Note di Matematica
Note Mat. 28 (2008), n. 2, 1-18
ISSN 1123-2536, e-ISSN 1590-0932
DOI 10.1285/i15900932v28n2p1

On a class of rational matrices and interpolating polynomials related to the discrete Laplace operator

Pierpaolo Vivo
ICTP - Abdus Salam International Centre for Theoretical Physics
Strada Costiera 11, 34151 Trieste, Italy
pvivo@ictp.it

Mario Casartelli
Dipartimento di Fisica, Universitá degli Studi di Parma
Parco Area delle Scienze 7a, 43100 Parma (Italy)
INFN, Gruppo collegato di Parma
CNR-INFM, Parma
casartelli@fis.unipr.it

Luca Dall'Asta
ICTP - Abdus Salam International Centre for Theoretical Physics
Strada Costiera 11, 34151 Trieste, Italy
dallasta@ictp.it

Alessandro Vezzani
Dipartimento di Fisica, Universitá degli Studi di Parma
Parco Area delle Scienze 7a, 43100 Parma (Italy), CNR-INFM, Parma.
Dipartimento di Fisica, Universitá di Modena e Reggio Emilia,
Via G. Campi 213A, 41000 Modena (Italy), CNR-INFM S3
alessandro.vezzani@fis.unipr.it

Received: 19/06/2006; accepted: 20/02/2007.

Abstract. Let $\tilde{\nabla}^2$ be the discrete Laplace operator acting on functions (or rational matrices) $f : \mathbf{Q}_L \to \mathbb{Q}$, where $\mathbf{Q}_L$ is the two dimensional lattice of size L embedded in $\mathbb{Z}_2$. Consider a rational $L \times L$ matrix $\mathcal{H}$, whose inner entries $\mathcal{H}_{ij}$ satisfy $\tilde{\nabla}^2 \mathcal{H}_{ij} = 0$. The matrix $\mathcal{H}$ is thus the classical finite difference five-points approximation of the Laplace operator in two variables. We give a constructive proof that $\mathcal{H}$ is the restriction to $\mathbf{Q}_L$ of a discrete harmonic polynomial in two variables for any $L > 2$. This result proves a conjecture formulated in the context of deterministic fixed-energy sandpile models in statistical mechanics.

Keywords: rational matrices, discrete Laplacian, discrete harmonic polynomials, sandpile

MSC 2000 classification: 11C99 (Polynomials and matrices)

Introduction and Motivation

An interesting class $\mathcal{M}_L$ of $L \times L$ matrices $\mathcal{H}$ with rational entries and a related vector space of polynomials in two variables arise in some theoretical physics models, the so-called *deterministic fixed-energy sandpiles* (DFES) with Bak-Tang-Wiesenfeld (BTW) toppling rule [3].

Introduced for the first time in [4] by imposing a global energy conservation constraint on its dissipative counterpart [2], DFES is a deterministic cellular automaton, in which two-dimensional configurations (represented by square matrices with integer elements $z_{ij}(t)$) evolve in discrete time steps t according to a precise parallel updating rule.

The main feature of DFES is that, in contrast with the dissipative model, only a small part of an a priori huge configuration space is dynamically explored, and the system enters a periodic orbit after a surprisingly short transient. This is a clear indication of the existence of many *hidden conservation laws* (HCL) which split the wide configuration space into dynamically intransitive, and thus much smaller subspaces [3].

Few of those HCL were identified in a non-systematic way in [1] and can be represented in the form:

$$\Phi_L[f](t) = \left[\sum_{i,j} f(i,j) z_{ij}(t) \right] \mod L \tag{1}$$

where the sum runs over the integer coordinates of the two-dimensional $L \times L$ lattice sites, $z_{ij}(t)$ is the integer value taken by the entry (i,j) at time t and $f(i,j)$ is a $L \times L$ matrix with rational entries. The interest is then in characterizing the *generating functions* (GF) of HCL, i.e. the class of inequivalent matrices f such that $\Phi_L[f](t)$ is a HCL ($\Phi_L[f](t+1) = \Phi_L[f](t)$ for all t).

Bagnoli et al. [1] gave the following three GF: $f_1 = i$, $f_2 = j$ and $f_3 = i^2 - j^2$. An intriguing observation is that, when thought as functions on the whole $\mathbb{R}^2$ ($f(x,y) : \mathbb{R}^2 \mapsto \mathbb{R}$), those three GF belong to a special vector space of polynomials in two variables, which we call *discrete harmonic polynomials* (see Def. 5). It is then appealing to conjecture that this should be a general feature of any GF of a HCL.

In fact, an exhaustive characterization of GF has been given in [3] from a completely different perspective, i.e. *without any reference to polynomials*, but working simply on the matrix representation of those GF.

It was proven in [3] that a functional of the form (1) is a HCL if and only if its GF is a *inner-harmonic matrix of size* L (see Def. 4)[1].

[1]In appendix B of [3] the necessary and sufficient condition is expressed in terms of K-harmonicity, and strictly speaking this is not equivalent to inner-harmonicity. However, it can be proved that for every K-harmonic function there exists an inner-harmonic function which belongs to the same equivalence class, i.e. generates an equivalent HCL. Thus, it is not restrictive to work with inner-harmonic matrices, as we will do from now on.

The purpose of this paper is to provide a rigorous link between the exact characterization of HCL in terms of matrices [3] and the conjectured polynomial form for any GF. More precisely, we will prove that every inner-harmonic matrix of size L (i.e. any GF of a HCL in the sandpile context) can be represented (non uniquely) as the restriction to the two-dimensional discrete lattice of a discrete harmonic polynomial in two variables.

The paper is organized as follows. In Section 1 we set up notations and basic definitions, providing in particular the notions of i) *inner-harmonic matrix of size* L (Def. 4), in terms of the well-known five-points formula for the discretization of the Laplace operator on a 2d lattice, and ii) *discrete harmonic polynomial* (Def. 5). In Section 2, we enunciate the main theorem and provide the algorithmic procedure for finding the discrete harmonic polynomial which interpolates any given inner-harmonic matrix of size $L \geq 3$. In the same section, we provide a stepwise example of application, together with pointers to subsequent lemmas needed for the proof. Section 3 is devoted to conclusive remarks and hints for future works, while a basis of discrete harmonic polynomials up to degree 9 is given in the Appendix.

1 Definitions

We define $\mathbf{Q}_L$ as the two dimensional lattice embedded in $\mathbb{Z}_2$, i.e.:

$$\mathbf{Q}_L = \{(i,j) \in \mathbb{Z}_2 | 0 \leq i,j \leq L-1\} \tag{2}$$

1 Definition. The *inner sublattice* $\mathbf{Q}_L^\dagger$ of $\mathbf{Q}_L$ is the set:

$$\mathbf{Q}_L^\dagger = \{(i,j) \in \mathbf{Q}_L | 1 \leq i,j \leq L-2\} \tag{3}$$

The discrete Laplace operator is defined as the classical finite difference five-points second order formula for the approximation of the Laplace operator:

2 Definition. Let $f : \mathbf{Q}_L \to \mathbb{Q}$. The discrete Laplace operator $\tilde{\nabla}^2$ acts on f as:

$$(\tilde{\nabla}^2 f^\dagger)(i,j) = 4f(i,j) - f(i-1,j) - f(i+1,j) - f(i,j-1) - f(i,j+1) \tag{4}$$

where $f^\dagger := f|_{\mathbf{Q}_L^\dagger}$.

The generalization to functions $f : \mathbb{R}^2 \to \mathbb{R}$ is straightforward (consider $f \equiv f^\dagger$ in this case).

3 Definition. Let $\mathcal{M}_L$ be the set of rational $L \times L$ matrices and $\mathfrak{F} = \{f | f : \mathbf{Q}_L \to \mathbb{Q}\}$. We define the invertible map $\Psi : \mathfrak{F} \to \mathcal{M}_L$ (*L-correspondence*) through the following:

$$\Psi(h) := \mathcal{H} \tag{5}$$

where $h(-1+j,-i+L) := \mathcal{H}_{i,j}$.

Through Ψ, the lower left corner of $\mathcal{H}$ is mapped to the point $(0,0)$.

4 Definition. A $L \times L$ rational matrix $\mathcal{H}_1$ is called *inner-harmonic matrix of size* L $(L > 2)$ if the following property holds $(h_1 = \Psi^{-1}(\mathcal{H}_1))$:

$$(\tilde{\nabla}^2 h_1^\dagger)(i,j) = 0 \tag{6}$$

as in the following example, where we restrict for simplicity to integer entries:

$$\tilde{\mathcal{H}} = \begin{pmatrix} 2 & 0 & 0 & 1 & 0 & 1 & 2 \\ 0 & 2 & 1 & 2 & 0 & 2 & 1 \\ 1 & 7 & 0 & 6 & -4 & 6 & 2 \\ 1 & 25 & -14 & 26 & -28 & 24 & 1 \\ 2 & 106 & -107 & 140 & -158 & 117 & 2 \\ 2 & 504 & -660 & 799 & -861 & 600 & 1 \\ 1 & 2568 & -3836 & 4577 & -4685 & 3143 & 0 \end{pmatrix} \tag{7}$$

5 Definition. A polynomial $P(x,y)$ is called *discrete harmonic polynomial* if $(\tilde{\nabla}^2 P)(x,y) = 0 \quad \forall (x,y) \in \mathbb{R}^2$.

Examples of discrete harmonic polynomials are $P_1(x,y) = x^2 - y^2, P_2(x,y) = x^3 - 3xy^2, P_3(x,y) = xy$.

The set of discrete harmonic polynomials of degree g will be denoted as $\mathbb{D}_g^\star$.

6 Definition. We say that a polynomial $P(x,y)$ *interpolates* a $L \times L$ matrix $\mathcal{H}$ if $P(i,j) = h(i,j)$, where $\Psi(h) = \mathcal{H}$. In this case, we write $P \doteq \mathcal{H}$.

Note that:

7 Remark. Discrete harmonic polynomials are generally *not* harmonic in $\mathbb{R}^2$, i.e. solutions of the continuum Laplace equation $\nabla^2 P = 0$. Generally speaking, every polynomial $\mathcal{P}(x,y)$ in two variables belongs to one of the following classes:

- $\mathcal{P}(x,y)$ is neither harmonic nor discrete harmonic. Example: $\mathcal{P}(x,y) = x^3 + y^3$
- $\mathcal{P}(x,y)$ is harmonic but not discrete harmonic. Example: $\mathcal{P}(x,y) = x^4 - 6x^2y^2 + y^4$
- $\mathcal{P}(x,y)$ is discrete harmonic but not harmonic. Example: $\mathcal{P}(x,y) = x^4 - 2x^2 - 6x^2y^2 + y^4$
- $\mathcal{P}(x,y)$ is both harmonic and discrete harmonic. Example: $\mathcal{P}(x,y) = xy$

8 Remark. Given a discrete harmonic polynomial $P(x,y)$, it obviously interpolates an inner-harmonic matrix $\mathcal{H}_L$ on $\mathbf{Q}_L$. For example, the polynomial $P(x,y) = x^3 - 3xy^2$ interpolates the following matrix on $\mathbf{Q}_7$:

$$\mathcal{H}_7 = \begin{pmatrix} 216 & 198 & 144 & 54 & -72 & -234 & -432 \\ 125 & 110 & 65 & -10 & -115 & -250 & -415 \\ 64 & 52 & 16 & -44 & -128 & -236 & -368 \\ 27 & 18 & -9 & -54 & -117 & -198 & -297 \\ 8 & 2 & -16 & -46 & -88 & -142 & -208 \\ 1 & -2 & -11 & -26 & -47 & -74 & -107 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \tag{8}$$

The converse is not trivial for any $L > 2$: while it is straightforward to find an interpolating polynomial $\Phi(x,y)$ for any given inner-harmonic matrix through any of the known Polynomial Interpolation formulas in two variables [5], the resulting Φ is generally *not* discrete harmonic in $\mathbb{R}^2$ (and incidentally neither harmonic). This can be seen easily by referring to the widely used Bilinear Interpolation formula (see e.g. [8]), the extension to the two-dimensional lattice of the well-known Lagrange interpolation formula in 1d:

$$\Phi(x,y) = \sum_{h,k} z_{hk} \prod_{\substack{j=0 \\ j\neq h}}^{L-1} \frac{x-j}{h-j} \prod_{\substack{r=0 \\ r\neq k}}^{L-1} \frac{y-r}{k-r} \tag{9}$$

where $z_{hk} = \mathcal{H}_{h,k}$, the sum runs over the sites of the matrix and the products over rows and columns respectively. Note that $\deg(\Phi) = 2(L-1)$.

It is then possible to interpolate the following simple inner-harmonic matrix of size $L = 4$:

$$\mathcal{H}_4 = \begin{pmatrix} 27 & 18 & -9 & -54 \\ 8 & 2 & -16 & -46 \\ 1 & -2 & -11 & -26 \\ -3 & 0 & 0 & 0 \end{pmatrix} \tag{10}$$

The bilinear interpolating polynomial is the following:

$$\begin{aligned} \Phi_{\mathcal{H}_4}(x,y) = -3 + \frac{11}{2}x - 3x^2 + \frac{1}{2}x^3 + \frac{11}{2}y - \frac{121}{12}xy + \\ \frac{5}{2}x^2y - \frac{11}{12}x^3y + -3y^2 + \frac{11}{2}xy^2 - 3x^2y^2 + \frac{1}{2}x^3y^2 \\ + \frac{3}{2}y^3 - \frac{11}{12}xy^3 + \frac{1}{2}x^2y^3 - \frac{1}{12}x^3y^3 \end{aligned} \tag{11}$$

and a straightforward calculation yields $\tilde{\nabla}^2(\Phi_{\mathcal{H}_4})(x,y) \neq 0$ in $\mathbb{R}^2$.

In the following section, we shall provide the enunciation of the main result, a stepwise example of application of the algorithm, and a constructive proof of the main theorem.

2 Interpolation by discrete harmonic polynomials: main result and algorithm

We enunciate our main result:

9 Theorem. *Let $\mathcal{H}$ be an inner-harmonic matrix of size $L > 2$. There exists a discrete harmonic polynomial $P(x, y)$ of degree less or equal to $2(L-1)$ such that P interpolates $\mathcal{H}$ on $\mathbf{Q}_L$.*

Before getting to the technical points, it is informative to provide an example of how our algorithmic procedure roughly works.

Let us consider the inner-harmonic matrix $\mathcal{H} := \mathcal{H}_4$ in (10).

`First step:`

Isolate the lower left (3×3) minor $\mathcal{H}^{(1)} \subset \mathcal{H}$:

$$\mathcal{H}^{(1)} = \begin{pmatrix} 8 & 2 & -16 \\ 1 & -2 & -11 \\ -3 & 0 & 0 \end{pmatrix} \tag{12}$$

`Second step:`

Apply Lemma 13 and find a discrete harmonic polynomial[2] $P^{(1)}(x, y) \doteq \mathcal{H}^{(1)}$:

$$\begin{aligned} P^{(1)}(x,y) = -3 + \frac{15}{4}x - \frac{1}{8}x^2 - \frac{3}{4}x^3 + \frac{1}{8}x^4 + \frac{15}{4}y + \\ - \frac{27}{4}xy - \frac{3}{4}x^2y - \frac{1}{8}y^2 + \frac{9}{4}xy^2 - \frac{3}{4}x^2y^2 + \frac{1}{4}y^3 + \frac{1}{8}y^4 \end{aligned} \tag{13}$$

`Third step:`

Evaluate $P^{(1)}(x, y)$ on the lattice $\mathbf{Q}_L \equiv \mathbf{Q}_4$, obtaining the matrix $\hat{\mathcal{H}}_4$:

$$\hat{\mathcal{H}}_4 = \begin{pmatrix} 24 & \boxed{18} & -9 & -57 \\ 8 & 2 & -16 & -46 \\ 1 & -2 & -11 & \boxed{-26} \\ -3 & 0 & 0 & -3 \end{pmatrix} \tag{14}$$

[2]Note that this polynomial does NOT coincide with the bilinear interpolating polynomial we would get for the same matrix.

Note that i) $\hat{\mathcal{H}}_4 \neq \mathcal{H}$ ii) the sites $(1,3) = 18$ and $(3,1) = -26$ are uniquely determined by the discrete harmonicity requirement and thus coincide in the two matrices.

`Fourth step:`
In order to amend the other mismatching entries along the border, compute the four (L)-Polynomials (Lemma 14):

$$\begin{aligned}
\xi_1(x,y) = {} & 384x - 656x^2 + 375x^3 - 65x^4 - 3x^5 + x^6 - 516y + 332xy + \\
& + 465x^2y - 440x^3y + 105x^4y - 6x^5y + 776y^2 - 1095xy^2 + \\
& + 360x^2y^2 + 30x^3y^2 - 15x^4y^2 - 225y^3 + 460xy^3 - 210x^2y^3 + \\
& + 20x^3y^3 - 55y^4 - 15xy^4 + 15x^2y^4 + 21y^5 - 6xy^5 - y^6 \qquad (15)\\
\xi_2(x,y) = {} & 240x - 386x^2 + 135x^3 + 25x^4 - 15x^5 + x^6 - 168y - 152xy + \\
& + 555x^2y - 280x^3y + 15x^4y + 6x^5y + 326y^2 - 255xy^2 + \\
& - 180x^2y^2 + 150x^3y^2 - 15x^4y^2 - 195y^3 + 260xy^3 - 30x^2y^3 + \\
& - 20x^3y^3 + 35y^4 - 75xy^4 + 15x^2y^4 + 3y^5 + 6xy^5 - y^6 \qquad (16)\\
\xi_3(x,y) = {} & 516x - 776x^2 + 225x^3 + 55x^4 - 21x^5 + x^6 - 348y - 332xy + \\
& + 1095x^2y - 460x^3y + 15x^4y + 6x^5y + 656y^2 - 465xy^2 + \\
& - 360x^2y^2 + 210x^3y^2 - 15x^4y^2 - 375y^3 + 440xy^3 - 30x^2y^3 + \\
& - 20x^3y^3 + 65y^4 - 105xy^4 + 15x^2y^4 + 3y^5 + 6xy^5 - y^6 \qquad (17)\\
\xi_4(x,y) = {} & 1644x - 2852x^2 + 1305x^3 - 35x^4 - 69x^5 + 7x^6 + \\
& - 1644y + 3225x^2y - 2130x^3y + 345x^4y + 2852y^2 + \\
& - 3225xy^2 + 690x^3y^2 - 105x^4y^2 - 1305y^3 + 2130xy^3 + \\
& - 690x^2y^3 + 35y^4 - 345xy^4 + 105x^2y^4 + 69y^5 - 7y^6 \qquad (18)
\end{aligned}$$

Those ξ_k have the remarkable properties to be i) discrete harmonic in $\mathbb{R}^2$ ii) almost everywhere 0 on $\mathbf{Q}_L$, except one single entry (two for ξ_4). In particular, $\xi_k \doteq \hat{\xi}_k$, where:

$$\hat{\xi}_1 = \begin{pmatrix} \gamma_1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \qquad (19)$$

$$\hat{\xi}_2 = \begin{pmatrix} 0 & 0 & 0 & \gamma_2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \qquad (20)$$

$$\hat{\xi}_3 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \gamma_3 \end{pmatrix} \tag{21}$$

$$\hat{\xi}_4 = \begin{pmatrix} 0 & 0 & \gamma_4 & 0 \\ 0 & 0 & 0 & -\gamma_4 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \tag{22}$$

where, for the particular choice of the basis polynomials used to build up the $\xi(x, y)$ (see Lemma 14 for details), we have

$$(\gamma_1, \gamma_2, \gamma_3, \gamma_4) = (-720, -720, 720, -720).$$

`Fifth step:`
Define the sought interpolating polynomial $P(x, y)$ for $\mathcal{H}$ as:

$$P(x, y) = P^{(1)}(x, y) + \sum_{k=1}^{4} z_k \xi_k(x, y) \tag{23}$$

where z_k are parameters to be determined, and compute $P(x, y)$ on $\mathbf{Q}_L$:

$$\hat{\mathcal{P}} = \begin{pmatrix} 24 - 720z_1 & 18 & -9 - 720z_4 & -57 - 720z_2 \\ 8 & 2 & -16 & -46 + 720z_4 \\ 1 & -2 & -11 & -26 \\ -3 & 0 & 0 & -3 + 720z_3 \end{pmatrix} \tag{24}$$

`Sixth step:`
Compute (z_1, z_2, z_3, z_4) by requiring $\hat{\mathcal{P}} \equiv \mathcal{H}$:

$$\begin{cases} 24 - 720z_1 & = 27 \\ -57 - 720z_2 & = -54 \\ -3 + 720z_3 & = 0 \\ -9 - 720z_4 & = -9 \end{cases}$$

which gives:

$$\begin{cases} z_1 & = -1/240 \\ z_2 & = -1/240 \\ z_3 & = 1/240 \\ z_4 & = 0 \end{cases} \tag{25}$$

Substituting (25) back into (23), the final result is obtained:

$$
\begin{aligned}
P(x,y) = -3 &+ \frac{69}{20}x + \frac{59}{60}x^2 - \frac{31}{16}x^3 + \frac{25}{48}x^4 - \frac{1}{80}x^5 - \frac{1}{240}x^6 + \frac{103}{20}y + \\
&- \frac{533}{60}xy - \frac{7}{16}x^2y + \frac{13}{12}x^3y - \frac{7}{16}x^4y + \frac{1}{40}x^5y - \frac{119}{60}y^2 + \\
&+ \frac{95}{16}xy^2 - 3x^2y^2 + \frac{1}{8}x^3y^2 + \frac{1}{16}x^4y^2 + \frac{7}{16}y^3 - \frac{7}{6}xy^3 + \frac{7}{8}x^2y^3 + \\
&+ \frac{1}{12}x^3y^3 + \frac{23}{48}y^4 - \frac{1}{16}xy^4 - \frac{1}{16}x^2y^4 - \frac{7}{80}y^5 + \frac{1}{40}xy^5 + \frac{1}{240}y^6
\end{aligned}
\tag{26}
$$

Note the difference between (26) and (11) although they interpolate the very same matrix (10). The degree of P is $6 \equiv 2(L-1)$ as stated in Theorem 9.

This procedure can be iterated without difficulties up to interpolating inner-harmonic matrices of any size through a repeated application of Lemma 15.

Hereafter we shall provide several preliminary lemmas which are essential for the proof of the main result and have been hinted previously.

10 Lemma. *Let $k > 0$. Then $\mathbb{D}_k^\star$ is a vector space of dimension* 2.

Proof. First, we easily prove the following statement: let $\mathbb{P}_N$ be the set of two variables polynomials up to degree N and let $P_1(x,y) \in \mathbb{P}_N$. Then $P_2(x,y) = \tilde{\nabla}^2 P_1(x,y) \in \mathbb{P}_{N-2}$.

In fact, we notice that the following properties hold:

$$\tilde{\nabla}^2(ax^n + by^m) = a\tilde{\nabla}^2(x^n) + b\tilde{\nabla}^2(y^m) \qquad \text{Linearity} \tag{27}$$

$$\tilde{\nabla}^2(x^n y^m) = x^n\tilde{\nabla}^2(y^m) + y^m\tilde{\nabla}^2(x^n) \qquad \text{Leibniz rule} \tag{28}$$

Furthermore, for every one-variable monomial in x (or y), it is straightforward to prove the following:

$$\tilde{\nabla}^2 x^n = \begin{cases} -2\sum_{k=0}^{(n-2)/2} \binom{n}{2k} x^{2k} & \text{if n is even} \\ -2\sum_{k=0}^{(n-3)/2} \binom{n}{2k+1} x^{2k+1} & \text{if n is odd} \end{cases} \tag{29}$$

Therefore, applying the Laplace operator to a one-variable monomial of degree n, we obtain a linear combination of one-variables monomials up to degree $n-2$. Thanks to (27) and (28), we can conclude that the same holds also for two-variables polynomials. $\boxed{QED}$

It is well-known that $\mathbb{P}_N$ is a linear vector space, with $\dim(\mathbb{P}_N) = \sum_{n=0}^{N}(n+1) = \frac{(N+1)(N+2)}{2}$. According to the previous results, we call $\Pi_N : \mathbb{P}_N \to \mathbb{P}_{N-2}$ the following linear map:

$$\Pi_N(P(x,y)) = (\tilde{\nabla}^2 P)(x,y) \tag{30}$$

Then, we call $\mathbb{D}_N = \ker(\Pi_N)$, i.e. the following vector subspace of $\mathbb{P}_N$:

$$\mathbb{D}_N = \{P(x, y) \in \mathbb{P}_N : \quad (\tilde{\nabla}^2 P)(x, y) = 0 \quad \forall (x, y) \in \mathbb{R}^2\} \tag{31}$$

The elements of $\mathbb{D}_N$ are discrete harmonic polynomials. The dimension of $\mathbb{D}_N$ can be found simply applying the Rank-nullity theorem to the map Π_N:

$$\begin{aligned} \dim(\mathbb{D}_N) &= \dim(\mathbb{P}_N) - \dim(\mathbb{P}_{N-2}) \\ &= \frac{(N+1)(N+2)}{2} - \frac{N(N-1)}{2} = 2N + 1 \end{aligned} \tag{32}$$

Let $\mathbb{D}_k^\star$ be the following vector subspace of $\mathbb{D}_N$:

$$\mathbb{D}_k^\star = \{P(x, y) \in \mathbb{D}_N | \quad \text{P's degree is exactly } k \leq N\} \tag{33}$$

Obviously, $\dim(\mathbb{D}_k^\star) = \dim(\mathbb{D}_k) - \dim(\mathbb{D}_{k-1}) = 2$.

Therefore, for $k > 0$ we can always find two (and not more) linearly independent discrete harmonic polynomials, i.e. elements of $\mathbb{D}_N$, with the same degree k.

Following the standard algebraic procedure, it is quite easy to build up a complete basis $\mathcal{B}^\star = \{e_1^\star, \ldots, e_{2N+1}^\star\}$ for $\mathbb{D}_N$, starting from the canonical basis in $\mathbb{P}_N$:

$$\mathcal{B}_N = \{1, x, y, x^2, xy, y^2, \ldots, y^N\}$$

Throughout this paper, we will refer to the basis $\{U_k(x, y)\}$ listed in the Appendix.

11 Lemma. *For every square matrix with an arbitrary fixed rational contour, there exists one and only one inner-harmonic completion.*

Proof. Let $F_\Omega(i, j) : \mathbf{Q}_L \to \mathbb{Q}$ and let its $(4L - 4)$ border sites be forced to assume rational values z_k belonging to the set Ω.

In matrix form, we have:

$$F_\Omega = \begin{pmatrix} z_1 & z_2 & z_3 & \cdots & \cdots & z_L \\ z_{4L-4} & x_1 & x_2 & \cdots & x_{L-2} & z_{L+1} \\ z_{4L-5} & x_{L-1} & x_L & \cdots & x_{2(L-2)} & z_{L+2} \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ z_{3L-2} & z_{3L-3} & \cdots & \cdots & z_{2L} & z_{2L-1} \end{pmatrix} \tag{34}$$

The nested $(L-2) \times (L-2)$ submatrix $F_\Omega^\dagger$ has unknown entries $x_j \in \mathbb{Q}$.

We prove that, for each set Ω, there exists one and only one submatrix $F_\Omega^\dagger$ with rational entries such that $F_\Omega(i, j)$ is inner-harmonic.

If we impose the inner-harmonicity condition on F_Ω, we get the linear system $\hat{\mathbf{A}}\vec{x} = \vec{\eta}(\{z_k\})$, where $\hat{\mathbf{A}}$ is the following $(L-2)^2 \times (L-2)^2$ matrix:

$$\hat{\mathbf{A}} = 4\hat{\mathbf{I}} - \hat{\mathbf{H}} = 4\hat{\mathbf{I}} - \begin{pmatrix} \mathbf{H} & \mathbf{I} & \mathbf{0} & \cdots & \mathbf{0} & \mathbf{0} \\ \mathbf{I} & \mathbf{H} & \mathbf{I} & \cdots & \cdots & \mathbf{0} \\ \mathbf{0} & \mathbf{I} & \mathbf{H} & \mathbf{I} & \cdots & \mathbf{0} \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ \vdots & \vdots & \vdots & \vdots & \ddots & \mathbf{I} \\ \mathbf{0} & \cdots & \cdots & \cdots & \mathbf{I} & \mathbf{H} \end{pmatrix} \tag{35}$$

$\hat{\mathbf{I}}$ is the identity matrix $(L-2)^2 \times (L-2)^2$, $\mathbf{I}$ is the identity matrix $(L-2)\times(L-2)$, $\mathbf{0}$ is the null matrix $(L-2)\times(L-2)$ and $\mathbf{H}$ is a well-known matrix describing the Hamiltonian of nearest-neighbor hopping on a one-dimensional lattice (see [7] and references therein):

$$\mathbf{H} = \begin{pmatrix} 0 & 1 & 0 & 0 & \cdots & 0 \\ 1 & 0 & 1 & 0 & \cdots & 0 \\ 0 & 1 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ \vdots & \vdots & \vdots & \vdots & \ddots & 1 \\ 0 & 0 & 0 & \cdots & 1 & 0 \end{pmatrix} \tag{36}$$

The vector $\vec{\eta}$ depends on the fixed contour values. Its entries are of the following forms:

$$\eta_j = \begin{cases} z_\alpha + z_\beta & \text{if } x_j \text{ is a corner site of } F_\Omega^\dagger \\ z_\gamma & \text{if } x_j \text{ is a border site of } F_\Omega^\dagger\text{, but not a corner site} \\ 0 & \text{otherwise} \end{cases}$$

Since the matrix (35) is diagonal predominant [6], the system admits one and only one solution in $\mathbb{Q}$. $\boxed{QED}$

12 Corollary. *If $\Omega = \{0, \ldots, 0\}$, then F_Ω is the null matrix $L \times L$.*

13 Lemma. *Given a 3×3 inner-harmonic matrix $\mathcal{A}$, it is always possible to find a discrete harmonic polynomial $P(x, y)$ with rational coefficients and degree 4 such that $P \doteq \mathcal{A}$ on $\mathbf{Q}_3$.*

Proof. For $L = 3$, there are 8 sites along the contour. Choose the following set of discrete harmonic polynomials[3] (see Appendix):

$$\{U_0(x, y), \ldots, U_6(x, y)\} \cup \{U_8(x, y)\} \tag{37}$$

[3] Obviously, infinitely many other choices are equally possible.

The sought polynomial $P(x, y)$ satisfying the Lemma may be written as a linear combination of the polynomials in (37), with unknown coefficients α_j ($j = 1, \dots, 8$).

The condition that $P \doteq \mathcal{A}$ translates into a linear system with 8 equations in the unknowns α_j, whose matrix of coefficient for the choice (37) is:

$$\mathbf{M} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 & 1 & -1 \\ 1 & 0 & 2 & 0 & 4 & 0 & 8 & 8 \\ 1 & 1 & 0 & 0 & -1 & 1 & 0 & 1 \\ 1 & 1 & 2 & 2 & 3 & -11 & 2 & -15 \\ 1 & 2 & 0 & 0 & -4 & 8 & 0 & 16 \\ 1 & 2 & 1 & 2 & -3 & 2 & -11 & -9 \\ 1 & 2 & 2 & 4 & 0 & -16 & -16 & -72 \end{pmatrix} \tag{38}$$

The determinant of $\mathbf{M}$ is nonzero. Thus the polynomial interpolating the contour (and for Lemma 11 also the central site) always exists and has degree 4. $\boxed{QED}$

14 Lemma. *For every $L \geq 3$, there exist four discrete harmonic* (L)-polynomials *ξ_1, ξ_2, ξ_3 and ξ_4, whose degree is less or equal to $2L$, such that $\xi_k \doteq Z^{(k)}$, $k = 1, 2, 3, 4$. The entries of the matrices $Z^{(k)}$, $k = 1, 2, 3, 4$ are all 0 except:*

(1) the entry $(0, L)$ for $Z^{(1)}$;

(2) the entry (L, L) for $Z^{(2)}$;

(3) the entry $(L, 0)$ for $Z^{(3)}$;

(4) the entries $(L - 1, L)$ and $(L, L - 1)$ for $Z^{(4)}$;

As it was evident from the example of application, the (L)-polynomials have the following effect. Given a $(L + 1) \times (L + 1)$ inner-harmonic matrix $\mathcal{G}$ and a discrete harmonic polynomial $P(x, y)$ interpolating the lower-left *minor* $(L \times L)$ of $\mathcal{G}$, those polynomials neutralize the mismatch between 4 sites along the border of $\mathcal{G}$ and the values assumed by $P(x, y)$ on $\mathbf{Q}_{L+1}$.

We prove now the existence of $\xi_1(x, y)$. For the others, the procedure is completely analogue.

Consider a set of $4L$ linearly independent discrete harmonic polynomials $P_{k,s}(x, y)$, where $k = 1, \dots, 2L$ is the degree, and $s = 1, 2$ which do not contain the constant term.

We write the sought $\xi_1(x, y)$ in the form

$$\xi_1(x, y) = \alpha_1 P_{1,1}(x, y) + \alpha_2 P_{1,2}(x, y) + \cdots + \alpha_{4L} P_{2L,2}(x, y) \tag{39}$$

To determine the $4L$ unknowns α_j, we require that $\xi_1(x,y)$ should be zero on i) the border sites of the lower-left minor $\mathcal{M}$ of $Z^{(1)}$ ii) four other points in $\mathbb{Z}_2$, precisely: $(L-1,L),(L,L),(L,0),(L+1,L)$.

This translates into a linear homogeneous system $\mathcal{S}$ in $4L$ equations for the $4L$ unknowns $\alpha_1,\ldots,\alpha_{4L}$. Note that the site $(L,L-1)$ is automatically zero due to the harmonicity condition.

The first row of the matrix of coefficients for $\mathcal{S}$ is given by:

$$P_{1,1}(0,0),P_{1,2}(0,0),\ldots,P_{2L,2}(0,0) \tag{40}$$

and these values are all zero because the polynomials do not contain the constant term.

Thus, the determinant is zero and the homogeneous system has an infinite non-zero solutions set $\{\alpha_1,\ldots,\alpha_{4L}\}$. Since the polynomials $P_{k,s}(x,y)$ are linearly independent, the obtained polynomial cannot be identically zero by definition.

Now, let ξ_1 be defined by a non-zero solution $\{\alpha_1,\ldots,\alpha_{4L}\}$. Being zero along the contour of $\mathcal{M}$, it is zero inside M because of the Corollary 12.

Proof. It is also zero by the discrete harmonicity relation on sites (k,L), $k=1,\ldots,L+1$, and sites (L,k), $k=0,\ldots,L+1$. Instead, it is required to be nonzero on the site $(0,L)$. Indeed, we can prove that this is the case by contradiction. Assume that $\xi_1(0,L)=0$. We have:

$$\xi_1(j,k)=0, \quad k=L, \quad j=0,1,\ldots,L+1\ . \tag{41}$$

Let:

$$n=L/2+1 \qquad m=L/2 \qquad \text{for even L} \tag{42}$$

$$n=(L+1)/2+1 \qquad m=(L+1)/2-1 \qquad \text{for odd L} \tag{43}$$

Due to the harmonicity relation, there is an integer J, $0<J<L+1$, such that:

$$\xi_1(J,L-1+i)=0 \qquad \text{for every } i=1,\ldots,n$$

$$\xi_1(J,-k)=0 \qquad \text{for every } k=1,\ldots,m$$

This means that the one variable polynomial $\eta(y)=\xi_1(J,y)$ has $2L+1$ zeros: but this is absurd, since its degree in y is at most $2L$. Therefore $\xi_1(0,L)\neq 0$. $\boxed{QED}$

15 Lemma. *Let A be a inner-harmonic matrix of order L, and A' the $(L-1)\times(L-1)$ lower-left inner-harmonic minor of A. Let $\chi(x,y)$ be a discrete harmonic polynomial of degree h interpolating A'. Then, it is possible to define a discrete harmonic polynomial $\sigma(x,y)$, of degree $k=\max[2(L-1),\ h]$, interpolating A.*

PROOF. Define:

$$\begin{cases} s_1 & := \text{Site } (0, L-1) \\ s_2 & := \text{Site } (L-2, L-1) \\ s_3 & := \text{Site } (L-1, L-1) \\ s_4 & := \text{Site } (L-1, L-2) \\ s_5 & := \text{Site } (L-1, 0) \end{cases}$$

and denote $\chi(s_k) := \chi_k$ and $A(s_k) := a_k$ for simplicity.

We write the sought $\sigma(x,y)$ in the form:

$$\sigma(x,y) = \chi(x,y) + \sum_{k=1}^{4} z_k \xi_k(x,y) \ , \tag{44}$$

where the $\xi_k(x,y)$ are (L-1)-polynomials as defined in Lemma 14, and z_k are coefficients to be determined. The degree of each of the ξ_k is at most $2(L-1)$, confirming the statement of the Lemma about the degree of σ.

We note that $\sigma(x,y) \equiv \chi(x,y)$ on the sites of A', since all the (L-1)-polynomials assume value 0 there.

The values assumed by the polynomial χ on the North and East borders of $\mathbf{Q}_L$ are uniquely constrained by the harmonicity condition, except the five sites s_k. In general, $a_k \neq \chi_k$.

For example, for $L = 5$ we have the following schematic situation (compare with (14)):

$$A = \begin{pmatrix} \square & \blacksquare & \blacksquare & \Diamond & \square \\ \cdot & \cdot & \cdot & \cdot & \Diamond \\ \cdot & \cdot & \cdot & \cdot & \blacksquare \\ \cdot & \cdot & \cdot & \cdot & \blacksquare \\ \cdot & \cdot & \cdot & \cdot & \square \end{pmatrix} \tag{45}$$

where:

$$\begin{cases} \cdot & \Rightarrow \text{Sites in } A', \text{ where } \chi \doteq A \\ \blacksquare & \Rightarrow \text{Sites where } \chi \doteq A \text{ by harmonicity} \\ \square & \Rightarrow \text{Sites } (s_1, s_3, s_5) \text{ where } \chi \not\doteq A \\ \Diamond & \Rightarrow \text{Sites } (s_2, s_4) \text{ where } \chi \not\doteq A, \text{ but mutually constrained by harmonicity} \end{cases}$$

Indeed, the discrete harmonicity condition, applied to A and χ, requires that:

$$\chi_2 + \chi_4 = a_2 + a_4 \tag{46}$$

Given that $\xi_k(s_1) = 0$ for $k \neq 1$ and $\xi_1(s_1) = \gamma \neq 0$ (Lemma 14), we get from equation (44):

$$\sigma(s_1) = \chi_1 + z_1\gamma = a_1 \tag{47}$$

This determines z_1 as $z_1 = (a_1 - \chi_1)/\gamma$.

The same procedure applies to the sites s_3 and s_5, determining z_2 and z_3: note the shift of indices, reflecting the fact that we have five sites and only four (L-1)-polynomials.

In fact, the polynomial ξ_4 has to be nonzero simultaneously on both sites s_2 and s_4, and by harmonicity $\xi_4(s_2) = -\xi_4(s_4)$. This constraint, however, is compatible with the correct definition of z_4 and therefore of $\sigma(x,y)$.

Indeed, define $\omega = \xi_4(s_2) = -\xi_4(s_4)$. Equation (44) requires evidently that $\sigma(s_2) = \chi_2 + z_4\omega = a_2$ and $\sigma(s_4) = \chi_4 - z_4\omega = a_4$. Both equations are obviously satisfied by $z_4 = (a_2 - \chi_2)/\omega$ thanks to (46).

Thus, the coefficients $z_1, \ldots, z_4$ in (44) are uniquely determined and the polynomial $\sigma(x,y)$ interpolating A exists. $\boxed{QED}$

We are now able to provide a proof of Theorem 9.

Proof of Theorem 9. We only need an iterative (or "telescopic") application of previous results: starting from $\mathcal{H}$, we drop the upper row and last column on the right, defining the minor $\mathcal{H}^{(1)}$.

If we can find a discrete harmonic polynomial $\chi(x,y) \doteq \mathcal{H}^{(1)}$, such that $\deg(\chi) \leq 2(L-1)$, the Theorem follows via Lemma 15; otherwise, we drop the upper row and last column on the right of $\mathcal{H}^{(1)}$ again, and restart the procedure.

This process is consistent, because the minors iteratively defined continue to be inner-harmonic.

Suppose that we have finally found the minor $\mathcal{H}^{(n)}$ (whose size is $L-n$) of $\mathcal{H}^{(n-1)}$, admitting an interpolating polynomial $\chi_{(n)}(x,y)$ such that $\deg(\chi_{(n)}) \leq 2(L-1)$. By Lemma 15, the minor $\mathcal{H}^{(n-1)}$ can be interpolated, and so on, up to interpolating $\mathcal{H}$.

Since at least for $L = 3$ the interpolating polynomial always exists (Lemma 13), in the worst possible case the telescopic algorithm will start from the 3×3 lower left minor of $\mathcal{H}$, and will eventually produce the desired result by repeated applications of Lemma 15. $\boxed{QED}$

3 Final remarks

In this note, we have developed a "telescopic" technique to interpolate an inner-harmonic matrix of size L by a discrete harmonic polynomial of degree less or equal to $2(L-1)$.

The solution we have presented proves a conjecture about hidden conservation laws in the context of some statistical mechanics models, namely the so called fixed-energy sandpiles with deterministic BTW toppling rule.

We remark that the algorithmic procedure we devised should be regarded as a mere tool for the proof, and by no means is meant to provide a computationally efficient and robust interpolator for inner-harmonic matrices.

As a final point, we wish to give here a short survey on other related questions and problems which have not been addressed in this paper and could be worthy of further investigations.

(1) *Discrete harmonic polynomials of minimal degree:* the constructive procedure outlined in section 2 does not lead to an uniquely defined interpolating polynomial. A natural question to ask is what the minimal attainable degree of such a polynomial is, and how to build it up.

(2) *A related combinatorial problem:* Another class of matrices ($\mathcal{M}_L^\star$) with *integer* entries and closely related to $\mathcal{M}_L$ emerges in [3] and proves to be connected to deep symmetries of the evolving rule of that model.
The main features of $\mathcal{M}_L^\star$ are:

- Condition (6) holds *modulus* the size L of the matrix.
- Cyclical border conditions are imposed and condition (6) holds for border sites as well.
- Entries are bounded by an integer M.

An interesting problem in analytical combinatorics, with many possible consequences on the underlying physical issue, is to count the number of those matrices for fixed L and M.

Acknowledgements. We thank Dr. Igor Krasovsky, Dr. Ilia Krasikov and Dr. Steven Noble (Brunel University) for helpful comments. We also thank Elisa Garimberti (Brunel University) for a careful revision of the manuscript.

Appendix: list of Discrete Harmonic Polynomials

We report here a basis of discrete harmonic polynomials up to degree 9 that we used repeatedly throughout the paper:

$$U_0(x, y) = 1 \tag{48}$$
$$U_1(x, y) = y \tag{49}$$

$$U_2(x,y) = x \tag{50}$$
$$U_3(x,y) = xy \tag{51}$$
$$U_4(x,y) = x^2 - y^2 \tag{52}$$
$$U_5(x,y) = -3x^2y + y^3 \tag{53}$$
$$U_6(x,y) = x^3 - 3xy^2 \tag{54}$$
$$U_7(x,y) = x^3y - xy^3 \tag{55}$$
$$U_8(x,y) = x^4 - 2x^2 - 6x^2y^2 + y^4 \tag{56}$$
$$U_9(x,y) = 5x^4y - 10x^2y^3 - 10x^2y + y^5 \tag{57}$$
$$U_{10}(x,y) = x^5 - 10x^3y^2 + 5xy^4 - 10xy^2 \tag{58}$$
$$U_{11}(x,y) = x^5y - \frac{10}{3}x^3y^3 - \frac{10}{3}xy^3 + xy^5 \tag{59}$$
$$U_{12}(x,y) = -15x^4y^2 - 10x^4 + 10x^2 + 15x^2y^4 + 30x^2y^2 - y^6 + x^6 \tag{60}$$
$$U_{13}(x,y) = 35x^4y^3 + 70x^4y - 21x^2y^5 - 70x^2y^3 - 70x^2y + y^7 - 7x^6y \tag{61}$$
$$U_{14}(x,y) = -21x^5y^2 - 70x^3y^2 + 35x^3y^4 - 7xy^6 + 70xy^4 - 70xy^2 + x^7 \tag{62}$$
$$U_{15}(x,y) = -7x^5y^3 + 7x^3y^5 - \frac{70}{3}x^3y^3 - \frac{70}{3}xy^3 + xy^7 + 14xy^5 + x^7y \tag{63}$$
$$\begin{aligned} U_{16}(x,y) &= -140x^4y^2 + 70x^4y^4 - 140x^4 + 166x^2 - 28x^2y^6 + 280x^2y^4 + \\ &\quad + 560x^2y^2 + y^8 - 28y^6 + x^8 - 28x^6y^2 \end{aligned} \tag{64}$$
$$\begin{aligned} U_{17}(x,y) &= 126x^5y^4 - 252x^5y^2 - 84x^3y^6 - 840x^3y^2 + 840x^3y^4 + 9xy^8 + \\ &\quad - 252xy^6 + 1260xy^4 - 1026xy^2 + x^9 - 36x^7y^2 \end{aligned} \tag{65}$$
$$\begin{aligned} U_{18}(x,y) &= 840x^4y^3 + 126x^4y^5 + 1260x^4y - 252x^2y^5 - 36x^2y^7 - 840x^2y^3 + \\ &\quad - 1026x^2y + y^9 + 9x^8y - 84x^6y^3 - 252x^6y \end{aligned} \tag{66}$$

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