R_1 = \{y = xm;\ m^{q+1} = 1\}$, we must in addition choose a pair of points for (0) and (∞) that are conjugate with respect to R_1; that is, the unique involution of the line at infinity that fixes each slope of R_1 must interchange (0) with (∞). It is clear that this can always be accomplished if the number of disjoint derivation sets is sufficiently small. Specifically,

10 Lemma. *If the derivation net Λ of Σ consists of $k+1$ disjoint derivation sets and*

$$k < \frac{q(q-1)}{2(q+1)},$$

then coordinates may be chosen so that one derivation set of Λ is represented by $R_1 = \{y = xm;\ m^{q+1} = 1\}$, while (0) with (∞) are slopes exterior to Λ that are conjugate with respect to R_1.

Proof. Note that there are exactly $q(q-1)/2$ conjugate pairs for R_1. A given derivation net can cover a single point of at most $(q+1)$ point pairs. We are given $k+1$ disjoint sets in Λ, so among the $q(q-1)$ infinite points outside R_1 there are $k(q+1)$ points covered by the nets of $\Lambda - R_1$, and $q(q-1) - k(q+1)$ uncovered points. Were there no conjugate pair for R_1 outside Λ, then $\Lambda - R_1$ must hit at least one point of each conjugate pair, in which case we would have

$$k(q+1) \geq q(q-1)/2.$$

Hence, if $k < q(q-1)/(2(q+1))$, the desired conjugate pair exists. Furthermore, since the stabilizer of R_1 in the automorphism group (namely $PGL(2, q^2)$) of the points at infinity of Σ is transitive on the $q(q-1)$ infinite points outside R_1, the slopes of any conjugate pair can be labeled (0) and (∞). $\boxed{QED}$

Combining the previous lemma with Lemma 8 of Section 9, we obtain the following theorem.

11 Theorem. *Let Λ be a set of $k+1$ disjoint derivation sets of the Desarguesian plane Σ such that $k < q(q-1)/(2(q+1))$. Assume that in the translation plane π obtained by the multiple derivation of Λ, there is an element $c \in GF(q^2)$ such that* (a) $c^q = -c$ *and* (b) *the pair of slopes* $\left(\frac{1}{c^{-1}-1}\right)$ *and* $\left(\frac{1}{1-c}\right)$ *lie outside* Λ.

Then π is a subregular translation plane admitting a Fano configuration.

11 The $1/4$-Theorem and the Lifting Theorem

We now study some of the subregular planes that satisfy the conditions of Theorem 11. Recall that all Baer subplanes of the derivation nets in Λ trivially intersect $x = 0$ (and $y = 0$) and hence may be represented in the form $y = x^q m + xn$, where $m \neq 0$; moreover, we denote the unique net containing $y = x^q m + xn$ as a Baer subplane by $B(m, n)$.

12 Lemma. *If $c^q = -c$ for $c \in GF(q^2)$ then the slope $\left(\frac{1}{1-c}\right)$ is in $B\left(\frac{1}{2}, \frac{1}{2}\right)$, while $\left(\frac{1}{c^{-1}-1}\right)$ is a slope in $B\left(\frac{1}{2}, -\frac{1}{2}\right)$.*

Proof. For the first claim we merely need to check that $y = x/(1-c)$ intersects $y = x^q/2 + x/2$ in a subline. This intersection occurs if and only if the equation $x/(1-c) = x^q/2 + x/2$ has a nonzero solution for x. Since $x^{(1-q)(1+q)} = 1$, this is equivalent to having a solution to

$$\left(\frac{1}{2} - \frac{1}{1-c}\right)^{q+1} = \frac{1}{2^{q+1}} = \frac{1}{4}.$$

But

$$\begin{aligned}
\left(\frac{1}{2} - \frac{1}{1-c}\right)^{q+1} &= \left(\frac{1}{2} - \frac{1}{1-c}\right)\left(\frac{1}{2} - \frac{1}{(1-c)^q}\right) \\
&= \left(\frac{1}{2} - \frac{1}{1-c}\right)\left(\frac{1}{2} - \frac{1}{(1+c)}\right) \\
&= \frac{1}{4} - \frac{1}{2}\left(\frac{1}{(1-c)} + \frac{1}{(1+c)}\right) + \frac{1}{1-c^2} \\
&= \frac{1}{4} - \frac{1}{2}\left(\frac{2}{1-c^2}\right) + \frac{1}{1-c^2} \\
&= \frac{1}{4}.
\end{aligned}$$

The remaining part of the lemma follows similarly. $\boxed{QED}$

Note that the Baer subplanes $y = x^q/2 \pm x/2$ intersect $y = 0$ in the points x for which

$$x^q = \mp x;$$

thus, each Baer subplane must intersect $y = 0$ in a Baer subline. The nets $B\left(\frac{1}{2}, \pm\frac{1}{2}\right)$, therefore, both share $y = 0$, whence neither net can be in Λ. In summary, we have

13 Lemma. i) $B\left(\frac{1}{2}, \frac{1}{2}\right)$ *consists of slopes* (0)*,* (1)*, and*

$$\left\{\left(\frac{1}{1-c}\right) \textit{ such that } c^q = -c\right\};$$

ii) $B\left(\frac{1}{2}, -\frac{1}{2}\right)$ *consists of slopes* (0)*,* (-1)*, and*

$$\left\{\left(\frac{1}{c^{-1}-1}\right) \textit{ such that } c^q = -c\right\};$$

and

iii) *both* $y = x$ *and* $y = -x$ *are in* R_1*.*

We are now in a position to prove two of our main results.

14 Theorem. *Let* Σ *be a Desarguesian affine plane of odd order* q^2*, and let* Λ *be a set of* $k+1$ *disjoint derivation sets and* π *denote the corresponding translation plane obtained by multiple derivation. If* $k + 1 < \frac{(q+3)}{4}$ *then the projective extension of* π *contains a Fano configuration.*

Proof. Since k is small enough, Theorem 11 will apply if we can find an appropriate value of $c \in GF(q^2)$. We have k nets disjoint from R_1; it is possible for each to intersect at most two components from $B(\frac{1}{2}, \frac{1}{2})$, and two from $B(\frac{1}{2}, -\frac{1}{2})$. We assume that $k < (q-1)/4$ and we seek a pair of slopes $(\frac{1}{1-c}), (\frac{1}{c^{-1}-1})$ that is missed by Λ. There are $q - 1$ nonzero candidates c for which $c^q = -c$, and each net of $\Lambda - R_1$ can contain at most four of the resulting slopes—two of each type. That accounts for strictly fewer than $4 \cdot (q-1)/4 = q - 1$ values of c, leaving at least one c and a corresponding pair of slopes $(\frac{1}{1-c}), (\frac{1}{c^{-1}-1})$ as desired. We can do a bit better when $(q-1)/2$ is odd: there must be some net of $\Lambda - R_1$ that shares at most three such slopes, instead of four. Thus, the desired element c will be guaranteed even if $k = (q-3)/4$. In summary, we have seen that when $(q-1)/2$ is even, the number of derivation nets should be less than $1 + (q-1)/4 = (q+3)/4$; when $(q-1)/2$ is odd, the number should be less than $1 + (q+1)/4 = (q+5)/4$. Hence, we have guaranteed the existence of the sought-for element c for any subregular plane constructed from no more than $(q+3)/4$ disjoint derivable nets. $\boxed{QED}$

As an immediate consequence of the previous theorem, we deduce a result that applies to any subregular plane.

15 Theorem. *Let π_1 be any subregular plane of odd order q^2. Then for any odd integer s, $s > 1$, there exist subregular planes π_s of order q^{2s} that contain π_1 as a subplane and whose projective extension admit Fano configurations.*

PROOF. π_1 may be constructed by the replacement of $k \leq q-2$ derivable nets. By Proposition 7, the corresponding translation plane π_s containing π_1 as a subplane may also be constructed by the replacement of k nets in the Desarguesian affine plane of order q^{2s}. Since $(q-2) \leq [(q^s-1)/4]$, for $s > 1$, it follows that π_s^+ admits Fano configurations by Theorem 14. $\boxed{QED}$

12 André Planes

When a subregular plane is an André plane, then the linear property of the replacement set used in its derivation (described in Section 6) allows the stronger version of the 1/4-Theorem that appears in Section 14. Here we investigate some immediate consequences of that property.

16 Lemma. *An André plane can be derived from a Desarguesian plane using no more than $(q-1)/2$ disjoint nets.*

PROOF. In Section 6 we observed that by starting with the Desarguesian affine plane Σ, multiple derivation by the maximum number of $q-1$ André nets produces a Desarguesian plane Σ'. Hence, if one is to replace a set of k disjoint nets in Σ, then this is equivalent to replacing $(q-1)-k$ in Σ'. $\boxed{QED}$

After several further preliminary results dealing with André planes, we will show that if the number of nets in a replacement set is no bigger than roughly $(3/8)q$, then any such plane of odd order admits Fano configurations. In other words, we improve the 1/4-Theorem by some $q/8$ nets, but still fall short of the maximum described in the previous lemma (namely, $4(q-1)/8$).

If we specialize the proof of the 1/4-Theorem (in the previous section) to an André plane, the question of the existence of Fano configurations boils down to asking when an André net

$$R_\delta = \left\{y = xm;\ m^{q+1} = \delta\right\}$$

intersects the nets $B\left(\frac{1}{2}, \frac{1}{2}\right)$ and $B\left(\frac{1}{2}, -\frac{1}{2}\right)$. Recall from Lemma 13 that R_1 shares $y = x$ with $B\left(\frac{1}{2}, \frac{1}{2}\right)$ and $y = -x$ with $B\left(\frac{1}{2}, -\frac{1}{2}\right)$. Also, $y = 0$ is a component of both of these nets. There are $q-2$ values of $\delta \neq 1$ (and thus $q-2$ other André nets R_δ) and there are $q-1$ further components in each of $B\left(\frac{1}{2}, \pm\frac{1}{2}\right)$.

In the proof preceding Lemma 13 we saw that $y = mx$ intersects $B\left(\frac{1}{2}, \frac{1}{2}\right)$ in a subline if and only if $1/4 = (1/2-m)^{q+1}$. In that case $m^{q+1} = \frac{1}{2}(m+m^q) = \delta$, whence $m^2 + \delta = 2\delta m$. The discriminant of $x^2 - 2\delta x + \delta = 0$ is $2\sqrt{\delta(\delta-1)}$, so that a solution to the quadratic requires that $\delta(\delta-1)$ be a square. For $\delta \neq 0, 1$

there are two distinct solutions whenever there is a solution. We have proved the following lemma.

17 Lemma. *The parallel classes of each of the nets $B\left(\frac{1}{2}, \pm\frac{1}{2}\right)$ are covered by $y = 0$, R_1, and exactly $(q-1)/2$ other André nets.*

By Lemma 16 we may assume that there are at most $(q-1)/2$ André nets in our replacement set. Note that $y = xm$ intersects $B\left(\frac{1}{2}, \frac{1}{2}\right)$ if and only if $y = -m$ intersects $B\left(\frac{1}{2}, -\frac{1}{2}\right)$. Since $(-m)^{q+1} = m^{q+1}$, the set of $(q-1)/2$ nets exterior to R_1 hitting $B\left(\frac{1}{2}, \frac{1}{2}\right)$ are exactly those hitting $B\left(\frac{1}{2}, -\frac{1}{2}\right)$. To cover at least one element of the pair of slopes $\left\{\left(\frac{1}{1-c}\right), \left(\frac{1}{c^{-1}-1}\right)\right\}$ when $c^q = -c$ requires that there be a δ in $GF(q)^*$ such that either $m^{q+1} = 1/(1-c^2) = \delta$ or $1/(1-c^{-2}) = \delta$. When $c^2 = -1$, these two expressions for δ are equal (namely, $\delta = 1/2$), and $R_{1/2}$ contains both $\left(\frac{1}{1-c}\right)$ and $\left(\frac{1}{c^{-1}-1}\right)$. Of course, if -1 is non-square, there exists an element $c \in GF(q^2) - GF(q)$ such that $c^2 = -1$, and this c also satisfies $c^q = -c$. In this way, we obtain the following lemma:

18 Lemma. *When -1 is non-square and there is no Fano configuration in a given André plane, then the André net $R_{1/2}$ must be in the replacement set Λ.*

Note that we may repeat all of this for any André net R_δ in place of R_1: Consider the collineation $(x, y) \longmapsto (x, yb)$ such that $b^{q+1} = \delta$. This map takes R_1 to R_δ. Furthermore, the nets $B\left(\frac{1}{2}, \pm\frac{1}{2}\right)$ map to $B\left(\frac{b}{2}, \pm\frac{b}{2}\right)$, slope $\left(\frac{1}{1-c}\right)$ maps to $\left(\frac{b}{1-c}\right)$, and $\left(\frac{1}{c^{-1}-1}\right)$ maps to $\left(\frac{b}{c^{-1}-1}\right)$. Let us now replace R_1 by $R_{1/2}$ in the proofs of the previous two lemmas; that is, we assume that the net $R_{1/2}$ is in Λ, and for $b^{q+1} = 1/2$ and $c^2 = -1$ we consider the slope $\left(\frac{b}{1-c}\right)$ in place of $\left(\frac{1}{1-c}\right)$. It will follow that either R_β is in Λ for $\beta = (\frac{b}{1-c})^{q+1} = 1/2^2$, or else we have a Fano configuration. Using this argument repeatedly, we see that for $\beta = 1/2^i$ there must be an André net R_β in Λ for all i. Hence, either a Fano configuration is obtained, or for $b^{q+1} = 2^{-1}$ the mapping $(x, y) \longmapsto (x, yb)$ is a collineation of the André plane.

19 Theorem. *Let π be an affine André plane of odd order q^2 where -1 is a non-square in $GF(q)$ (that is, $q \equiv -1 \bmod 4$).*

One of the following two possibilities must occur:

(1) the projective extension of π admits a Fano configuration, or

(2) there is a representation of π so that for $t^{q+1} = 2$, $\tau : (x, y) \longmapsto (x, ty)$ is a collineation of π.

In particular,

20 Corollary. *In the above situation, if there is no Fano configuration in* π *and if* R_δ *is an André net of the associated Desarguesian plane used in the multiple derivation to produce* π*, then so is* $R_{2\delta}$*.*

PROOF. If τ is a collineation of π then note that τ is also a collineation of the associated Desarguesian plane Σ, implying that $y = xm$ maps to $y = xtm$. Hence, if $m^{q+1} = \delta$ then $(tm)^{q+1} = 2\delta$. $\boxed{QED}$

We now consider a more general version of the previous theorem that applies when $c \neq -1$, but first we need two lemmas.

21 Lemma. *If for some* $c \in GF(q^2)$ *that satisfies* $c^q = -c$*, the slopes* $(1+c)$ *and* $(-(c^{-1}+1))$ *both lie outside the replacement net* Λ *of an André plane, then that plane has a Fano configuration.*

PROOF. The mapping $\sigma\colon (x,y) \longmapsto (y^q, x^q)$ fixes each point that satisfies $y = x^q$, interchanges $x = 0$ with $y = 0$, and maps the slopes $(1/(1-c))$ to $(1+c)$ and $(1/(c^{-1}-1))$ to $(-(c^{-1}+1))$. Moreover, because $m^{q+1} = m^{q(q+1)}$, σ leaves invariant every André net; in particular, it takes Λ to Λ. Observe what σ does to quadrangle $ABCD$ of the Basic Construction of Section 9: the slopes of the sides of the image quadrangle are $(0), (\infty)$, $(1+c)$ and $(-(c^{-1}+1)$; by assumption they will all lie outside Λ. Also, the image of the diagonal triangle will still lie in the replaced component $y = x^q$. That is, the image quadrangle will generate a Fano subplane of the derived plane. $\boxed{QED}$

22 Lemma. *For* $c^q = -c$*, the André net* R_δ *contains a slope in*

$$\left\{\left(\frac{1}{1-c}\right), \left(\frac{1}{c^{-1}-1}\right)\right\}$$

if and only if the André net $A_{1/\delta}$ *contains a slope in*

$$\left\{(1+c), (-(c^{-1}+1))\right\}.$$

PROOF. The André net R_β hits the pair $\left\{\left(\frac{1}{1-c}\right), \left(\frac{1}{c^{-1}-1}\right)\right\}$ if and only if either $\left(\frac{1}{1-c}\right)^{q+1} = \beta$ or $\left(\frac{1}{c^{-1}-1}\right)^{q+1} = \beta$; this occurs if and only if $1 - 1/\beta \in \{c^2, c^{-2}\}$. Similarly, it hits the pair $\left\{(1+c), (-(c^{-1}+1))\right\}$ if and only if $1-\beta \in \{c^2, c^{-2}\}$. $\boxed{QED}$

23 Proposition. *If* R_δ *is an André net in the replacement set* Λ *and*

- *there exists an element* $c \in GF(q^2)$ *such that* $c^q = -c$*, and either* $c^2 = 1 - 1/\delta$ *or* $c^{-2} = 1 - 1/\delta$*, and*
- *either both* R_{δ^2} *and* $R_{\delta(1-\delta)}$ *are not in* Λ*, or both* $R_{1/\delta}$ *and* $R_{1/(1-\delta)}$ *are not in* Λ*,*

then there is a Fano configuration.

PROOF. If $b^{q+1} = \delta$, then either we obtain a Fano configuration or the André nets cover at least one of the pair $\left\{\left(\frac{b}{1-c}\right), \left(\frac{b}{c^{-1}-1}\right)\right\}$ for all c's for which $c^q = -c$. In particular, we have $(b/(1-c))^{q+1} = \delta^2$ when $c^2 = 1 - 1/\delta$, while it equals $\delta(1-\delta)$ when $c^{-2} = 1 - 1/\delta$; thus, the first slope is covered if and only if $R_{\delta^2} \in \Lambda$ or $R_{\delta(1-\delta)} \in \Lambda$. The same two possibilities arise should the second slope be covered. In other words, if neither of these two nets is in Λ, then there would have to be a Fano configuration in the resulting André plane.

To prove the claim for the second pair of possibilities, we observe that according to the proof of Lemma 22, the assumption that $c^2 = 1 - 1/\delta$ is equivalent to the slope $(1+c)$ being in $R_{1/\delta}$ (because $(1+c)^{q+1} = 1-c^2 = 1-\left(1-\frac{1}{\delta}\right) = \frac{1}{\delta}$), and the slope $(-(1+c^{-1}))$ being in $R_{1/(1-\delta)}$ (because $(-(1+c^{-1}))^{q+1} = \frac{1}{1-\delta}$). Consequently, if neither $R_{1/\delta}$ nor $R_{1/(1-\delta)}$ are in Λ, we deduce from Lemma 21 that there is a Fano configuration. One sees easily that the assumption $c^{-2} = 1-1/\delta$ leads by a similar path to the same conclusion. $\boxed{QED}$

13 André Planes of Order p^2

We can specialize the previous results to when $q = p$ is a prime.

24 Theorem. *Assume that -1 is a non-square, p is prime, and the order of 2 is $(p-1)/2$ or $(p-1)$. Then any non-Desarguesian André plane of order p^2 admits a Fano configuration.*

PROOF. According to Theorem 19, if there were no Fano configuration, then there would exist a t with $t^{p+1} = 2$ such that $\tau : (x, y) \longmapsto (x, ty)$ is a collineation of the André plane. If R_δ is in Λ then its images under repeated applications of τ would all be in Λ, implying that there are at least $|\tau| \geq (p-1)/2$ André nets in Λ. But there can be no more than $(p-1)/2$ André nets (by Lemma 16), so the proof is complete unless there are exactly $(p-1)/2$ André nets in Λ. However, this final alternative cannot occur: because these nets would come in pairs from $(p-1)/4$ elements of the form $(b/(1-c))^{q+1} = \delta/(1-c^2)$, where $c^p = -c$ (as in the proof of the previous proposition); but (1) is always a slope in the derivation set (by our Basic Construction), yet $\delta/(1-c^2)$ cannot be 1, whence there are at least $(p+1)/2$ nets in Λ, which is a contradiction. $\boxed{QED}$

25 Corollary. *If p is an odd prime such that $(p-1)/2$ is prime then any non-Desarguesian André plane of order p^2 admits a Fano configuration.*

PROOF. Since $(p-1)/2$ is a prime, the order of 2 in $GF(p)^*$ is either $(p-1)/2$ or $(p-1)$. $\boxed{QED}$

There are exactly twenty-four odd primes between 1 and 100. Of these, the primes p such that $(p-1)/2$ is odd are in

$$\{3, 7, 11, 19, 23, 31, 43, 47, 59, 67, 71, 79, 83\}.$$

Note that $2^5 = 32 \equiv 1 \bmod 31$, so that the order of 2 in $GF(31)^*$ is $5 = (p-1)/6$. However, for the remaining primes p in the list, the order of 2 is at least $(p-1)/2$. In particular, for $p = 3$, 7, 11, 23, 47, 59, 79, and 83, we have $(p-1)/2$ is a prime (so that the order of 2 is at least $(p-1)/2$). We must deal independently with the other four p's in the list.

For $p = 19$, we have $(p-1)/2 = 9$; but $2^3 = 8 < 19$, so the order of 2 is at least 9.

For $p = 43$, $(p-1)/2 = 21$, and $2^7 = 64 \cdot 2 = 21 \cdot 2 = 42 \equiv -1 \bmod 43$.

If $p = 67$, $(p-1)/2 = 33$, but $2^{11} = 64 \cdot 32 \equiv -3 \cdot 32 \equiv -96 \equiv 41 \bmod 67$.

Finally, for $p = 71$, $(p-1)/2 = 35$, but both $2^5 = 32$ and $2^7 = 64 \cdot 2 \equiv -14 \bmod 71$. In summary,

26 Corollary. *Any non-Desarguesian André plane of order*

$$p^2 \in \left\{3^2, 7^2, 11^2, 19^2, 23^2, 43^2, 47^2, 59^2, 67^2, 71^2, 79^2, 83^2\right\}$$

admits a Fano configuration.

27 Remark. Exactly half of the twenty-four odd primes between 1 and 100 are covered by the corollary. Similarly, one may go through a list of primes to 200 with a similar conclusion: our results apply to about half of these primes.

14 A Combinatorial Problem and the 3/8-Theorem for André Planes

28 Problem. Let Ω be a subset of $GF(q) - \{0\}$, q odd, that contains 1, and let $t^q = -t$ such that $t^2 = \theta$, where θ is a non-square in $GF(q)$. Assume that we have the following two properties:

(1) (i) for each $\alpha \in GF(q) - \{0\}$ and for each $\beta \in \Omega$, either $\frac{\beta}{1-\alpha^2\theta}$ or $\frac{\beta}{1-(\alpha\theta)^{-2}\theta}$ is in Ω,

(2) (ii)and for each $\alpha \in GF(q) - \{0\}$, and for each $\beta \in \Omega$, either $\beta(1-\alpha^2\theta)$ or $\beta(1-(\alpha\theta)^{-2}\theta)$ is in Ω.

Problem: Determine the conditions on q so that any Ω that contains at least two elements will necessarily contain more than $(q-1)/2$ elements.

29 Theorem. *Any non-Desarguesian André plane of odd order q^2 to which the combinatorial problem applies has a Fano configuration.*

Proof. The expressions such as $\frac{\beta}{1-\alpha^2\theta}$ in properties (i) and (ii) correspond to values of $\delta = m^{q+1}$ that arise from slopes like $m = \frac{b}{1-c}$ (where $c^2 = \alpha^2\theta$), as described in the proof of Proposition 23. A solution to the combinatorial problem would guarantee more than $(q-1)/2$ values of δ, whence an assumption of "no Fano configuration" would contradict the condition in Lemma 16 that Λ has no more than $(q-1)/2$ nets R_δ. $\boxed{QED}$

30 Example. Suppose that $(q-1)/2$ is odd. Choose $\theta = -1$, a non-square in $GF(q)$. Then choosing α above to be 1, we have $\frac{\beta}{1-\alpha^2\theta} = \frac{\beta}{1-(\alpha\theta)^{-2}\theta} = \beta/2$. Hence, $\beta/2 \in \Omega$ for all $\beta \in \Omega$, in which case $\beta/2^h \in \Omega$ for all integers h. Thus, we must have $\beta 2^h \in \Omega$ for all h. That is, $2^h \in \Omega$ and if $\beta \in \Omega$, then the coset $\beta\langle 2\rangle \subseteq \Omega$. Furthermore, for each $\alpha \in GF(q) - \{0\}$, $\frac{\beta 2^h}{1-\alpha^2\theta}$ or $\frac{\beta 2^h}{1-(\alpha\theta)^{-2}\theta} \in \Omega$, and $\beta 2^h(1-\alpha^2\theta)$ or $\beta 2^h(1-(\alpha\theta)^{-2}\theta)$ is in Ω, for each $\beta \in \Omega$ and integer h.

31 Lemma. $1/(1-\alpha^2\theta) = 1-\rho^2\theta$ *if and only if* $(\alpha^2\theta - 1)$ *is a square in* $GF(q)$.

Proof. $(1-\alpha^2\theta)(1-\rho^2\theta) = 1$, for some $\rho \in GF(q)$, if and only if $(\alpha^2 + \rho^2)\theta = (\alpha^2\rho^2)\theta^2$. This holds, in turn, if and only if $\rho^2(\alpha^2\theta - 1) = \alpha^2$, if and only if $\rho^2 = \alpha^2/(\alpha^2\theta - 1)$, if and only if $\alpha^2\theta - 1$ is a square. $\boxed{QED}$

32 Remark. $(\alpha^2\theta - 1)$ is a square about half of the time. There are $(q-1)/2$ elements $\alpha^2\theta$, and if 4 divides $(q-1)$, then we get $(q-1)/4$ possible squares. Otherwise, if 4 does not divide $(q-1)$ we obtain either $((q-1)/2-1)/2 = (q-3)/4$ squares, or that number plus one. Depending on this divisibility condition, we get either $(q-1)/4$ or at least $(q-3)/4$ non-squares. Hence, $A \cap B$, where

$$A = \left\{\frac{\beta}{1-\alpha^2\theta}, \frac{\beta}{1-(\alpha\theta)^{-2}\theta};\ \alpha \in GF(q) - \{0\}\right\} \text{ and}$$

$$B = \left\{\beta(1-\alpha^2\theta), \beta(1-(\alpha\theta)^{-2}\theta;\ \alpha \in GF(q) - \{0\}\right\},$$

contains at most $(q-1)/4$ elements when 4 divides $q-1$, or at most $(q+1)/4$ when 4 does not divide $q-1$.

We are now ready for the promised improvement of the 1/4-Theorem for André planes.

33 Theorem. *Let* π *be an André plane of odd order* q^2 *constructed from a Desarguesian affine plane by replacement of a set* Λ *of*

$$|\Lambda| \leq \frac{3(q-1)}{8}$$

disjoint nets. Then π *admits Fano configurations.*

PROOF. We interpret the preceding observation in terms of the replacement set Λ of André nets R_δ: let $\Omega = \{\delta;\ R_\delta \in \Lambda\}$. Note that because $\alpha = \pm\beta$ and $\alpha = \pm\beta^{-1}$ all produce the same pair of elements in A or in B, one element of Ω would suffice to cover the pairs from these four sources. For there to be no Fano configuration we would need Ω to contain one element from each pair in A — that is, for each value of α either $\frac{\beta}{1-\alpha^2\theta} \in \Omega$ or $\frac{\beta}{1-(\alpha\theta)^{-2}\theta} \in \Omega$ — and one from each pair in B. That would require Ω to contain $(q-1)/4$ elements from A and at least another $(q-1)/8$ from B. Recall that $1 \in \Omega$ but not in sets A or B. Thus, we would need at least $1 + (q-1)/4 + (q-1)/8$ nets to block our Fano configuration. $\boxed{QED}$

A more precise count yields the following additional information.

34 Corollary. *In the following situations, the André plane π admits Fano configurations:*

(1) *If* 4 *divides* $(q-1)$ *but* 8 *does not divide* $q-1$*, and*

$$|\Lambda| \leq 3 + \frac{3(q-5)}{8},$$

or

(2) *If* 4 *does not divide* $(q-1)$ *but* 8 *divides* $(q-3)$*, and*

$$|\Lambda| \leq 1 + \frac{3(q-3)}{8},$$

or

(3) *If* 4 *does not divide* $(q-1)$ *and* 8 *does not divide* $(q-3)$*, and*

$$|\Lambda| \leq 3 + \frac{3(q-7)}{8}.$$

15 Proof of The Avoidance Theorems

We note that in our Basic Construction, as long as we have the appropriate slopes outside the replacement net, the number of disjoint derivable nets is irrelevant. This leads us to consider a different point of view. Suppose we have a set Λ of disjoint derivable nets. We now wish to multiply derive a subset of these nets, as opposed to the entire set. We still must assume that there are two components, which we call $x = 0, y = 0$, that are exterior to the set of parallel classes of the subset, and that this pair is a common conjugate pair for at least one of its derivable nets. (For example, this condition is always valid in André planes.) Since there are at most two of the disjoint nets of Λ that can contain the slopes $\left(\frac{1}{c^{-1}-1}\right)$ and $\left(\frac{1}{1-c}\right)$, we merely must avoid choosing such derivable nets in the subset in question. When we consider the problem in this way, we see that there are many more subregular planes that admit Fano configurations than those strictly satisfying the hypothesis of the 1/4-Theorem. This argument proves the Avoidance Theorem:

35 Theorem. [The Avoidance Theorem] *Let Λ be a set of k disjoint derivable nets of a Desarguesian affine plane of odd order q^2. Assume that for any subset Λ^* of Λ, there is a pair of components disjoint from Λ that is a conjugate pair for a derivation net of Λ^*. Then of the $2^k - 1$ possible nonempty subsets of Λ that produce subregular planes, at least $2^{k-2} - 1$ contain Fano configurations.*

36 Corollary. *For any Desarguesian affine plane of odd order q^2, there are $2^{(q-1)/2} - 1$ possible non-Desarguesian André planes obtained by the replacement of a subset of the set of $(q-1)$ disjoint André nets. Of these planes at least $2^{(q-5)/2} - 1$ admit Fano configurations.*

37 Remark. In the corollary we are not trying to sort out the isomorphism classes of André planes admitting Fano configurations. A quick analysis based on Foulser's description of the full collineation group of an André plane which is not Desarguesian and not Hall [11], leads us to estimate that for $q = p^r$ there are at least $(2^{(q-5)/2})/r(q-1)$ nonisomorphic André planes admitting Fano configurations.

16 The Subregular Nearfield Plane and Other Maximum André Planes

We know from Lemma 16 that any André plane of order q^2 can be obtained from a Desarguesian plane by deriving $(q-1)/2$ or fewer disjoint derivation nets. Let us call a plane that requires all $(q-1)/2$ nets a ***maximum André plane***. The subregular nearfield plane is an important member of this family: The *subregular nearfield plane* of odd order is an André plane whose replacement net Λ consists of the non-square André nets (that is, $\Lambda = \{R_\delta;\ \delta$ a non-square in $GF(q)\}$. Its lines (in terms of the Desarguesian plane from which it is derived) are $x = 0$, $y = 0$, and $y = x^{q^i}m$, where $i = 0$ if m^{q+1} is a nonzero square and $i = 1$ if m^{q+1} is non-square. For these nearfield planes, when -1 is a non-square the regularity of the nets will again imply the existence of Fano configurations. On the other hand, when -1 is a square our methods will fail, and we will content ourselves with determining other examples of maximum André planes where Fano configurations exist.

For these planes we modify our Basic Construction by taking $B = (0, b)$; that is, the fundamental quadrangle is $A = (\infty)$, $B = (0, b)$, $C = (c, 0)$ and

$D = (1, 0)$, whose diagonal points are

$$P = AB \cap CD = (0, 0), \quad Q = AC \cap BD = (c, b(1 - c)), \quad \text{and}$$

$$R = AD \cap BC = \left(1, \frac{b(c-1)}{c}\right).$$

For our Fano configuration, we want P, Q, and R to be in one of the replaced subplanes, namely $y = x^q m$ for some m for which m^{q+1} is a non-square. Moreover, we want the six lines AD: $x = 1$, AB: $x = 0$, AC: $x = c$, CD: $y = 0$, BD: $y = -xb + b$, and BC: $y = -x\frac{b}{c} + b$ of the Desarguesian plane to remain lines in the nearfield plane (with slope (∞) or with a slope whose $q + 1$st power is a square). The first requirement (applied to points Q and R) tells us

$$c^q m = b(1 - c) \quad \text{and} \quad m = b(c - 1)/c,$$

from which it follows that

$$c^q = -c \quad \text{and} \quad b = \frac{mc}{(c-1)}.$$

Our requirement that the slopes $(-b/c)$ and $(-b)$ correspond to lines in the unreplaced part implies that the elements $(-b/c)^{q+1}$ and $(-b)^{q+1}$ are both square. After the next lemma we shall see that the requirement is easily achieved when -1 is a square, and impossible to achieve otherwise.

We shall write the elements of $GF(q^2)$ in terms of a fixed element $\mathbf{i} \in GF(q^2) - GF(q)$ for which $\mathbf{i}^2 = \theta$ is a non-square in $GF(q)$. Of course, any element of $GF(q^2)$ is represented uniquely as $\alpha\mathbf{i} + \beta$ for some $\alpha, \beta \in GF(q)$; moreover, $(\alpha\mathbf{i} + \beta)^q = -\alpha\mathbf{i} + \beta$. In this notation any $c \in GF(q^2)$ for which $c^q = -c$ must satisfy $c = \alpha\mathbf{i}$ for some $\alpha \in GF(q)$, because $c^{q+1} = -c^2$ exactly when $c = \alpha\mathbf{i}$. It is important to observe that since $c^{q+1} = -c^2 = -\alpha^2\theta$,

> c^{q+1} *is a nonzero square in* $GF(q)$ *if and only if* -1 *is a non-square.*

38 Lemma. *In* $GF(q)$*,*

(1) if -1 *is a non-square then* $1 - \alpha^2\theta$ *is a non-square for* $(q + 1)/2$ *values of* α*;*

(2) if -1 *is a square then* $1 - \alpha^2\theta$ *is a square different from* 0 *and* 1 *for* $(q - 1)/2$ *nonzero values of* α*.*

Proof. We work in the affine Desarguesian plane over $GF(q)$. For the first claim let λ, λ' be non-squares in $GF(q)$. Then $1 = \lambda x^2 + \lambda'^2$ represents an ellipse (with $q+1$ affine points); it is therefore satisfied by $(q+1)/2$ values of $x = \alpha$. For the second claim, $1 = \lambda x^2 + y^2$ represents an ellipse (with $q-1$ points for which $x \neq 0$) satisfied by $(q-1)/2$ nonzero values of $x = \alpha$. $\boxed{QED}$

39 Theorem. *It is always possible to construct an André plane of order q^2 that admits Fano configurations using $(q-1)/2$ derivation nets. Specifically,*

(1) If -1 is a non-square in $GF(q)$ then the subregular nearfield plane of order q^2 obtained from a Desarguesian affine plane by the derivation of the set of all $(q-1)/2$ non-square André nets admits Fano configurations.

(2) If -1 is a square in $GF(q)$ then for any non-square γ let π_γ denote the plane obtained by the multiple derivation of the André net Λ_γ that consists of the $(q-3)/2$ non-square André nets different from R_γ, namely

$$\Lambda_\gamma = \{R_\delta;\ \delta \in GF(q) - \{\gamma\} \text{ and } \delta \text{ is a non-square.}\}$$

Then there are at least $(q-3)/2$ different choices of André nets R_β with β a nonzero square of $GF(q)$ such that the derivation of π_γ by R_β results in a maximum André plane that admits Fano configurations.

Proof. By the previous lemma, for the claim in (1) we are guaranteed $(q+1)/2$ values of α for which $1 - \alpha^2\theta$ is a non-square (of $GF(q)$). (Recall that $\mathbf{i}^2 = \theta$ is a non-square.) Letting $c = \alpha\mathbf{i}$ for one of them, we have $c^q = -c$, so that c^{q+1} is a square while $(c-1)^{q+1} = 1 - c^2 = 1 - \alpha^2\theta$ is a non-square. For some non-square δ we choose any slope (m) for which $m^{q+1} = \delta$. Then for $b = \frac{mc}{(c-1)}$ we have b^{q+1} is a square (since it is the product of an even number of non-squares), which implies that $(-b)^{q+1}$ and $(-b/c)^{q+1}$ are both nonzero squares. In this way all the conditions for the existence of a Fano configuration have been satisfied.

The claim in (2) proceeds similarly, except that when -1 is a square, c^{q+1} is necessarily a non-square. Since again we choose a slope m for which $\delta = m^{q+1}$ is also non-square, when $b = \frac{mc}{(c-1)}$ we have b^{q+1} is a square if and only if α is chosen so that $(c-1)^{q+1}$ is a square. We can therefore arrange that $(-b)^{q+1}$ be square while $(-b/c)^{q+1}$ is a non-square, or vice versa. However, we cannot arrange that they both be squares, so we cannot arrange, using our construction, for the resulting nearfield plane to have a Fano configuration. Let us therefore choose α so that $(-b)^{q+1} = \gamma$ is a non-square, and instead of using R_γ in the replacement net, use R_β for any square $\beta \neq 0, (-b/c)^{q+1}$. In this way the replacement net has $(q-1)/2$ nets that include δ but exclude the slopes $(-b)$ and $(-b/c)$, as required for the existence of a Fano configuration. The plane multiply derived from this replacement net will be the desired maximum André plane. $\boxed{QED}$

40 Remark. Although the last theorem shows that many maximum André planes of order 25 admit Fano configurations, we remain unable to prove that the nearfield plane of that order is among them. Likewise, $p = 31$ remains open. For $p = 31$, $2^5 \equiv 1 \bmod 31$, implying that we need at least 5 André nets to block a Fano configuration. However, by the corollary to the 3/8-theorem, we require at least $1 + 3 + 3(31 - 7)/8 = 13$ nets to block. But these nets are in an orbit under a group of order 5 acting on the net (the order of 2). Hence, this would force at least 15 nets. Hence, we see that for order 31^2, either the André plane admits Fano configurations or exactly $(p - 1)/2 = 15$ nets have been replaced. Such nets would have to be of the form $R_{\beta_i 2^j}$, for $i = 1, 2, 3$ and for $j = 1, 2, 3, 4, 5$.

17 Avoidance in André Planes

To represent an André plane π of order q^2, we may, of course, agree **not** to derive the standard André net, whose partial spread is $R_1 = \{y = xm; m^{q+1} = 1\}$. In this section, we show that either the given André plane admits a Fano configuration or it has a derivate that does. Specifically,

41 Theorem. *An André plane of order q^2 either admits a Fano configuration, or it has an André net $\mathcal{D}$ such that the André plane $\mathcal{D}$-derived from it admits a Fano configuration.*

Proof. Every subregular translation plane admits as a collineation group the kernel homology group of order $q^2 - 1$ that fixes all components of the associated Desarguesian affine plane Σ, and acts transitively on the $q^2 - 1$ nonzero points of each such component. Furthermore, the André planes admit a collineation group of order $(q^2 - 1)^2$, whose elements are

$$(x, y) \to (xa, yb) \text{ for } a, b \in GF(q^2) - \{0\}.$$

We start with the given André plane π and its associated Desarguesian plane Σ, and we describe the nets that are replaced to obtain π from Σ. Should π not contain a convenient Fano configuration, we then determine one further net $\mathcal{D}$ whose replacement guarantees that the André plane derived from π, call it π', necessarily contains a Fano configuration.

For a quadrangle $ABCD$ in Σ, take $A = (\infty)$ and, without loss of generality (by the transitivity conditions of the associated group), take $B = (0, -1)$ and $D = (1, 0)$, and let $C = (c, 0)$. Again, we want the six lines AD: $x = 1$, AB:

$x = 0$, AC: $x = c$, CD: $y = 0$, BD: $y = x - 1$, and BC: $y = x\frac{1}{c} - 1$ to remain lines in both π and π'. The diagonal points are

$$P = AB \cap CD = (0,0), \quad Q = AC \cap BD = (c, (c-1)), \quad \text{and}$$

$$R = AD \cap BC = \left(1, \frac{(1-c)}{c}\right).$$

These diagonal points are incident with $y = x^q\left(\frac{1-c}{c}\right)$, provided $c^q = -c$. For $ABCD$ to generate a Fano configuration we wish that the André net given by $R_{c^{-(q+1)}}$ not be among the derived nets, while $R_{\left(\frac{1-c}{c}\right)^{q+1}}$ is in the set of derived nets. Let γ be a non-square of $GF(q)$. If we let $c^2 = \gamma^{-1}$, then $c^{-(q+1)} = -\gamma$ and $\left(\frac{1-c}{c}\right)^{q+1} = -\gamma + 1$. As γ runs through the non-squares, there are exactly $(q-1)/2$ pairs of André nets $R_{-\gamma}, R_{-\gamma+1}$. Choose any such pair and consider the four possibilities for the multiple derivation of π from Σ:

(1) $replace, replace,$

(2) $don't\ replace, replace,$

(3) $replace, don't\ replace,$

(4) $don't\ replace, don't\ replace.$

In case (2) $ABCD$ generates a Fano configuration in the given André plane π. In the other cases, we will see that we need only change at most one net to attain a Fano configuration. In particular, in case (1) we define π' to be related to π by having the first listed net not replaced; in case (4), we obtain π' from π by replacing the second listed net.

This leaves case (3). We can reduce it to one of the other cases unless π is defined so that for every non-square $\gamma \in GF(q)$, we replace a net $R_{-\gamma}$ if and only if we do not replace the net $R_{-\gamma+1}$. We can therefore assume that we cannot reduce the case to another; that is, we assume that

for all non-square $\gamma \in GF(q)$, $\gamma \neq -1$, the nets $R_{-\gamma}$ are replaced.

First assume that -1 is a square, in which case each $R_{-\gamma}$ is a non-square net. Then we have replaced all non-square nets, so we have a nearfield plane; we know from the previous section that to construct an André plane that admits Fano configurations we may choose one of these non-square nets not to replace.

So, assume that -1 is a non-square, in which case $R_{-\gamma}$ is a square net. By our assumptions, to construct π we have replaced all $(q-3)/2$ square nets other than R_1. It is possible, moreover, that we could have replaced just one further net, necessarily non-square, since the maximum number of replaced nets is $(q-1)/2$. With these assumptions we start over with the general quadrangle

$ABCD$, where $A = (\infty)$, $B = (0, b)$, $C = (c, 0)$ and $D = (1, 0)$, whose diagonal points are

$$P = AB \cap CD = (0,0), \quad Q = AC \cap BD = (c, b(1-c)), \quad \text{and}$$
$$R = AD \cap BC = \left(1, \frac{b(c-1)}{c}\right).$$

Just as before, we want P, Q, and R to be in one of the replaced subplanes, namely $y = x^q m$ for some m for which m^{q+1} is a non-square in $GF(q)$. Moreover, we want the six lines AD: $x = 1$, AB: $x = 0$, AC: $x = c$, CD: $y = 0$, BD: $y = -xb + b$, and BC: $y = -x\frac{b}{c} + b$ of the Desarguesian plane to remain lines in the nearfield plane (with slope (∞) or with a slope whose $q+1$st power is a square). The first requirement (applied to points Q and R) tells us

$$c^q m = b(1-c) \quad \text{and} \quad m = b(c-1)/c,$$

from which it follows that

$$c^q = -c \quad \text{and} \quad b = \frac{mc}{(c-1)}.$$

We know that $c^{q+1} = -\gamma$ is a square, and we have replaced all $(q-3)/2$ square nets except R_1. Therefore, the requirement now is that the slopes $(-b/c)$ and $(-b)$ correspond to lines in the unreplaced part. Since there is at most one non-square net which has been replaced, this implies that the elements $(-b/c)^{q+1}$ and $(-b)^{q+1}$ could be any non-squares with at most one exception (namely, avoid the non-square that defines a replaced non-square net). Since there are then at least $(q-3)/2$ suitable non-squares, we require $(q-3)/2 \geq 1$ so that $q \geq 5$, which may be assumed by Corollary 26. Hence, it remains to show that there exists an element c such that both c^{q+1} and $((c-1)/c)^{q+1}$ are squares. Equivalently, we want a non-square γ for which $1 - \gamma$ is a square.

By Lemma 38, we know that if θ is a fixed non-square and -1 is also a non-square, then $1 - \alpha^2\theta$ is a non-square for $(q+1)/2$ values of α; it is therefore a square for $(q-3)/2$ values. Therefore, if all $(q-3)/2$ square nets except R_1 are replaced (plus perhaps one additional non-square), we obtain a Fano configuration in the given André plane π. $\boxed{QED}$

42 Remark. We note that a Desarguesian plane of odd order q^2 is an André plane that does not admit a Fano configuration. However, if the Desarguesian plane is derived then the associated André plane, namely the Hall plane of order q^2, does admit a Fano configuration. This example shows that until it is known whether or not all non-Desarguesian André planes admit Fano configurations, the previous theorem is the best possible general theorem regarding the existence of Fano configurations in arbitrary André planes.

18 The Subregular Spreads

We turn to specific examples to determine which of the known subregular spreads admit Fano configurations as a consequence of our investigation. We first ask which of the known classes satisfy the 1/4-condition, which then admit Fano configurations by the 1/4-Theorem 14. In fact, most of these do admit Fano configurations. We offer a few remarks on the known classes.

18.1 Ostrom, Johnson-Seydel spread of order 3^6 with $k = 7$ Reguli

Since $3^3 \equiv -1 \bmod 4$, any subregular plane constructed with $\leq (3^3 + 1)/4$ reguli admits Fano configurations and since $(3^3 + 1)/4 = 7$, this translation plane admits Fano configurations (see [20], [17]).

18.2 Ostrom and Foulser planes of order v^{4t} with $v(v-1)/2$-Reguli

There are a variety of subregular translation planes of order q^4 for both q even and odd with various interesting properties. For example, there is the Ostrom subregular plane of order q^4 admitting $SL(2, q)$, where $q = p^r$, for p odd, where the p-elements are elations. In this case, $k = q(q-1)/2$, which does not satisfy the $(q^2 - 1)/4$ bound.

On the other hand, Foulser [12] discussed the complete set of subregular spreads of order q^4 whose replacing reguli are fixed by $\sigma : (x, y) \longmapsto (x^q, y^q)$. He has shown that for any Desarguesian plane of order v^{4t}, it is possible to construct subregular planes admitting $SL(2, v)$, where $v = p^z$, and the p-elements are elations. In this setting, there are $v(v-1)/2$ reguli and since $v(v-1)/2 \leq (v^{2t} - 1)/4$, for $t > 1$, any such translation plane admits Fano configurations.

18.3 Bruck Triple Planes

In [5], Bruck gives a more-or-less complete determination of the subregular planes that may be obtained by the multiple derivation of three reguli. Since $3 \leq (q-1)/4$ for $q \geq 13$, we see that any Bruck triple plane of order q^2 for $q \geq 13$ admits Fano configurations.

18.4 The Ebert Quadruple Planes

Ebert ([9] and [7]) has given a theory of subregular spreads obtained from a Desarguesian spread by the replacement of exactly four disjoint reguli. There are a tremendous number of nonisomorphic translation planes of odd order (asymptotically equal to q^6/Λ, where $16 \leq \Lambda \leq 3072$ of order q^2). Since $4 \leq (q-1)/4$ for $q \geq 17$, we see that any Ebert quadruple plane of order q^2 for $q \geq 17$ will admit Fano configurations.

18.5 Prohaska Planes

Prohaska [22] constructs several classes of subregular translation planes, some of which admit S_4 or A_5 acting on the reguli. Any of these planes can be lifted, producing planes admitting similar groups. Many of these planes are constructed using fewer than 7 reguli and since $6 \leq (q-1)/4$ provided $q \geq 25$, most of these planes admit Fano configurations.

18.6 Ebert Quintuple Planes

In [8], Ebert provides some examples of planes of orders p^2, for $p = 11, 19, 29, 31$ that may be constructed by replacing five reguli. We call these planes the *Ebert quintuple planes*. These subregular spreads are of interest in that they do not have the two conditions required for the triviality of the group fixing each circle of the set of five. Clearly the planes of orders 29^2 and 31^2 will admit Fano configurations.

18.7 Abatangelo and Larato Planes

In [1], there are constructions of subregular planes of order q^2 admitting $SL(2,5)$ by multiple derivation, for $q \equiv 1 \bmod 4$ and $q \equiv -1 \bmod 5$. All of these planes are similar to the Prohaska planes and most of these admit Fano configurations.

18.8 Bonisoli-Korchmáros-Szőnyi Planes

In [4], the three authors construct a class of subregular $SL(2,5)$-planes of odd order q^2, where $q^2 \equiv 4 \bmod 5$. There are three types: (I) five reguli and A_5 acting in its natural representation on the set, for $q \equiv 2 \bmod 3$, (II) ten reguli and A_5 having orbit lengths of $1, 6, 3$, $q \equiv 1 \bmod 3$ and $q \equiv 1 \bmod 4$, (III) ten reguli and A_5 again having orbit lengths of $1, 6, 3$ but the set of reguli are distinct from that of (II), $q \equiv -1 \bmod 4$, $3 \nmid q$ and $q \equiv 3, 5$ or $6 \bmod 7$, and (IV) fifteen reguli, where A_5 has orbit lengths of $1, 2, 4, 8$, $q \equiv 2 \bmod 3$, $q \equiv 1 \bmod 4$, $q \equiv 1, 2$ or $4 \bmod 7$.

In any case, the maximum number of reguli used is 15 so for any translation plane of order q^2 for $q \geq 61$, there are Fano configurations.

18.9 Charnes and Dempwolff Planes

In [6], sections 3 and 4, there are constructions of spreads in $PG(3,q)$ admitting $SL(2,5) * C$, a subgroup of a $GL(2,q^2)$ of an associated Desarguesian spread, where $*$ denotes a central product and C, the kernel homology group of order q^2-1. In particular, we have $(q,30) = 1 = (q^2-1,5)$. In (4.1) and (4.2), there are constructions of spreads arising from a G-orbit O_i, for $i = 1,2$ of length $10(q+1)$ and $5(q+1)$, respectively for $q \equiv 1 \bmod 12$ and $q \equiv -1 \bmod 3$. This spreads turn out to be subregular and constructed by multiple derivation of at most 10 reguli, so for $q \geq 41$, any such plane admits Fano configurations.

18.10 The Orr Spread

Orr [19] shows that the elliptic quadric of $PG(3,q)$ for $q = 5$ or 9 has a partial flock of deficiency one, which cannot be extended to a flock. What this means is there is a translation plane of order 9^2 constructed by the replacement of 8 reguli from a Desarguesian spread which is not an André plane. Since this number does not satisfy the 1/4-bound, it is not known if these planes admit a Fano configuration.

19 Final Remarks

We have shown that a wide variety of subregular planes of odd orders admit Fano configurations. In fact, with the possible exception of the Ostrom-Foulser plane of order q^4 admitting $SL(2,q)$, and a few small-order planes, all known non-André subregular planes admit Fano configurations. We have not tried to apply our analysis to the Ostrom-Foulser plane, but conceivably this plane also admits Fano configurations. For André planes, we have shown that most of these do, in fact, admit Fano configurations. Hence, we offer the following as a conjecture:

43 Conjecture. *All non-Desarguesian subregular planes of odd order admit Fano configurations.*

We have not dealt with other types of translation planes having spreads in $PG(3,q)$ nor have we considered translation planes of larger dimension or of infinite translation planes of characteristic not 2. Hence, we end with a problem.

44 Problem. What classes of large-dimension translation planes of odd order or of characteristic not 2 admit Fano configurations?

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