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Brezis-Browder Principle Revisited

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Abstract. Most of the known (sequential) maximality principles are logical equivalents of the Brezis-Browder's [Adv. Math., 21 (1976), 355-364]. But, for at least one of these, the inclusional relation cannot be reversed. It is our aim to put this (metrical) statement in its natural (abstract) framework. Some applications of these facts to Zorn maximality principles are then given.

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MSC 2000 classification: Primary 49J53. Secondary 47J30.

1 Introduction

Let M be a nonempty set. Take a *quasi-order* $(\leq)$ (i.e.: reflexive and transitive relation) over it, as well as a function $x \mapsto \psi(x)$ from M to $R_+ := [0, \infty[$. Call the point $z \in M$, *$(\leq, \psi)$-maximal* when: $w \in M$ and $z \leq w$ imply $\psi(z) = \psi(w)$. A basic result about the existence of such points is the 1976 Brezis-Browder ordering principle [6]:

1 Proposition. *Suppose that*

(1a) $(M, \leq)$ *is sequentially inductive:*
each ascending sequence has an upper bound (modulo $(\leq)$*)*

(1b) ψ *is* $(\leq)$*-decreasing* $(x \leq y \Longrightarrow \psi(x) \geq \psi(y))$.

Then, for each $u \in M$ *there exists a* $(\leq, \psi)$*-maximal* $v \in M$ *with* $u \leq v$.

This statement, including the well known Ekeland's variational principle [11], found some useful applications to convex and nonconvex analysis (cf. the above references). So, it cannot be surprising that many extensions of Proposition 1 were proposed. Among these, we quote the 1982 contribution in Altman [1] or the 1987 paper in Anisiu [3]; see also Bae, Cho and Yeom [4]. The obtained results are interesting from a technical viewpoint. However, we must emphasize that, in all concrete situations when a maximality principle of this type is to be applied, a substitution of it by the Brezis-Browder's is always possible. This (cf. Bao and Khanh [5]) raises the question of to what extent are these enlargements of Proposition 1 effective. As we shall see below (in Section 2) the answer is negative for most of these. On the other hand, there do exist metrical maximality principles which are not comparable with Proposition 1; see the 1990 paper in Kang and Park [17]. It is our second aim in this exposition to show (cf. Section 3) that all such statements may be viewed as particular cases of an "asymptotic" type version of Proposition 1 (which seems to include it in a strict sense). Finally, in Section 4, an application of these facts is given to (standard) Zorn maximality principles. Further aspects will be delineated elsewhere.

2 Logical equivalents of Proposition 1

Let M be some nonempty set; and $(\leq)$, some quasi-order on it. Further, let $x \mapsto \varphi(x)$ stand for a function between M and $R_+ \cup \infty = [0, \infty]$.

2 Proposition. *Assume (1a) and (1b) are true, as well as*

(2a) ($(M, \leq)$ is almost regular (modulo φ))
$\forall x \in M, \forall \varepsilon > 0, \exists y = y(x, \varepsilon) \geq x : \varphi(y) \leq \varepsilon.$

Then, for each $u \in M$ there exists $v \in M$ with $u \leq v$ and $\varphi(v) = 0$ (hence v is $(\leq, \varphi)$-maximal).

Proof. By (2a), there must be some $z \geq u$ with $\varphi(z) < \infty$. Clearly, (1a)-(1b) apply to $M(z, \leq) := \{x \in M; z \leq x\}$ and $(\leq, \varphi)$. So, for the starting point $z \in M(z, \leq)$ there exists $v \in M(z, \leq)$ with **i)** $z \leq v$ (hence $u \leq v$) and **ii)** v is $(\leq, \varphi)$-maximal in $M(z, \leq)$. Suppose by contradiction that $\gamma := \varphi(v) > 0$; and fix some β in $]0, \gamma[$. By (2a) again, there must be $y = y(v, \beta) \geq v$ (hence $y \in M(z, \leq)$) with $\varphi(y) \leq \beta < \gamma (= \varphi(v))$. This cannot be in agreement with the second conclusion above. Hence, $\varphi(v) = 0$; and we are done. QED

Clearly, Proposition 2 is a logical consequence of Proposition 1. But, the converse inclusion is also true; to verify it, we need some conventions. By a (generalized) *pseudometric* over M we shall mean any map $d : M \times M \to R_+ \cup \infty$. Suppose that we introduced such an object, which is also *reflexive* $[d(x, x) = 0, \forall x \in M]$. Call the point $z \in M$, $(\leq, d)$-*maximal*, if: $u, v \in M$ and $z \leq u \leq v$ imply $d(u, v) = 0$. Note that, if d is (in addition) *sufficient* $[d(x, y) = 0 \Longrightarrow x = y]$, the $(\leq, d)$-maximal property becomes: $w \in M, z \leq w \Longrightarrow z = w$ (and reads: z is *strongly* $(\leq)$-*maximal*). So, existence results involving such points may be viewed as "metrical" versions of the Zorn maximality principle (cf. Moore [20, Ch 4, Sect 4]). To get sufficient conditions for these, one may proceed as below. Let (x_n) be an ascending sequence in M. The d-Cauchy property for it is introduced in the usual way $[\forall \varepsilon > 0, \exists n(\varepsilon)$ such that $n(\varepsilon) \leq$

$p \leq q \Longrightarrow d(x_p, x_q) \leq \varepsilon]$. Also, call (x_n), *d-asymptotic* when $d(x_n, x_{n+1}) \to 0$, as $n \to \infty$. Clearly, each (ascending) d-Cauchy sequence is d-asymptotic too. The reverse implication is also true when all such sequences are involved; i.e., the global conditions below are equivalent

(2b) each ascending sequence is d-Cauchy

(2c) each ascending sequence is d-asymptotic.

By definition, either of these will be referred to as $(M, \leq)$ is *regular* (modulo d). Note that this property implies its relaxed version

(2d) ($(M, \leq)$ is weakly regular (modulo d))
$\forall x \in M, \forall \varepsilon > 0, \exists y = y(x, \varepsilon) \geq x$: $y \leq u \leq v \Longrightarrow d(u, v) \leq \varepsilon$.

The following ordering principle is then available (cf. Kang and Park [17]):

3 Proposition. *Assume that $(M, \leq)$ is sequentially inductive and weakly regular (modulo d). Then, for each $u \in M$ there exists a $(\leq, d)$-maximal $v \in M$ with $u \leq v$.*

Proof. Let us introduce the function (from M to $R_+ \cup \infty$)

(a2) $\varphi_d(x) = \sup\{d(u, v); x \leq u \leq v\},\ x \in M.$

Clearly, (1b) holds for this object, as well as (2a) (if one takes (2d) into account). Hence, Proposition 2 is applicable to M and $(\leq, \varphi_d)$. This, added to $[\varphi_d(z) = 0$ iff z is $(\leq, d)$-maximal$]$ gives the desired conclusion. QED

As a direct consequence of this, we get the maximality principle in Turinici [26] (see also Conserva and Rizzo [9]):

4 Proposition. *Assume that $(M, \leq)$ is sequentially inductive and regular (modulo d). Then, conclusion of Proposition 3 is holding.*

So far, Proposition 4 is a logical consequence of Proposition 1. The reciprocal of this is also true, by simply taking $d(x, y) = |\psi(x) - \psi(y)|, x, y \in M$ (where ψ is the above one). We therefore established the inclusional chain Prop 1 $\Longrightarrow$ Prop 2 $\Longrightarrow$ Prop 3 $\Longrightarrow$ Prop 4 $\Longrightarrow$ Prop 1. Hence, all these ordering principles are nothing but logical equivalents of the Brezis-Browder's [6] (Proposition 1). (This also includes the related statements in Szaz [24] and Tataru [25]; which extend the one in Dancs, Hegedus and Medvegyev [10]). Further aspects may be found in Hamel [14, Ch 4]; see also Hyers, Isac and Rassias [15, Ch 5]. Some basic applications of these to flow invariance theory for evolution equations may be found in Cârjă and Vrabie [8].

3 Asymptotic extensions

The developments in the preceding section raise the (delicate) question of whether or not extensions of Proposition 1 (or its variants) exist without being reducible to it. Any attempt of solving it must begin from the sequential inductivity condition (1a). Precisely, an examination of the argument in Proposition 3 shows that one may impose it asymptotically (i.e., to sequences (x_n) with $\varphi_d(x_n) \to 0$) for the written conclusion be retainable. So, it is natural to ask whether this has a general character. A positive answer to this may be given under the lines below. Let again M be some nonempty set. Take a quasi-order $(\leq)$ over it, as well as a function $\varphi : M \to R_+ \cup \infty$. The following counterpart of Proposition 2 is now available.

5 Theorem. *Assume that (1b) and (2a) are true, as well as*

(3a) $(M, \leq)$ is sequentially inductive (modulo φ): each ascending sequence (x_n) with $\varphi(x_n) \to 0$ has an upper bound (modulo $(\leq)$).

Then, for each $u \in M$ there exists $v \in M$ with $u \leq v$ and $\varphi(v) = 0$ (hence v is $(\leq, \varphi)$-maximal).

Proof. By (2a), it is not hard to construct an ascending (modulo $(\leq)$) sequence (u_n) with ($u \leq u_0$ and) $\varphi(u_n) \leq 2^{-n}, \forall n$ (hence $\varphi(u_n) \to 0$). Let v stand for an upper bound (modulo $(\leq)$) of this sequence (assured by (3a)). This element has all properties we need. QED

Now, (1a) is a particular case of (3a). This tells us that Proposition 2 (hence Proposition 1 as well) is a particular case of Theorem 5. The reciprocal question (Prop 2 $\Longrightarrow$ Th 5) remains open; we conjecture that the answer is negative. To explain our position, it will be useful to consider

6 Example. Let $R^2 = R \times R$ stand for the cartesian plane; and $(\leq)$ denote the partial order induced by the convex cone R^2_+. Further, put $M = A \cup B$, where $A = \{u_n := (n, 0); n \geq 0\}$, $B = \{v_n := (n, 2^{-n}); n \geq 0\}$; and take the function (from M to $R_+ \cup \infty$): $\varphi(z) = \infty$, if $z \in A$ and $\varphi(z) = 0$, if $z \in B$. For the moment, (1b) is retainable, because each point of B is (strongly) $(\leq)$-maximal (cf. Section 2). Moreover, (2a) is retainable too, in view of $u_n \leq v_n$, for all $n \geq 0$. Unfortunately, the structure $(M, \leq)$ cannot satisfy (1a); for, e.g., the ascending sequence (u_n) is not bounded above; so that, Proposition 2 is not applicable to $(M, \leq)$ and φ. Nevertheless (in compensation to this), $(M, \leq)$ fulfills (3a); wherefrom, Theorem 5 applies to the same data. (The argument is based on the above property of B; we do not give details).

Summing up, Theorem 5 includes in a strict sense Proposition 2; but, this is realized at the level of *the same* structure. For a genuine answer to the posed question, a variant of Example 6 involving *many* sub-structures of $(M, \leq)$ is needed. Concerning this aspect, notice that roughly speaking, (1a) acts as a *global* completeness of $(M, \leq)$; while (3a), as a *local* completeness of the same, with respect to the function φ. So, if the latter property is strictly larger than the former one (modulo these sub-structures), we are done. This tells us that a promising way of constructing such examples is related to completeness type techniques, as developed in Sempi [22] and Wolk [29]; see also Amato [2], Jinag and Cho [16], Liu [19] and Sullivan [23].

A basic particular case of these facts corresponds to the construction in Section 2. Precisely, let $d : M \times M \to R_+ \cup \infty$ be a reflexive (generalized) pseudometric (over M); and $\varphi_d : M \to R_+ \cup \infty$, its associated by (a2) function. Clearly, (1b) holds in this context; and the almost regularity (modulo φ_d) condition (2a) is just the one in (2d). Putting these together, it results the following maximality statement involving these data.

7 Theorem. *Assume that $(M, \leq)$ is sequentially inductive (modulo φ_d) and weakly regular (modulo d). Then, for each $u \in M$ there exists a $(\leq, d)$-maximal $v \in M$ with $u \leq v$.*

As before, the sequential inductivity (modulo φ_d) holds under (1a); wherefrom, Theorem 7 includes Proposition 3. An interesting question is that of the reciprocal inclusion being also retainable; further aspects will be treated elsewhere.

Now, the pseudometric setting above is also appropriate for discussing the sequential inductivity (modulo φ_d) condition. This will necessitate some conventions. Denote by $\mathcal{S}(\mathcal{M})$, the class of all sequences (x_n) in M. By a (sequential) *convergence structure* on M we mean, as in Kasahara [18], any part $\mathcal{C}$ of $\mathcal{S}(\mathcal{M}) \times \mathcal{M}$ with the properties

(a3) $x_n = x, \forall n \in N \Longrightarrow ((x_n); x) \in \mathcal{C}$

(b3) $((x_n); x) \in \mathcal{C} \Longrightarrow ((y_n); x) \in \mathcal{C}$, for each subsequence (y_n) of (x_n).

In this case, $((x_n); x) \in \mathcal{C}$ will be denoted $x_n \xrightarrow{\mathcal{C}} x$; and referred to as: x is the $\mathcal{C}$-*limit* of (x_n). When x is generic in this convention, we say that (x_n) is $\mathcal{C}$-*convergent*. Assume that we fixed such an object and let $(\leq, d)$ be taken as before. Call the subset Z of M, $(\leq)$-*closed* (modulo $\mathcal{C}$) when the $\mathcal{C}$-limit of each ascending sequence in Z is an element of it. Further, let us say that $(\leq)$ is *self-closed* (modulo $\mathcal{C}$) when $M(x, \leq)$ is $(\leq)$-closed (modulo $\mathcal{C}$), for each $x \in M$; or, equivalently: the $\mathcal{C}$-limit of each ascending sequence is an upper bound of it. Finally, term the (reflexive) pseudometric d, $(\leq)$- *complete* (modulo $\mathcal{C}$) when each ascending d-Cauchy sequence is $\mathcal{C}$-convergent.

We may now give an appropriate answer to the posed question.

8 Theorem. *Suppose that $(\leq)$ is self-closed (modulo $\mathcal{C}$), d is $(\leq)$-complete (modulo $\mathcal{C}$) and $(M, \leq)$ is weakly regular (modulo d). Then, conclusions of Theorem 7 are retainable.*

Proof. We claim that, under the accepted conditions, Theorem 7 is applicable to $(M, \leq; d)$; precisely, that $(M, \leq)$ is sequentially inductive (modulo φ_d). Let (x_n) be an ascending sequence with $\varphi_d(x_n) \to 0$. In particular, it is an ascending d-Cauchy sequence; so that (by the $(\leq)$-completeness (modulo $\mathcal{C}$) of d) $x_n \xrightarrow{\mathcal{C}} y$, for some $y \in M$. Combining with the self-closedness (modulo $\mathcal{C}$) of $(\leq)$ yields $x_n \leq y$, for all n; and this proves the claim. QED

Now, a good choice for our convergence structure is $\mathcal{C} = (\xrightarrow{d})$ [introduced as: $x_n \xrightarrow{d} x$ whenever $d(x_n, x) \to 0$ as $n \to \infty$; and called the *primal convergence* structure attached to d]. For, if (in addition) d is *triangular* $[d(x,z) \leq d(x,y) + d(y,z), \forall x, y, z \in M]$, Theorem 8 includes the statement by Kang and Park [17]; which, in turn, includes the maximality principle by Granas and Horvath [13]. Note incidentally, that all applications (based on Theorem 8) discussed by these authors may be also handled via Ekeland's variational principle [11]. Further aspects of structural nature may be found in Gajek and Zagrodny [12]; see also Brunner [7] and Turinici [27].

4 Zorn maximality principles

Let us now return to the setting of Section 2. Precisely, given the nonempty set M, take a reflexive (generalized) pseudometric $(x, y) \mapsto d(x, y)$ over it. Remember that, when d is (in addition) sufficient, the point $v \in M$ assured by Proposition 3 is strongly $(\leq)$-maximal. In the absence of this property, we may ask whether the weaker counterpart of this concept is holding: $w \in M, z \leq w \Longrightarrow z \geq w$ (referred to as: z is *$(\leq)$-maximal*). To establish a maximality result of this type, we need some conventions. Let $\text{dist}(.,.)$ stand for the associated (to d) point to set distance function $[\text{dist}(x, Z) = \inf\{d(x, z); z \in Z\}, x \in M, Z \subseteq M]$. The working hypothesis to be considered is

(4a) $((M, \leq)$ is almost weakly regular (modulo d)) $\forall x \in M$, $\forall \varepsilon > 0$, $\exists y = y(x, \varepsilon) \geq x$: $y \leq u \leq v \Longrightarrow \text{dist}(u, M(v, \leq)) \leq \varepsilon$.

This is nothing else than the condition (2a) with respect to the function

(a4) $\psi_d(x) = \sup\{\text{dist}(u, M(v, \leq)); x \leq u \leq v\}, \quad x \in M.$

Further, let $(\xleftarrow{d})$ stand for the *dual convergence* attached to d [introduced as: $x \xleftarrow{d} x_n$ if and only if $d(x, x_n) \to 0$ as $n \to \infty$].

9 Corollary. *Assume that $(M, \leq)$ is sequentially inductive (modulo ψ_d) and almost weakly regular (modulo d); and $(\leq)$ is self-closed (modulo $(\overset{d}{\longleftarrow})$). Then, for each $u \in M$ there exists a $(\leq)$-maximal $v \in M$ with $u \leq v$; i.e., $(\leq)$ appears as a Zorn quasi-order.*

PROOF. By the admitted hypotheses (on $(M, \leq)$) it follows via Theorem 5 that, for the starting point $u \in M$ there exist another one $v \in M$ with $u \leq v$ and $\psi_d(v) = 0$ (hence v is $(\leq, \psi_d)$-maximal). We now claim that the generic implication is valid: $(\forall z \in M)$ $\psi_d(z) = 0 \Longrightarrow z$ is $(\leq)$-maximal. (And from this, the conclusion is clear). For, take some $w \geq z$. Since $[\text{dist}(z, M(y, \leq)) = 0, \forall y \geq w]$, it is not hard to construct an ascending sequence (x_n) in $M(w, \leq)$ with $z \overset{d}{\longleftarrow} x_n$. But then, the choice of $(\leq)$ yields $x_n \leq z, \forall n$; hence $w \leq z$, as claimed. QED

The following completion of this fact is to be noted. Call the (ascending) sequence (x_n), *eventually d-asymptotic* when $[\forall n, \forall \varepsilon > 0, \exists (p, q)\colon n \leq p < q, d(x_p, x_q) < \varepsilon]$. This is a weaker form of the d-asymptotic property introduced in Section 2. Precisely, the generic implication is clear: (for each sequence) d-asymptotic $\Longrightarrow$ eventually d-asymptotic; but, the converse is not in general valid. Let us now consider the condition

(4b) ($(M, \leq)$ is eventually regular (modulo d))
each ascending sequence is eventually d-asymptotic.

We claim that this is a sufficient one for (4a) above. In fact, assume this were not true; then, there must be some pair $x \in M, \varepsilon > 0$ with $[\forall y \geq x, \exists (u, v) : y \leq u \leq v, \text{dist}(u, M(v, \leq)) \geq \varepsilon]$. Put $x_0 = x$; with $y = x_0$ we get a couple (x_1, x_2) with $x_0 \leq x_1 \leq x_2$, $\text{dist}(x_1, M(x_2, \leq)) \geq \varepsilon$. Further, with $y = x_2$ there exist (x_3, x_4) with $x_2 \leq x_3 \leq x_4$, $\text{dist}(x_3, M(x_4, \leq)) \geq \varepsilon$; and so on. This finally gives us an ascending sequence (x_n) with: $d(x_{2p+1}, x_k) \geq \varepsilon$, for all $k > 2p + 1$ and all $p \geq 0$. So, for the ascending sequence $(y_n = x_{2n+1})$ we must have $d(y_p, y_q) \geq \varepsilon$, for all $p, q \geq 1$ with $p < q$; in contradiction with the eventual d-asymptotic property of it, assured by (4b); hence the claim. As a direct consequence, we have (cf. Turinici [26]):

10 Corollary. *Assume that $(M, \leq)$ is sequentially inductive and eventually regular (modulo d); and $(\leq)$ is self-closed (modulo $(\overset{d}{\longleftarrow})$). Then, conclusions of Corollary 9 are retainable.*

In particular, assume that

(4c) M is $(\leq, d)$-compact (modulo $(\overset{d}{\longleftarrow})$): each ascending sequence
has a d-Cauchy convergent (modulo $(\overset{d}{\longleftarrow})$) subsequence.

The first half of this (related to the d-Cauchy property) gives at once (4b). And the second half of the same (involving the convergence (modulo $(\overset{d}{\longleftarrow})$) property) gives (1a) if one admits the self-closedness (modulo $(\overset{d}{\longleftarrow})$) of $(\leq)$. We therefore deduced:

11 Corollary. *Assume that M is $(\leq, d)$-compact and $(\leq)$ is self-closed (modulo $(\overset{d}{\longleftarrow})$). Then, for each $u \in M$ there exists a $(\leq)$-maximal $v \in M$ with $u \leq v$.*

Note that, when d is (in addition) triangular and *symmetric* $[d(x, y) = d(y, x), \forall x, y \in M]$, the regularity condition (4c) reads in the standard way

(4d) M is $(\leq, d)$-compact:
each ascending sequence has a convergent subsequence;

and the corresponding version of Corollary 11 includes the metrical portion of a related statement in Ward [28]. Some applications of these to mapping theory may be found in Park and Yie [21].

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